Kinematics of Particle Beams

Werner Herr, CERN

(http://cern.ch/Werner.Herr/CAS/CAS2009_Divonne/lectures/rel.pdf)

Kinematics of Particle Beams

SPECIAL RELATIVITY

(in less than 60 minutes ...)

Werner Herr, CERN

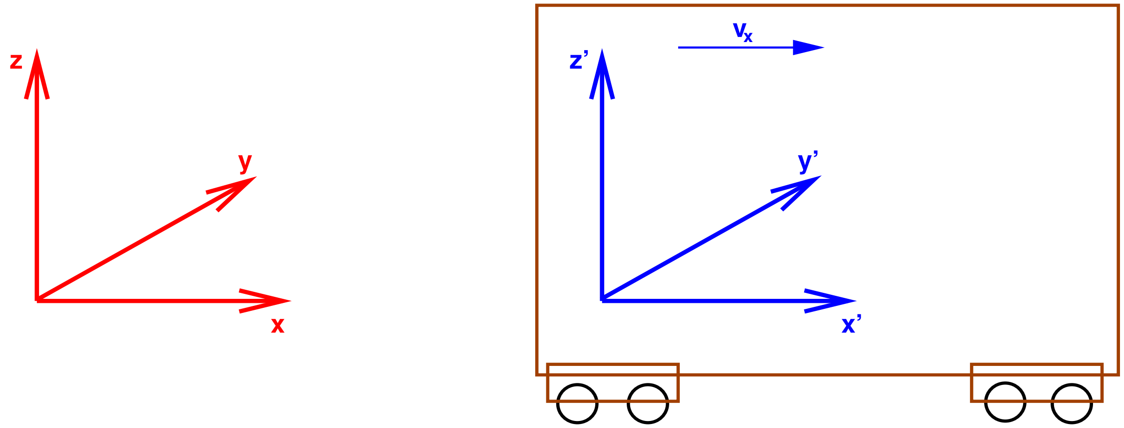
(http://cern.ch/Werner.Herr/CAS/CAS2009_Divonne/lectures/rel.pdf)

Small history

- **1678 (Römer, Huygens): Speed of light c is finite ($c \approx 3 \cdot 10^8$ m/s)**
- **1687 (Newton): Principles of Relativity**
- **1863 (Maxwell): Electromagnetic theory, light are waves moving through static ether**
- **1887 (Michelson, Morley): Speed c independent of direction, end of ether theory**
- **1905 (Einstein): Principles of Special Relativity**
- **1907 (Minkowski): Concepts of Spacetime**

Principles of Relativity (Newton)

- **Assume a frame at rest (F) and another frame moving in x-direction (F') with constant velocity $\vec{v} = (v_x, 0, 0)$**



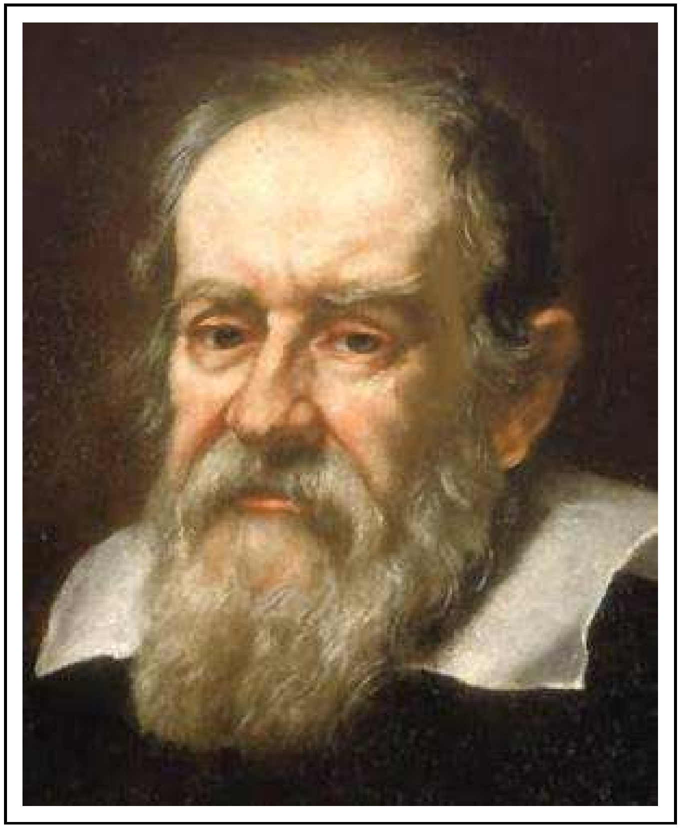
Principles of Relativity (Newton)

- **Assume a frame at rest (F) and another frame moving in x-direction (F') with constant velocity $\vec{v} = (v_x, 0, 0)$**
  - **Classical laws (mechanics) are the same in all frames**
  - **No absolute space possible, but absolute time**
  - **Coordinates of space are transformed through Galilei transformation**

Why transformations ?

- **To study physical laws in different frames:**
  - **How is a physical process in F' described (measurement, observations) in the rest frame F ?**
  - **Need transformation of coordinates (x, y, z) to describe (translate) results of measurements and observations to the moving system (x', y', z').**
  - **For Newton's principle of relativity need Galilei transformation**

Galilei transformation



$$x' = x - v_x t$$
$$y' = y$$
$$z' = z$$
$$(t' \equiv t)$$

Consequences of Galilei transformation

- **Velocities can be added**
- **From Galilei transformation, take derivative:**

$$x' = x - v_x t$$

$$\dot{x}' = \dot{x} - v_x \quad \rightarrow \quad v' = v - v_x$$

- **A car moving with speed v' in a frame moving with speed v_x we have in rest frame $v = v' + v_x$**
- **But: if $v' = \mathbf{0.75c}$ and $v_x = \mathbf{0.75c}$ do we get $v = \mathbf{1.5c}$?**

Problems with Galilei transformation

- Maxwell's equations are wrong when Galilei transformations are applied (because they predict the speed of light)
  - First solution: introduction of "ether"
  - But: speed of light the same in all frames and all directions (no "ether")
  - Need other transformations for Maxwell's equations
- Introduced principles of special relativity

Principles of Special Relativity (Einstein)

- **A frame moving with constant velocity is called an "inertial frame"**
- **All (not only classical) physical laws in related frames have equivalent forms, in particular:**

  speed of light c the same in all frames

- **Cannot distinguish between inertial frames**
  - **In particular, cannot determine absolute speed of an inertial frame**
  - **No absolute space, no absolute time**

Coordinates must be transformed differently

- **Transformation must keep speed of light constant**
- **Time must be changed by transformation as well as space coordinates**
- **Transform** $(x, y, z),\ t \quad \rightarrow \quad (x', y', z'),\ t'$

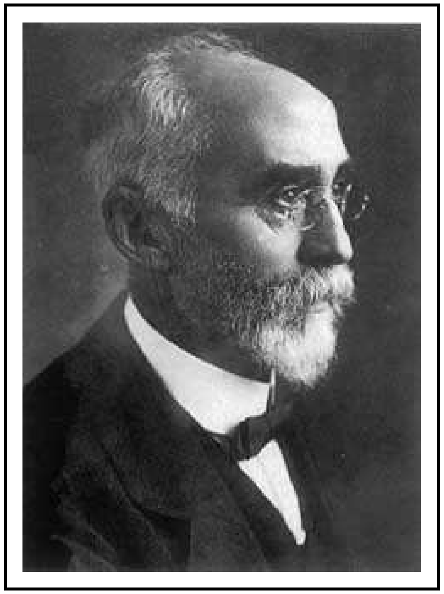
Constant speed of light requires:

$$x^2 + y^2 + z^2 - c^2t^2 = 0 \;\Rightarrow\; x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$$

(front of a light wave)

- **Defines the Lorentz transformation**

Lorentz transformation



$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

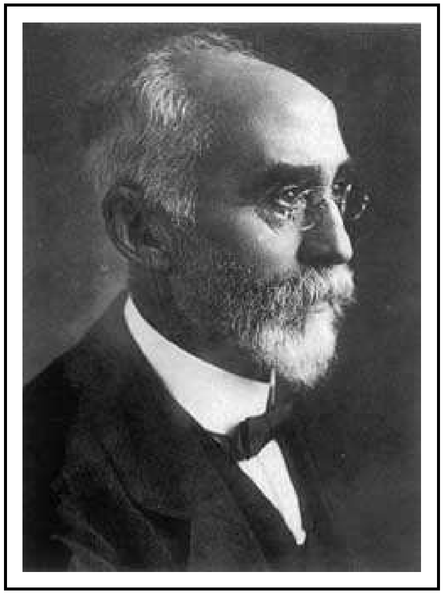
$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

- **Transformation for constant velocity along x-axis**

Lorentz transformation



$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} = \gamma \cdot (t - \frac{v \cdot x}{c^2})$$

- **Transformation for constant velocity along x-axis**

Definitions: relativistic factors

$$\beta_r = \frac{v}{c}$$

$$\gamma = \frac{1}{\sqrt{(1 - \frac{v^2}{c^2})}} = \frac{1}{\sqrt{(1 - \beta_r^2)}}$$

- β_r **relativistic speed:** $\beta_r = [\mathbf{0},\ \mathbf{1}]$
- γ **relativistic factor:** $\gamma = [\mathbf{1},\ \infty]$

(unfortunately, you will also see other β **and** γ **... !)**

Einstein's contributions

$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

$$(x, y, z) \quad \rightarrow \quad (x, y, z, ct)$$

- **Physical laws unchanged under Lorentz transformations**
- **Combine dimension of time with 3 dimensions of space**
- **Simultaneity has no absolute meaning in independent frames**

Other important contributors

$$x' = \frac{x - vt}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{t - \frac{v \cdot x}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

- **Lorentz transformation was known by H. Poincaré**
- **First thoughts about problem with simultaneity and spacetime**

Consequences of Einstein's interpretation

- Relativistic phenomena:
  - Simultaneity of events in independent frames
  - Lorentz contraction
  - Time dilatation
- Formalism with four-vectors introduced
  - Invariant quantities
  - Mass - energy relation

Simultaneity between moving frames

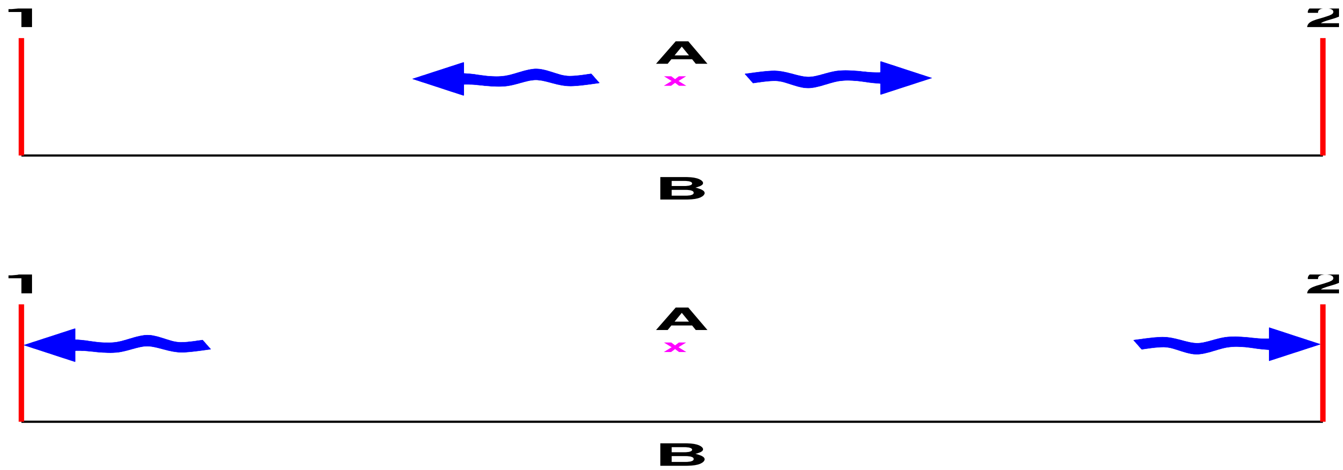
- **Assume two events in frame F at positions x_1 and x_2 happen simultaneously at times $t_1 = t_2$:**

$$t_1' = \frac{t_1 - \frac{v \cdot x_1}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}} \qquad \text{and} \qquad t_2' = \frac{t_2 - \frac{v \cdot x_2}{c^2}}{\sqrt{(1 - \frac{v^2}{c^2})}}$$

  implies that $t_1' \neq t_2'$

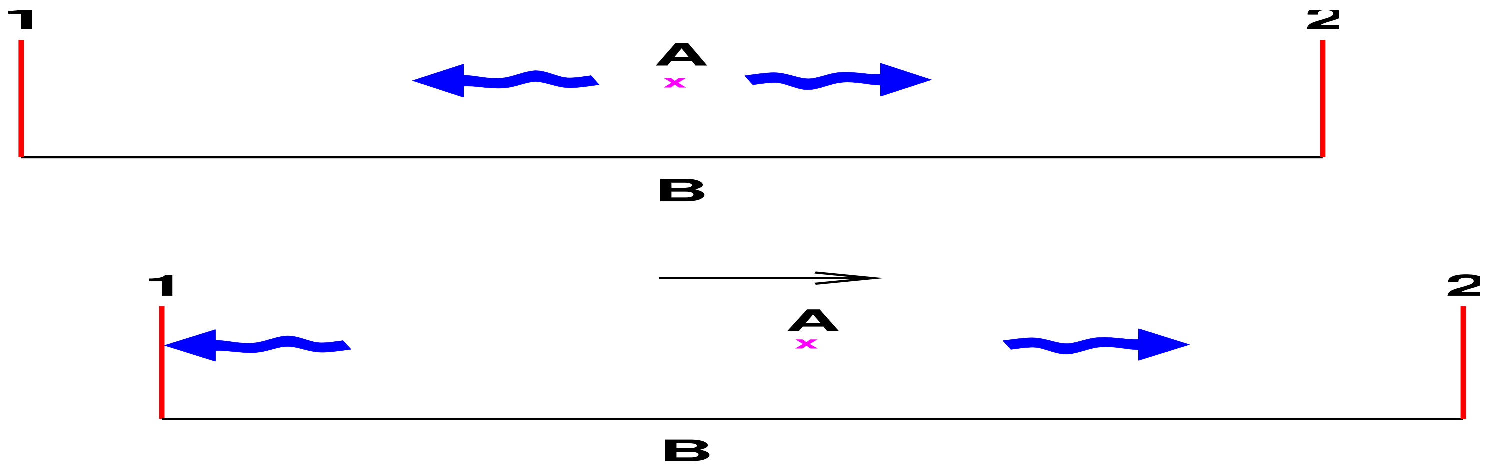
- **Two events simultaneous at positions x_1 and x_2 in F are not simultaneous in F'**

Simultaneity between moving frames



- System with a light source (x) and detectors (1, 2) and one observer (A) in this frame, another (B) outside
- System at rest → observation the same in A and B
- What if system with A is moving ?

Simultaneity between moving frames



- For A: both flashes arrive simultaneously in 1,2
- For B: flash arrives first in 1, later in 2
- A simultaneous event in F is not simultaneous in F'
- Why do we care ??

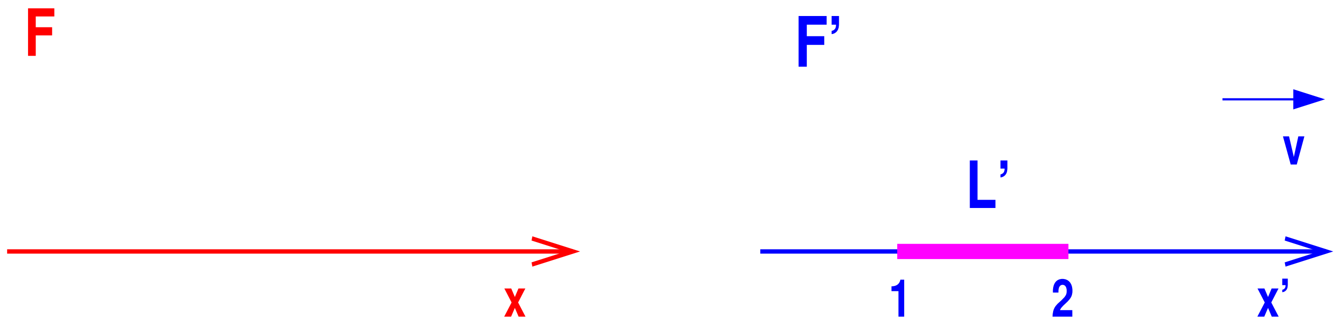
Why care about simultaneity ?

- **Simultaneity is not frame independent**
- **This is a key in special relativity**
- **Most paradoxes are explained by that !**

Why care about simultaneity ?

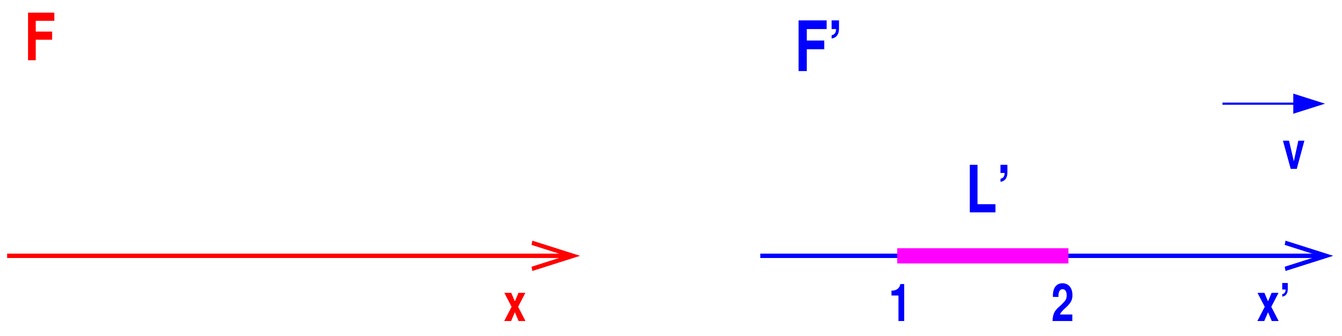
- **Simultaneity is not frame independent**
- **This is a key in special relativity**
- **Most paradoxes are explained by that !**
- **More important: sequence of events can change !**
  - **For $t_1 < t_2$ we may find (not always !) a frame where $t_1 > t_2$ (concept of before and after depends on the observer)**
  - **Requires introduction of "antiparticles" in relativistic quantum mechanics**
  - **Physical "reason" for antiparticles**

Consequences: length measurement



Length of a rod in F' is $L' = x'_2 - x'_1$, measured simultaneously at a fixed time t', what is the length L seen in F ??

Consequences: length measurement



We have to measure simultaneously the ends of the rod at a fixed time t in frame F →

$$x'_1 = \gamma \cdot (x_1 - vt) \quad \text{and} \quad x'_2 = \gamma \cdot (x_2 - vt)$$

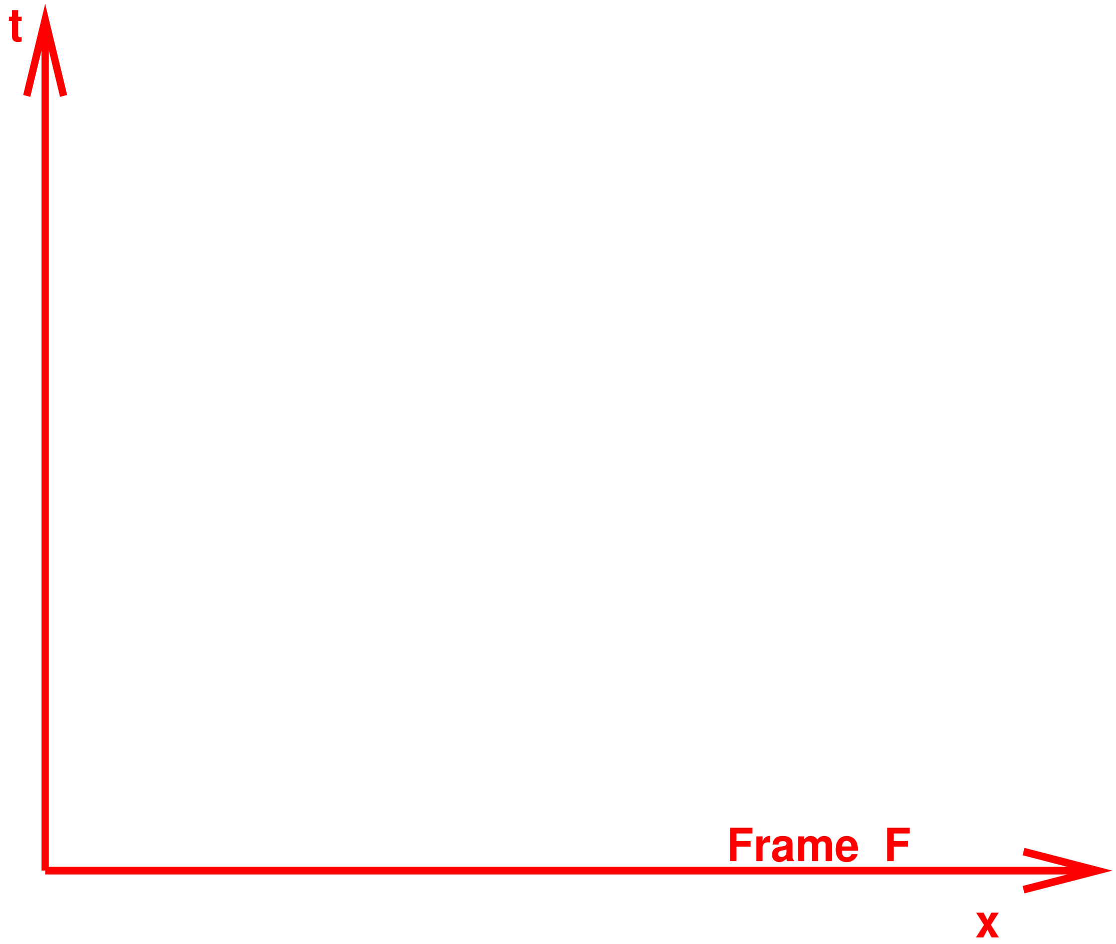
$$L' = x'_2 - x'_1 = \gamma \cdot (x_2 - x_1) = \gamma \cdot L$$

→ $$L = L'/\gamma$$

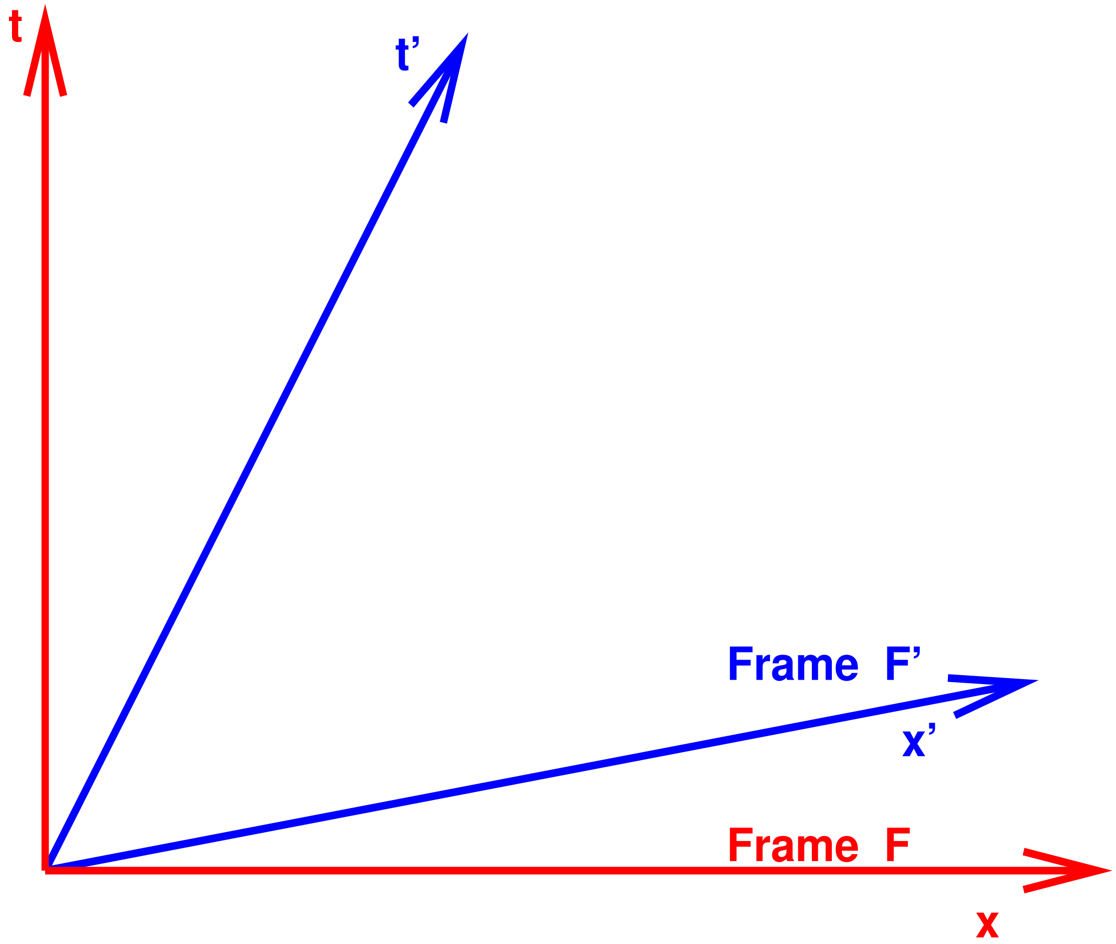
Lorentz contraction

- **In moving frame an object has always the same length (our principle !)**
- **From stationary frame moving objects appear contracted by a factor γ (Lorentz contraction)**
- **Why do we care ?**
- **Turn the argument around: assume length of a proton bunch appears always at 0.1 m in laboratory frame (e.g. in the RF bucket), what is the length in its own (moving) frame ?**
  - **At 5 GeV ($\gamma \approx 5.3$) $\rightarrow$ L' = 0.53 m**
  - **At 450 GeV ($\gamma \approx 480$) $\rightarrow$ L' = 48.0 m**

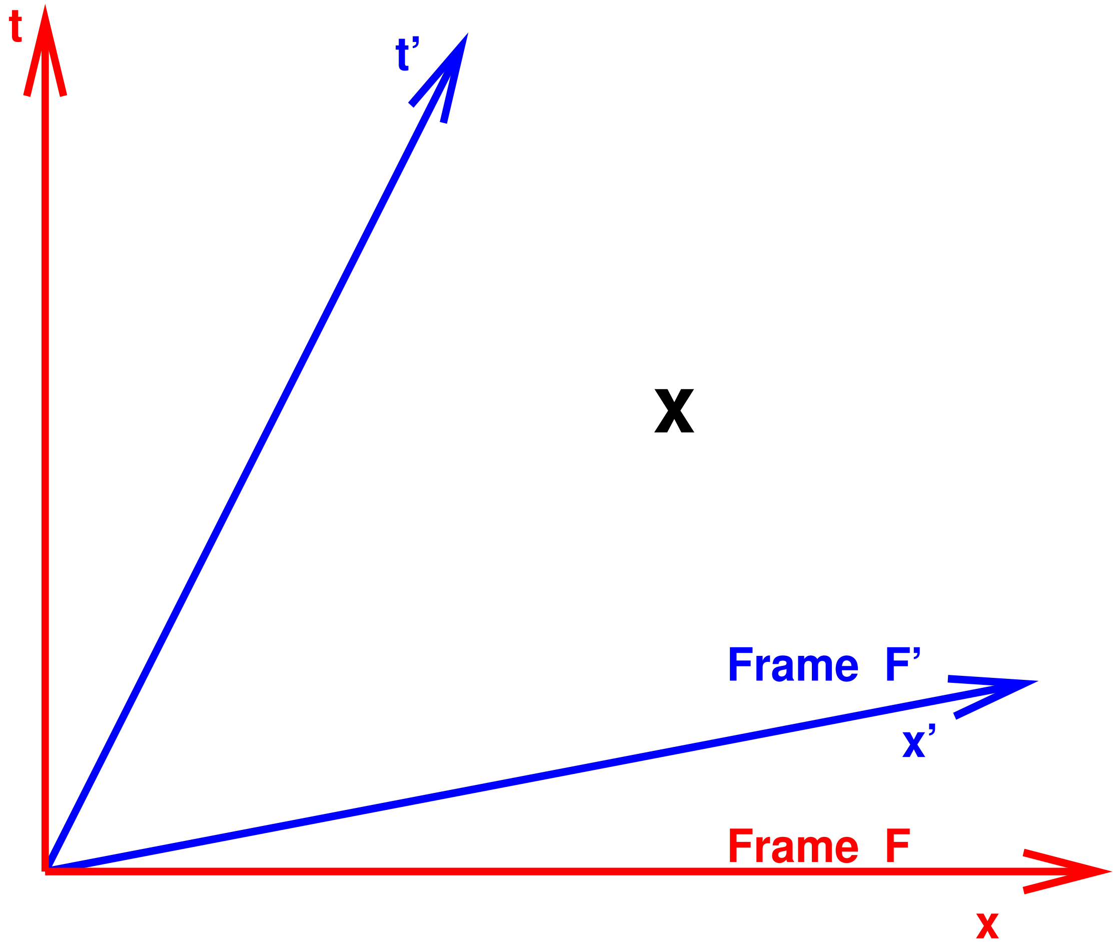
Lorentz transformation - schematic



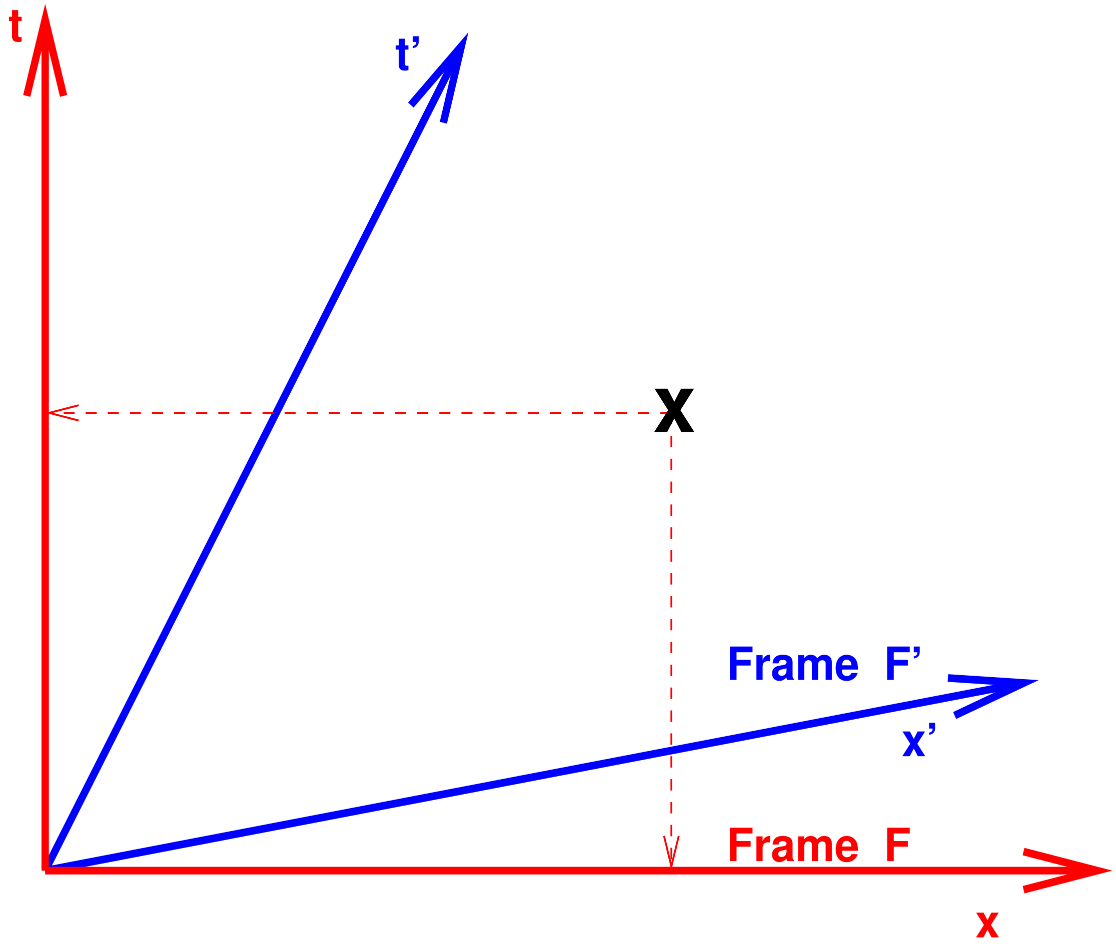
Lorentz transformation - schematic



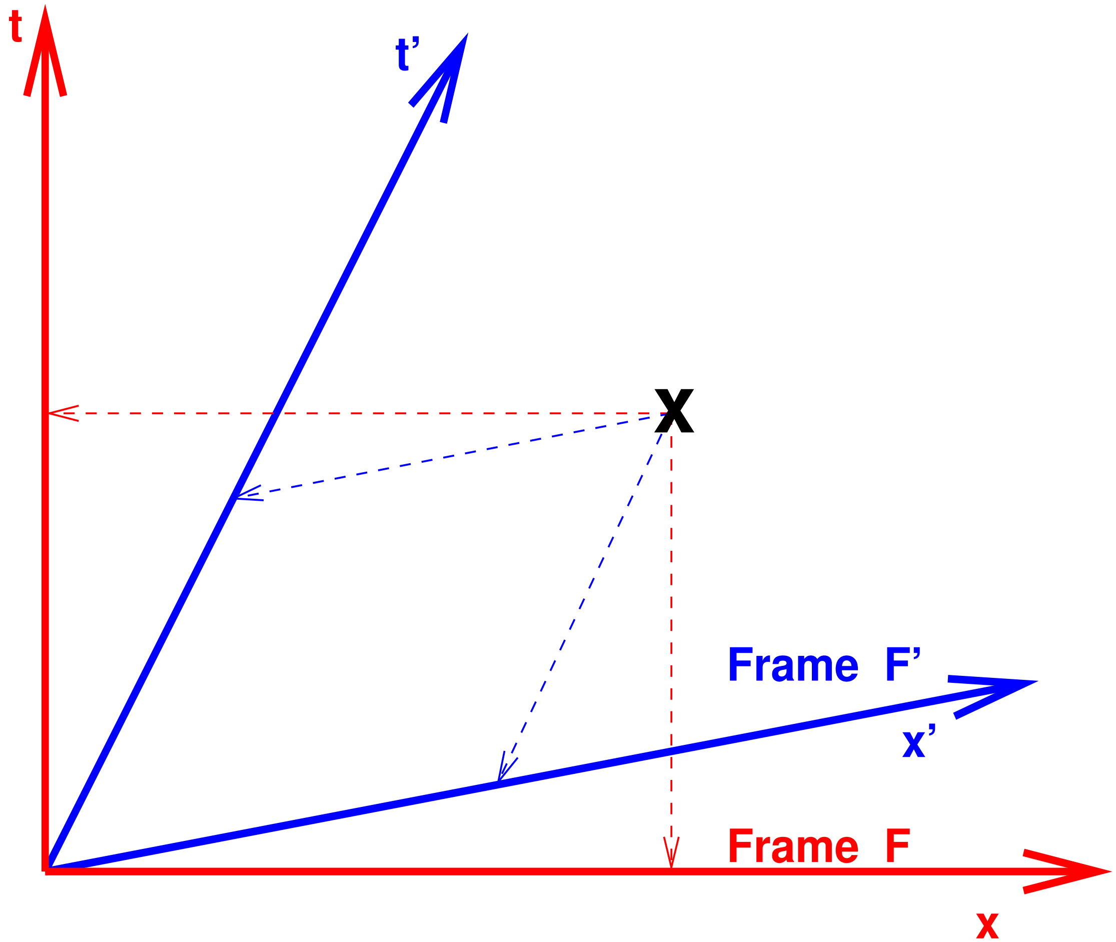
Lorentz transformation - schematic



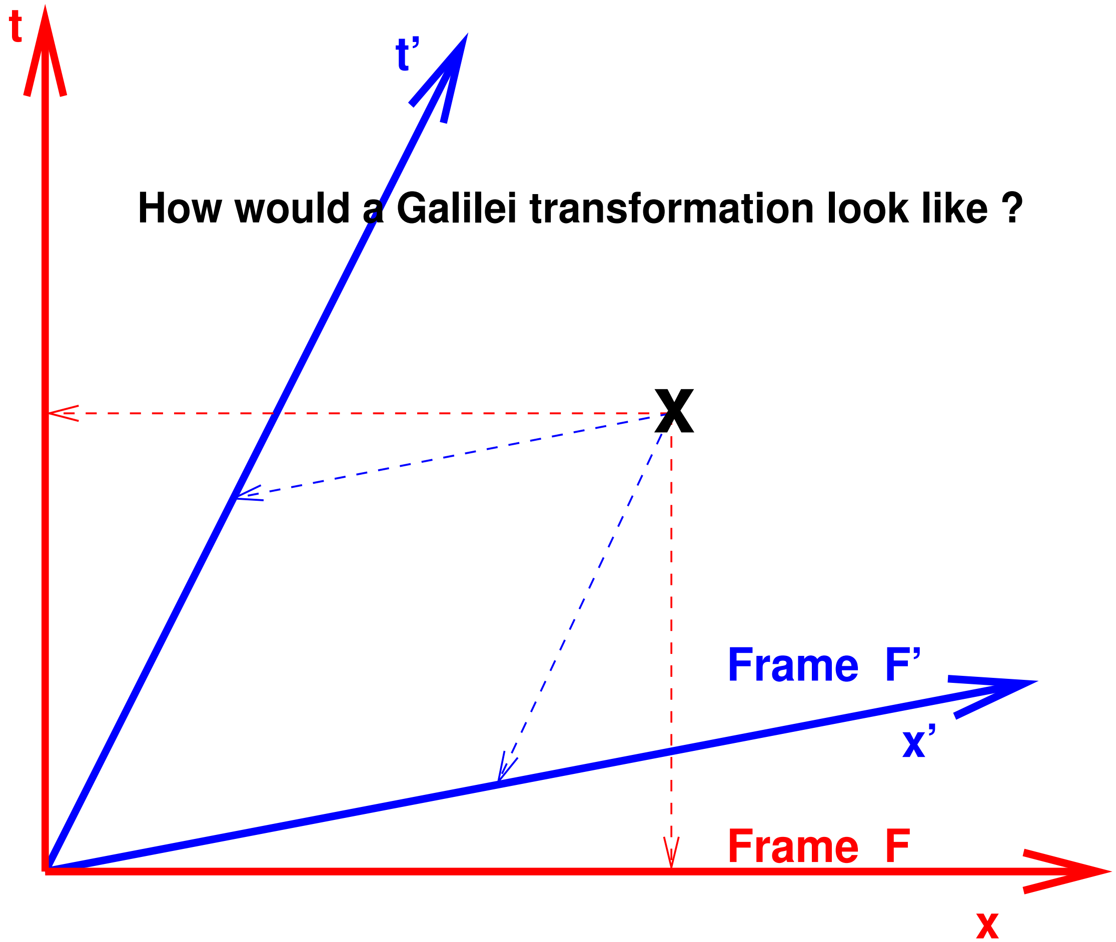
Lorentz transformation - schematic



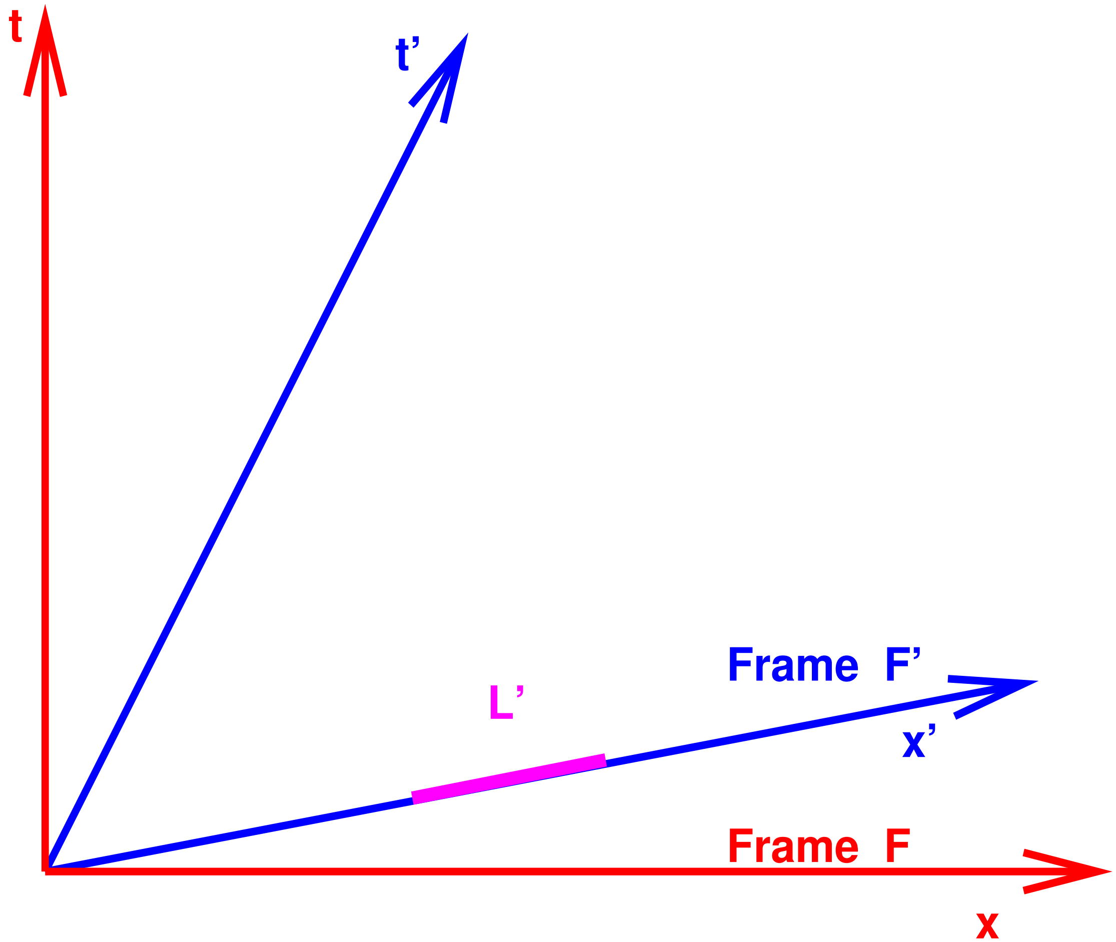
Lorentz transformation - schematic



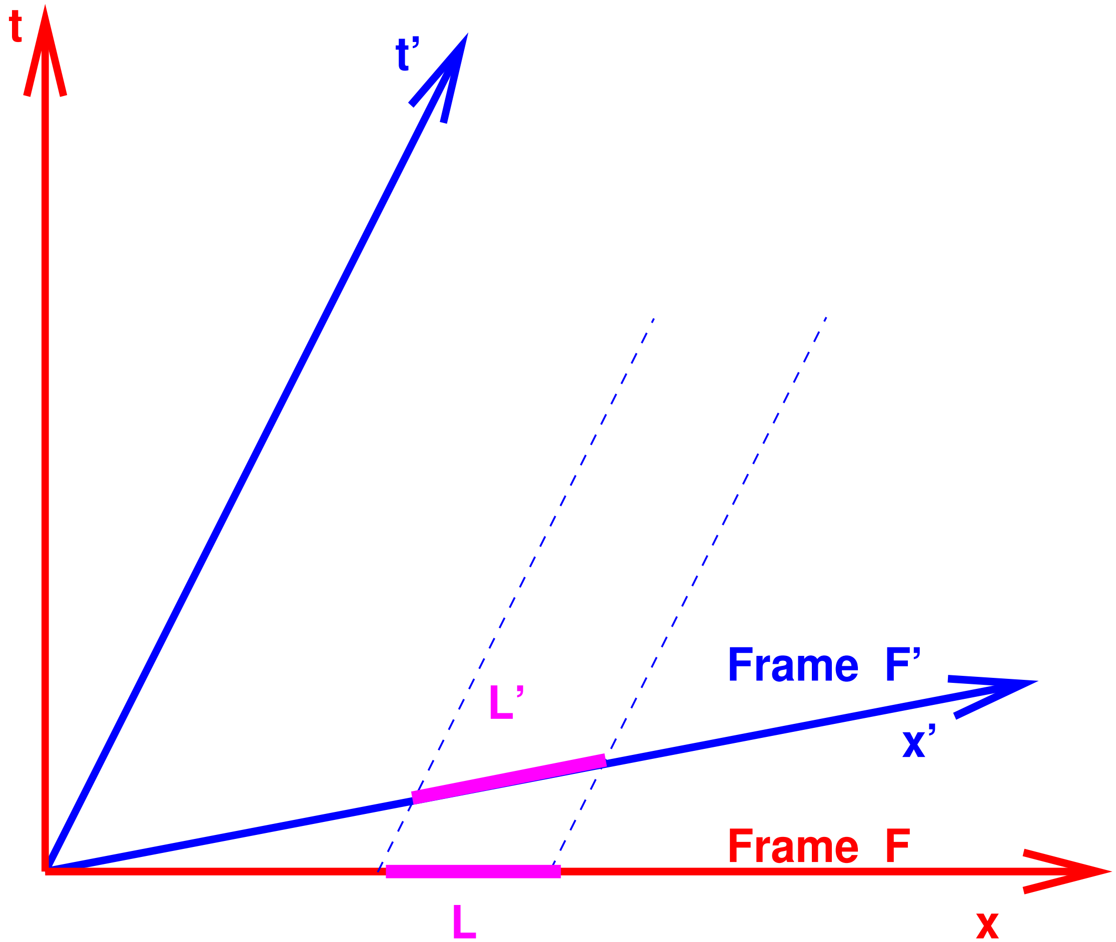
Lorentz transformation - schematic



Lorentz contraction - schematic



Lorentz contraction - schematic



Lorentz contraction

For the coffee break and lunch:

Can you "see" (visually) a Lorentz contraction ??

Time dilatation

A clock measures time difference $\Delta t = t_2 - t_1$ in frame F, measured at fixed position x, what is the time difference Δt' as measured from the moving frame F' ??

For Lorentz transformation of time in moving frame we have:

$$t'_1 = \gamma(t_1 - \frac{v \cdot x}{c^2}) \quad \text{and} \quad t'_2 = \gamma(t_2 - \frac{v \cdot x}{c^2})$$

$$\Delta t' = t'_2 - t'_1 = \gamma \cdot (t_2 - t_1) = \gamma \cdot \Delta t$$

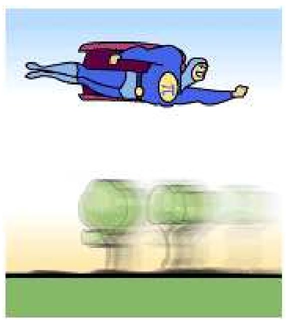
$$\rightarrow \Delta t' = \gamma \Delta t$$

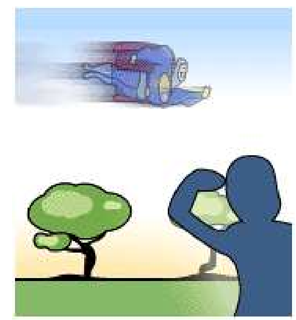
Time dilatation

- **In moving frame time appears to run slower**
- **Why do we care ?**
  - μ **have lifetime of 2** μ**s** ($\equiv$ **600 m**)
  - **For** $\gamma \; \geq \;$ **150, they survive 100 km to reach earth from upper atmosphere**
  - **They can survive more than 2** μ**s in a** μ**-collider**
  - **Generation of neutrinos from the SPS beams**

Of course ALL inertial frames are equivalent

- **Length contraction observed in F' from F is the same as observed in F from F'**
- **Time dilatation observed in F' from F is the same as observed in F from F'**





- **No contradiction: the same reality can look very different from different perspectives**

Addition of velocities

- **Galilei:** $v = v_1 + v_2$
- **With Lorentz transform we have:**

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}} \qquad \text{or equivalently :} \qquad \beta = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$$

for $\beta = 0.5$ **we get:**

0.5c + 0.5c = 0.8c

0.5c + 0.5c + 0.5c = 0.93c

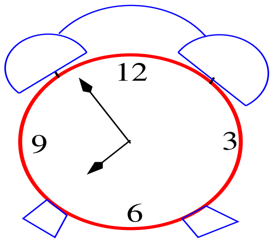
0.5c + 0.5c + 0.5c + 0.5c = 0.976c

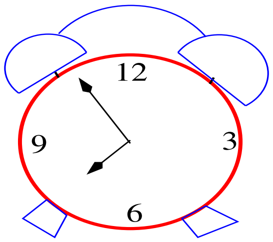
0.5c + 0.5c + 0.5c + 0.5c + 0.5c = 0.992c

First summary

- Constant speed of light requires Lorentz transformation
- No absolute space or time
- Speed of light is maximum possible speed
- Moving objects appear shorter
- Moving clocks seem to go slower

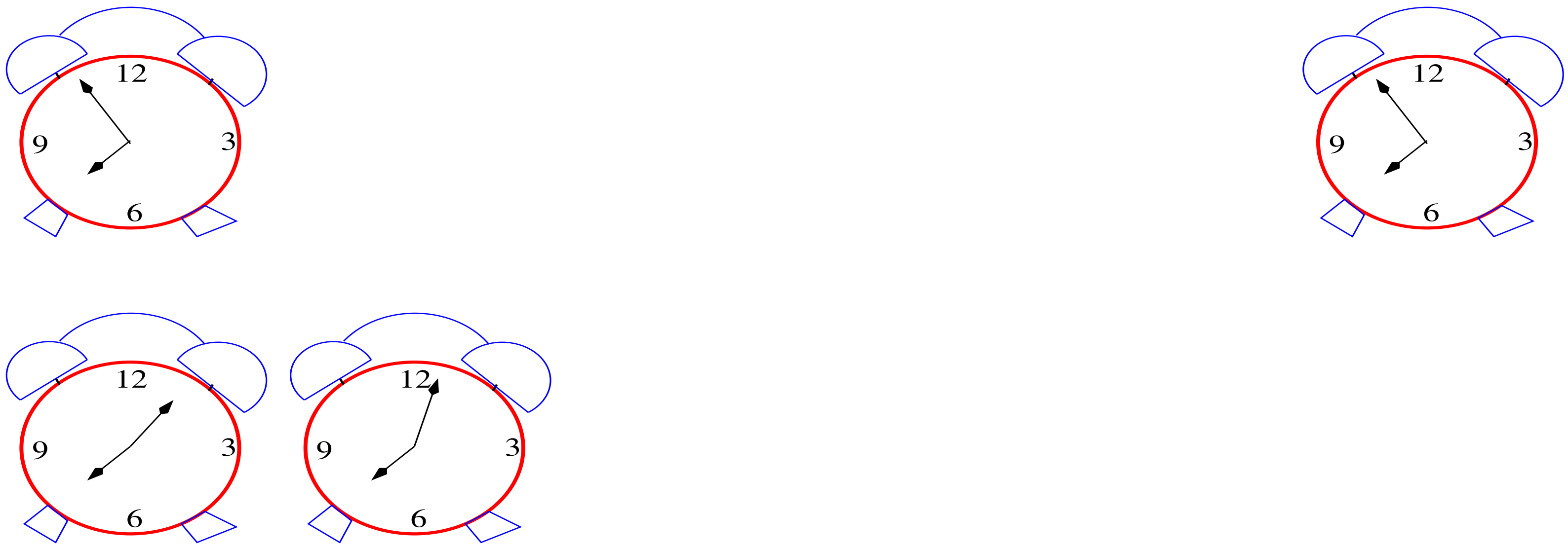
Moving clocks go slower





Moving clocks go slower

Moving clocks go slower



Nota bene: try to live near the equator, not the poles .....

Introducing four-vectors*)

Position four-vector X: $X = (ct, x, y, z) = (ct, \vec{x})$

This mathematical setting is called Minkowski space and Lorentz transformation can be written in matrix form:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix}_{s_2} = \begin{pmatrix} \gamma & \frac{-\gamma v}{c} & 0 & 0 \\ \frac{-\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}_{s_1}$$

$$X' = M_L \circ X$$

*) **definition for four-vectors not unique ! (I use PDG 2008)**

Introducing four-vectors

Define an invariant product*) like: $X \diamond Y$

$$X = (x_0, \vec{x}), \qquad Y = (y_0, \vec{y}) \quad \Rightarrow \quad X \diamond Y = x_0 \cdot y_0 - \vec{x} \cdot \vec{y}$$

For example try $X \diamond X$**:**

$$X \diamond X \; = c^2 t^2 \; - \; x^2 \; - \; y^2 \; - \; z^2$$

This product is an invariant, i.e.:

$$X \diamond X \; = \; c^2 t^2 - x^2 - y^2 - z^2 \; = \; X' \diamond X' \; = \; c^2 t'^2 - x'^2 - y'^2 - z'^2$$

Quantities which are invariant have the same value in all inertial frames

*) **definition of product not unique ! (I use PDG 2008)**

Introducing four-vectors

It describes a distance **in the spacetime between two points**
X_1 **and** X_2**:** $\Delta X = X_2 - X_1 = (ct_2 - ct_1, x_2 - x_1, y_2 - y_1, z_2 - z_1)$

$$\Delta s^2 \; = \; \Delta X \diamond \Delta X \; = \; c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Δs^2 **can be positive (timelike) or negative (spacelike)**
Special case (time interval $\vec{x_2} = \vec{x_1} + \vec{v}\Delta t$**):**

$$c^2\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \; = \; c^2\Delta t^2(1 - \frac{v^2}{c^2}) \; = \; c^2(\frac{\Delta t}{\gamma})^2 \; = \; c^2\Delta\tau^2$$

- $\Delta\tau$ **is the time interval measured in the moving frame**
- τ **is a fundamental time:** proper time τ

The meaning of "proper time"

$\Delta\tau$ is the time interval measured in the moving frame

Back to μ-decay

- μ **lifetime is** $\approx$ **2** μ**s**
- μ **decay in** $\approx$ **2** μ**s in their frame, i.e. using the "proper time"**
- μ **decay in** $\approx$ $\gamma\cdot$ **2** μ**s in the laboratory frame, i.e. earth**
- μ **appear to live longer than 2** μ**s in the laboratory frame, i.e. earth**

The meaning of "proper time"

- **How to make neutrinos ??**
- **Let pions decay:** $\pi \rightarrow \mu + \nu_\mu$
  - **π-mesons have lifetime of 2.6 $\cdot$ 10^{-8} s (i.e. 7.8 m)**
  - **For 40 GeV π-mesons: γ = 288**
  - **In laboratory frame: decay length is 2.25 km (required length of decay tunnel)**

More four-vectors

Position four-vector X:

$$X = (ct, x, y, z) = (ct, \vec{x})$$

Velocity four-vector V:

$$V = \frac{dX}{d\tau} = \gamma \frac{dX}{dt} = \gamma \dot{X} = \gamma(\frac{d(ct)}{dt}, \dot{x}, \dot{y}, \dot{z}) = \gamma(c, \vec{\dot{x}}) = \gamma(c, \vec{v})$$

Please note that:

$$V \diamond V = \gamma^2(c^2 - \vec{v}^2) = c^2!!$$

- **c is an invariant (of course)**

More four-vectors

Momentum four-vector P:

$$P = m_0 V = m_0 \gamma (c, \vec{v}) = (\mathbf{m}c, \vec{p})$$

using:

m_0 **(mass of a particle)**

$\mathbf{m} \equiv m_0 \cdot \gamma$ **(relativistic mass)**

$\vec{p} = \mathbf{m} \cdot \vec{v} = m_0 \gamma \vec{v}$ **(relativistic 3-momentum)**

We can get another invariant: $P \diamond P = m_0^2 (V \diamond V) \;=\; m_0^2 c^2$
and the derivatives:

$$P \diamond \frac{dP}{d\tau} \;=\; 0 \quad \Rightarrow \quad V \diamond \frac{dP}{d\tau} \;=\; 0$$

More four-vectors

Momentum four-vector P:

$$P = m_0 V = m_0 \gamma (c, \vec{v}) = (\mathbf{m} c, \vec{p})$$

using:

m_0 **(mass of a particle)**

$\mathbf{m} \equiv m_0 \cdot \gamma$ **(relativistic mass)**

$\vec{p} = \mathbf{m} \cdot \vec{v} = m_0 \gamma \vec{v}$ **(relativistic 3-momentum)**

We can get another invariant: $P \diamond P = m_0^2 (V \diamond V) \ = \ m_0^2 c^2$
and the derivatives:

$$P \diamond \frac{dP}{d\tau} \ = \ 0 \quad \Rightarrow \quad V \diamond \frac{dP}{d\tau} \ = \ 0$$

Still more four-vectors

Force four-vector F:

$$F = \frac{dP}{d\tau} = \gamma \frac{dP}{dt} = \gamma \frac{d}{dt}(mc, \vec{p}) = \gamma (c \frac{dm}{dt}, \vec{f})$$

since:

$$V \diamond \frac{dP}{d\tau} \ = \ V \diamond F \ = \ 0 \quad \rightarrow \quad \frac{d}{dt}(mc^2) - \vec{f}\vec{v} = 0$$

$\vec{f}\vec{v}$ is rate of change of kinetic energy dT/dt
after integration:

$$T = \int \frac{dT}{dt} dt = \int \vec{f}\vec{v} dt = \int \frac{d(mc^2)}{dt} dt = mc^2 + const.$$

$$T = mc^2 + const. = mc^2 - m_0 c^2$$

Still more four-vectors

Force four-vector F:

$$F = \frac{dP}{d\tau} = \gamma \frac{dP}{dt} = \gamma \frac{d}{dt}(mc, \vec{p}) = \gamma (c \frac{dm}{dt}, \vec{f})$$

since:

$$V \diamond \tfrac{dP}{d\tau} \;=\; V \diamond F \;=\; 0 \quad \rightarrow \quad \tfrac{d}{dt}(mc^2) - \vec{f}\vec{v} = 0$$

$\vec{f}\vec{v}$ **is rate of change of kinetic energy** dT/dt

after integration:

$$T = \int \frac{dT}{dt} dt = \int \vec{f}\vec{v} dt = \int \frac{d(mc^2)}{dt} dt = mc^2 + const.$$

$$T = mc^2 + const. = mc^2 - m_0 c^2$$

Relativistic energy

Interpretation:

$$E = mc^2 = T + m_0 c^2$$

- **Total energy** E **is** $E = mc^2$
- **Sum of kinetic energy plus rest energy**
- **Energy of particle at rest is** $E_0 = m_0 c^2$

$$\boxed{E = m \cdot c^2 = \gamma m_0 \cdot c^2}$$

using the definition of relativistic mass again: $m = \gamma m_0$

Still more four-vectors

Equivalent four-momentum vector:

$$P = (mc, \vec{p}) \rightarrow (E/c, \vec{p})$$

then:

$$P \diamond P = m_0^2 c^2 = \frac{E^2}{c^2} - \vec{p}^2$$

follows:

$$E^2 = \vec{p}^2 c^2 + m_0^2 c^4$$

Relativistic energy

These units are not very convenient:

$m_p = 1.672 \ \cdot \ 10^{-27}$ **Kg**

$\rightarrow \ m_p c^2 = 1.505 \ \cdot \ 10^{-10}$ **J**

$\rightarrow \ m_p c^2 = 938$ **MeV** $\rightarrow \ m_p = 938$ **MeV/c²**

$\rightarrow \ m_p c^2 \cdot \gamma(7 \ TeV) = 1.123 \ \cdot \ 10^{-6}$ **J**

$\rightarrow \ m_p c^2 \cdot \gamma(7 \ TeV) \cdot 1.15 \cdot 10^{11} \cdot 2808 = 360 \ \cdot \ 10^{6}$ **J**

Why did I write = 938 MeV/c² ??

Relativistic energy

in particle physics: omit c and dump it into the units:

$$[\mathbf{E}] = \mathbf{eV} \quad [\mathbf{p}] = \mathbf{eV/c} \quad [\mathbf{m}] = \mathbf{eV/c^2}$$

Four-vectors get an easier form:

$$P = (m, \vec{p}) = (E, \vec{p})$$

and from $P \diamond P \ = \ E^2 - p^2 \ = \ m_0^2$ **follows directly:**

$$E^2 = \vec{p}^2 + m_0^2 \quad (= m^2 = \gamma^2 m_0^2)$$

Relativistic energy

Note:

$$E = mc^2 = \gamma \cdot m_0 c^2 \quad \rightarrow \quad E = \gamma m_0$$

$$p = m_0 \gamma v = \gamma m_0 \cdot \beta c \quad \rightarrow \quad p = \gamma m_0 \cdot \beta$$

$$T = m_0(\gamma - 1) \cdot c^2 \quad \rightarrow \quad T = \gamma m_0 - m_0$$

- **for large β: numerical values very similar**
- ⚠ **careful for low energies (i.e. small β) ..... !**

Interpretation of relativistic energy

- **For any object, $m \cdot c^2$ is the total energy**
  - **Object can be composite, like proton ..**
  - **m is the mass (energy) of the object "in motion"**
  - **m_0 is the mass (energy) of the object "at rest"**
- **For discussion: what is the mass of a photon ?**

Relativistic mass

The mass of a fast moving particle is increasing like:

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

assume a 75 kg heavy man:

- **Rocket at 100 km/s, γ = 1.00000001, m = 75.000001 kg**
- **PS at 26 GeV, γ = 27.7, m = 2.08 tons**
- **LHC at 7 TeV, γ = 7642, m = 573.15 tons**
- **LEP at 100 GeV, γ = 196000, m = 14700 tons**

Relativistic mass

- **Why do we care ?**

$$m = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

- **Particles cannot go faster than c !**
- **What happens when we accelerate ?**

Relativistic mass

When we accelerate:

- **For v $\ll$ c:**
  - **E, m, p, v increase ...**
- **For v $\approx$ c:**
  - **E, m, p increase, but v does not !**
  - **Remember that for later**

Relativistic energy

Since we remember that:

$$T = m_0(\gamma - 1)c^2$$

therefore:

$$\gamma = 1 + \frac{T}{m_0 c^2}$$

we get for the speed v, i.e. β:

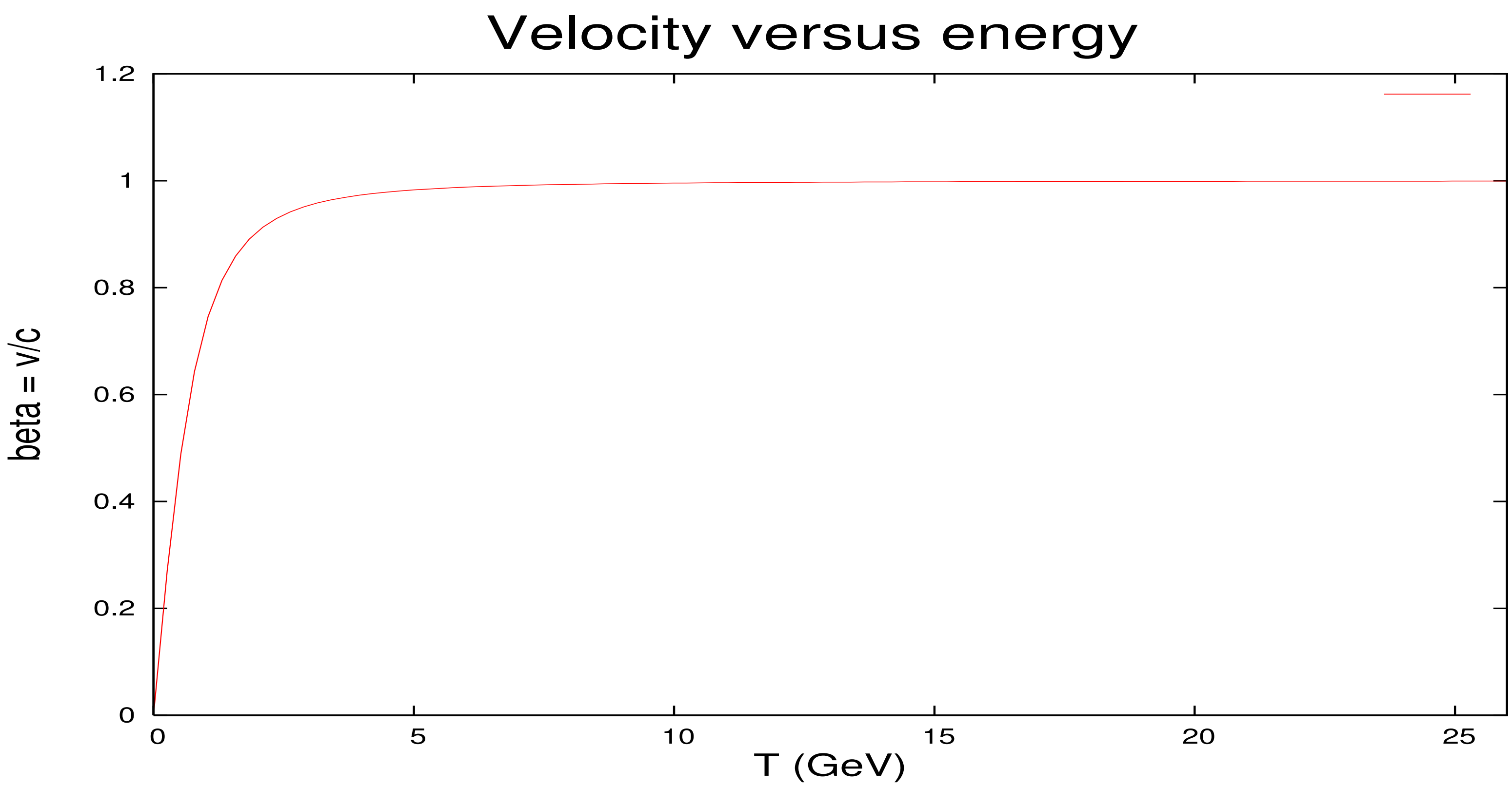
$$\beta = \sqrt{1 - \frac{1}{\gamma^2}}$$

Energy versus velocity

E (GeV)	v (km/s)	γ	β
1	103848.6	1.066	0.34640164
26	299597.3	27.72	0.99934902
450	299791.82	479.74	0.99999787
7000	299792.455	7462.7	0.99999999
∞	299792.458	∞	1.00000000

- **Q: which type of particle have I used ?**

Velocity versus energy (protons)



Why do we care ??

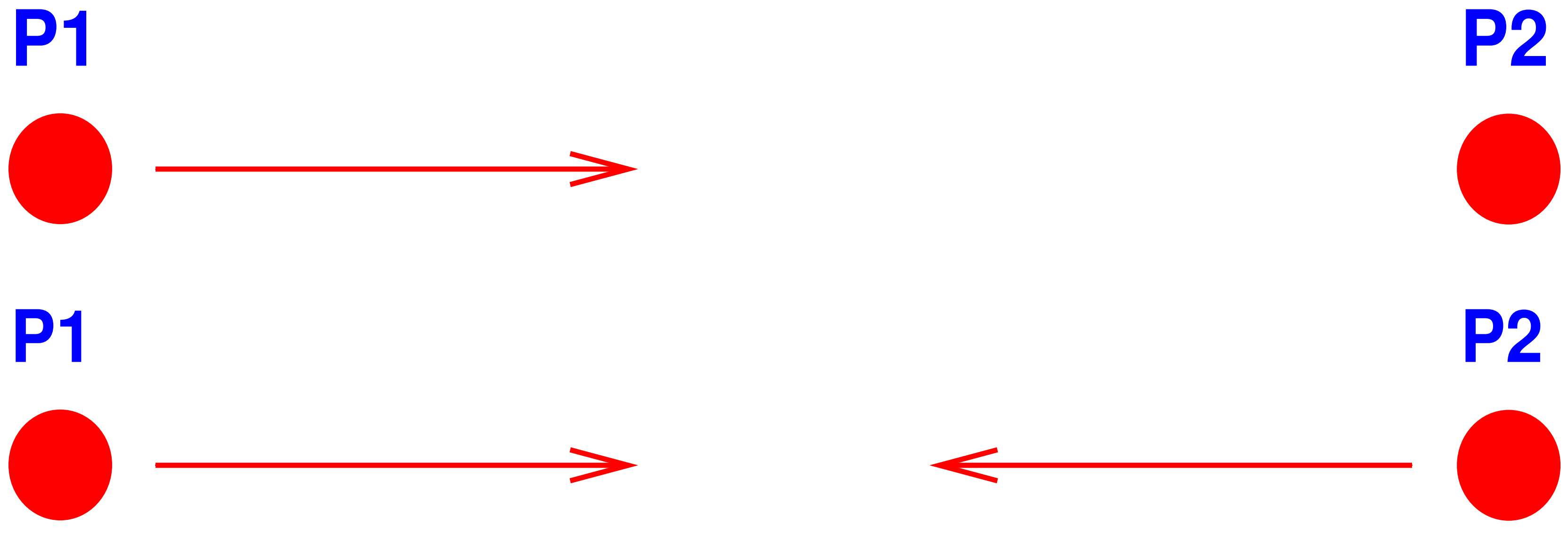
E (GeV)	v (km/s)	γ	β	T (LHC)
450	299791.82	479.74	0.99999787	88.92465 μs
7000	299792.455	7462.7	0.99999999	88.92446 μs

- **For identical circumference very small change in revolution time**
- **If path for faster particle slightly longer, the faster particle arrives later !**

Four vectors

- Use of four-vectors simplify calculations significantly
- Follow the rules and look for invariants
- In particular kinematic relationships, e.g.
  - Particle decay
  - Particle collisions →

Particle collisions



- What is the available collision energy ?

Particle collisions - collider

Assume identical particles and beam energies, colliding head-on

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (E, -\vec{p})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

Particle collisions - collider

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + E, \vec{p} - \vec{p}) = (2E, \vec{0})$$

The square of the total available energy s **in the centre of mass system is the momentum invariant:**

$$s^2 = P^* \diamond P^* = 4E^2$$

$$s = \sqrt{P^* \diamond P^*} = 2E$$

i.e. in a (symmetric) collider the total energy is twice the beam energy

Particle collisions - fixed target

The four momentum vectors are:

$$P1 = (E, \vec{p}) \qquad P2 = (m_0, \vec{0})$$

The four momentum vector in centre of mass system is:

$$P^* = P1 + P2 = (E + m_0, \vec{p})$$

Particle collisions - fixed target

With the above it follows:

$$P^* \diamond P^* = E^2 + 2m_0E + m_0^2 - \vec{p}^2$$

since $E^2 - \vec{p}^2 = m_0^2$ **we get:**

$$s^2 = 2m_0E + m_0^2 + m_0^2$$

if E **much larger than** m_0 **we find:**

$$s = \sqrt{2m_0E}$$

Particle collisions - fixed target

Homework: try for $E1 \neq E2$ **and** $m1 \neq m2$

Examples:

collision	beam energy	s (collider)	s (fixed target)
pp	315 (GeV)	630 (GeV)	24.3 (GeV)
pp	7000 (GeV)	14000 (GeV)	114.6 (GeV)
e+e−	100 (GeV)	200 (GeV)	0.320 (GeV)

Forces and fields

Motion of charged particles in electromagnetic fields $\vec{E}, \vec{B}$ determined by Lorentz force

$$\vec{f} = \frac{d}{dt}(m_0 \gamma \vec{v}) = q \cdot (\vec{E} + \vec{v} \times \vec{B})$$

or as four-vector:

$$F = \frac{dP}{d\tau} = \gamma \left(\frac{\vec{v} \cdot \vec{f}}{c}, \vec{f} \right) = \gamma \left(\frac{1}{c} \frac{dE}{dt}, \frac{d\vec{p}}{dt} \right)$$

Field tensor

Electromagnetic field described by field-tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \begin{pmatrix} 0 & \frac{-E_x}{c} & \frac{-E_y}{c} & \frac{-E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

derived from four-vector $A_\mu = (\Phi, \vec{A})$ like:

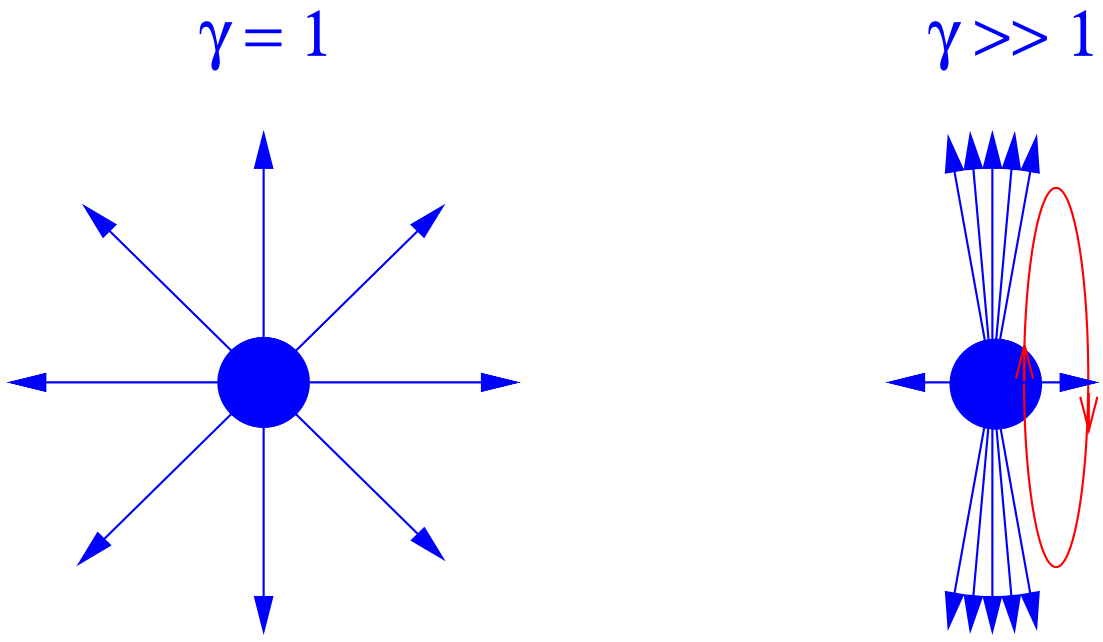
$$F^{\mu\nu} = \delta^\mu A^\nu - \delta^\nu A^\mu$$

Lorentz transformation of fields

$$
\begin{aligned}
\vec{E}'_{\perp} &= \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}) \\
\vec{B}'_{\perp} &= \gamma\left(\vec{E}_{\perp} - \frac{\vec{v} \times \vec{E}}{c^2}\right) \\
\vec{E}'_{\parallel} &= \vec{E}_{\parallel} \\
\vec{B}'_{\parallel} &= \vec{B}_{\parallel}
\end{aligned}
$$

- **Field perpendicular to movement transform**

Lorentz transformation of fields



- **In rest frame purely electrostatic forces**
- **In moving frame $\vec{E}$ transformed and $\vec{B}$ appears**

Kinematic relations

We have already seen a few, e.g.:

- $T = E - E_0 = (\gamma - 1)E_0$
- $E = \gamma \cdot E_0$
- $E_0 = \sqrt{E^2 - c^2p^2}$
- **etc. ...**

Very useful for everyday calculations →

Kinematic relations

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
$\mathbf{cp} =$	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
$\mathbf{T} =$	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	**T**	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations

	cp	**T**	**E**	γ
$\beta =$	$\frac{1}{\sqrt{(\frac{E_0}{cp})^2+1}}$	$\sqrt{1-\frac{1}{(1+\frac{T}{E_0})^2}}$	$\sqrt{1-(\frac{E_0}{E})^2}$	$\sqrt{1-\gamma^{-2}}$
$\mathbf{cp} =$	cp	$\sqrt{T(2E_0+T)}$	$\sqrt{E^2-E_0^2}$	$E_0\sqrt{\gamma^2-1}$
$E_0 =$	$\frac{cp}{\sqrt{\gamma^2-1}}$	$T/(\gamma-1)$	$\sqrt{E^2-c^2p^2}$	E/γ
$\mathbf{T} =$	$cp\sqrt{\frac{\gamma-1}{\gamma+1}}$	$\mathbf{T}$	$E-E_0$	$E_0(\gamma-1)$
$\gamma =$	$cp/E_0\beta$	$1+T/E_0$	E/E_0	γ

Kinematic relations

- **Example: CERN Booster**

At injection: T = 50 MeV

→ **E = 0.988 GeV, p = 0.311 GeV/c**

→ γ **= 1.0533,** β **= 0.314**

At extraction: T = 1.4 GeV

→ **E = 2.338 GeV, p = 2.141 GeV/c**

→ γ **= 2.4925,** β **= 0.916**

Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$(1+\frac{1}{\gamma})\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

Kinematic relations - logarithmic derivatives

	$\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{dE}{E} = \frac{d\gamma}{\gamma}$
$\frac{d\beta}{\beta} =$	$\frac{d\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{dp}{p}$	$\frac{1}{\gamma(\gamma+1)}\frac{dT}{T}$	$\frac{1}{(\beta\gamma)^2}\frac{d\gamma}{\gamma}$
$\frac{dp}{p} =$	$\gamma^2\frac{d\beta}{\beta}$	$\frac{dp}{p}$	$[\gamma/(\gamma+1)]\frac{dT}{T}$	$\frac{1}{\beta^2}\frac{d\gamma}{\gamma}$
$\frac{dT}{T} =$	$\gamma(\gamma+1)\frac{d\beta}{\beta}$	$(1+\frac{1}{\gamma})\frac{dp}{p}$	$\frac{dT}{T}$	$\frac{\gamma}{(\gamma-1)}\frac{d\gamma}{\gamma}$
$\frac{dE}{E} =$	$(\beta\gamma)^2\frac{d\beta}{\beta}$	$\beta^2\frac{dp}{p}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$
$\frac{d\gamma}{\gamma} =$	$(\gamma^2-1)\frac{d\beta}{\beta}$	$\frac{dp}{p} - \frac{d\beta}{\beta}$	$(1-\frac{1}{\gamma})\frac{dT}{T}$	$\frac{d\gamma}{\gamma}$

Example LHC (7 TeV): $\frac{\Delta p}{p} \approx 10^{-4}$ → $\frac{\Delta\beta}{\beta} = \frac{\Delta v}{v} \approx 2 \cdot 10^{-12}$

Summary

- **Relativistic effects vital in accelerators:**
  - **Lorentz contraction**
  - **Time dilatation**
  - **Relativistic mass effects**
  - **Modification of electromagnetic field**
- **Find back in later lectures ...**