1. Reminder: the ideal world

\[ \begin{pmatrix} x \\ x' \end{pmatrix}_s = M \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_0 \]

\[ M_{\text{foc}} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|}s) \\ -\sqrt{|K|} \sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0 \]
Beam parameters of a typical high energy ring: \( I_p = 100 \text{ mA} \)

particles per bunch: \( N \approx 10^{11} \)

Example: HERA Bunch pattern

... question: do we really have to calculate some \( 10^{11} \) single particle trajectories?

\[
x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\psi(s) + \phi)
\]

\[
\psi(s) = \int_0^s \frac{ds}{\beta(s)}
\]
Beam Emittance and Phase Space Ellipse

equation of motion: \[ x''(s) - k(s) x(s) = 0 \]

general solution of Hills equation: \[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \varphi) \]

beam size: \[ \sigma = \sqrt{\varepsilon \beta} \approx \text{"mm"} \]

\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \]

* \( \varepsilon \) is a constant of the motion … it is independent of „s“
* parametric representation of an ellipse in the \( x \ x' \) space
* shape and orientation of ellipse are given by \( \alpha, \beta, \gamma \)

\[ \alpha(s) = \frac{-1}{2} \beta'(s) \]
\[ \gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)} \]
13.) Liouville during Acceleration

$$\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)$$

Beam Emittance corresponds to the area covered in the $x, x'$ Phase Space Ellipse

**Liouville:** Area in phase space is constant.

**But so sorry ...** $\varepsilon \neq \text{const}$!

**Classical Mechanics:**

phase space = diagram of the two canonical variables
position & momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = \text{kin. Energy} - \text{pot. Energy}$$
According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

\[
q = \text{position} = x \\
p = \text{momentum} = \gamma mv = mc\gamma \beta_x
\]

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}
\]

Liouville's Theorem:
\[\int p\, dq = \text{const}\]

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

\[
x' = \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{\beta_x}{\beta}
\]

where \(\beta_x = \frac{v_x}{c}\)

\[
\int p\, dq = mc \int \gamma \beta_x\, dx
\]

\[
\int p\, dq = mc\gamma \beta \int x'\, dx
\]

\[
\Rightarrow \quad \varepsilon = \int x'\, dx \propto \frac{1}{\beta\gamma}
\]

the beam emittance shrinks during acceleration \(\varepsilon \sim 1/\gamma\)
Nota bene:

1.) A proton machine ... or an electron linac ... needs the highest aperture at injection energy !!!
   as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

$$\sigma = \sqrt{\varepsilon \beta}$$

2.) At lowest energy the machine will have the major aperture problems,
   → here we have to minimise $\hat{\beta}$

3.) we need different beam optics adopted to the energy:
   A Mini Beta concept will only be adequate at flat top.

---

**LHC injection optics at 450 GeV**

**LHC mini beta optics at 7000 GeV**
Example: HERA proton ring

injection energy: 40 GeV \( \gamma = 43 \)
flat top energy: 920 GeV \( \gamma = 980 \)

emittance \( \varepsilon \) (40 GeV) = \( 1.2 \times 10^{-7} \)
\( \varepsilon \) (920 GeV) = \( 5.1 \times 10^{-9} \)

7 \( \sigma \) beam envelope at \( E = 40 \text{ GeV} \)

... and at \( E = 920 \text{ GeV} \)
14.) The „Δp / p ≠ 0“ Problem

Linear Accelerator

Energy Gain per „Gap“:

\[ W = q U_0 \sin \omega_{RF} t \]

*RF Acceleration: multiple application of the same acceleration voltage; brilliant idea to gain higher energies ... but changing acceleration voltage
Problem: panta rhei !!!
(Heraklit: 540-480 v. Chr.)

Example: HERA RF:

Bunch length of Electrons $\approx 1\text{cm}$

$\nu = 500\text{MHz}$
$c = \lambda\nu$
$\lambda = 60\text{cm}$

$\sin(90^\circ) = 1$
$\sin(84^\circ) = 0.994$

$\frac{\Delta U}{U} = 6.0 \times 10^{-3}$

Typical momentum spread of an electron bunch:

$\frac{\Delta p}{p} \approx 1.0 \times 10^{-3}$
15.) Dispersion: trajectories for $\Delta p / p \neq 0$

**Force acting on the particle**

$$F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B_y v$$

*remember: $x \approx m \rho, \rho \approx m \rightarrow$ develop for small $x$

$$m \frac{d^2x}{dt^2} - \frac{mv^2}{\rho} (1 - \frac{x}{\rho}) = e B_y v$$

*consider only linear fields, and change independent variable: $t \rightarrow s$

$$B_y = B_0 + x \frac{\partial B_y}{\partial x}$$

$$x'' - \frac{1}{\rho} \left(1 - \frac{x}{\rho}\right) = e B_0 + e \frac{x t g}{mv}$$

$p = p_0 + \Delta p$

... but now take a small momentum error into account !!!!
**Dispersion:**

Develop for small momentum error

\[ \Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} \approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2} \]

\[
\begin{align*}
x'' - \frac{1}{\rho} + \frac{x}{\rho^2} & \approx \frac{e}{p_0} B_0 - \frac{\Delta p}{p_0^2} eB_0 + \frac{\text{xeg}}{p_0} - \text{xeg} \frac{\Delta p}{p_0^2} \\
& - \frac{1}{\rho} \\
& = - \frac{1}{\rho} \\
& \approx 0
\end{align*}
\]

\[
\begin{align*}
x'' + \frac{x}{\rho^2} & \approx \frac{\Delta p}{p_0} \frac{\text{xeg}(-eB_0)}{p_0} + k \times x = \frac{\Delta p}{p_0} \frac{1}{\rho} + k \times x \\
& \approx \frac{1}{\rho}
\end{align*}
\]

\[
x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \frac{1}{\rho} \quad \Rightarrow \quad x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p_0} \frac{1}{\rho}
\]

**Momentum spread** of the beam adds a term on the r.h.s. of the equation of motion.

\(\Rightarrow\) *inhomogeneous differential equation.*
General solution:

\[ x(s) = x_h(s) + x_i(s) \]

Normalise with respect to \( \Delta p/p \):

\[ D(s) = \frac{x_i(s)}{\Delta p/p} \]

**Dispersion function** \( D(s) \)

* is that special orbit, an ideal particle would have for \( \Delta p/p = 1 \)

* the orbit of any particle is the sum of the well known \( x_\beta \) and the dispersion

* as \( D(s) \) is just another orbit it will be subject to the focusing properties of the lattice
**Dispersion**

*Example: homogenous dipole field*

Matrix formalism:

\[
x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}
\]

\[
x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \frac{\Delta p}{p}
\]

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\
  x'
\end{pmatrix}_0 + \frac{\Delta p}{p} \begin{pmatrix} D \\
  D' \end{pmatrix}
\]
\[
\begin{pmatrix}
  x \\
  x' \\
  \Delta p/p_s
\end{pmatrix} = \begin{pmatrix}
  C & S & D \\
  C' & S' & D' \\
  0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
  x \\
  x' \\
  \Delta p/p_0
\end{pmatrix}
\]

Example HERA

\[
x_\beta = 1 \ldots 2 \, \text{mm}
\]

\[
D(s) \approx 1 \ldots 2 \, \text{m}
\]

\[
\Delta p/p \approx 1 \cdot 10^{-3}
\]

Amplitude of Orbit oscillation
contribution due to Dispersion \(\approx\) beam size

Calculate \(D, D'\)

\[
D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}
\]
Example: Drift

\[ M_{\text{Drift}} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(\tilde{s}) d\tilde{s} = 0 \]

Example: Dipole

\[ M_{\text{Dipole}} = \begin{pmatrix} \cos \frac{l}{\rho} & \rho \sin \frac{l}{\rho} \\ \frac{1}{\rho} \sin \frac{l}{\rho} & \cos \frac{l}{\rho} \\ -\frac{1}{\rho} \sin \frac{l}{\rho} & \frac{1}{\rho} \cos \frac{l}{\rho} \end{pmatrix} \]

\[ D(s) = \rho \cdot (1 - \cos \frac{l}{\rho}) \]

\[ D'(s) = \sin \frac{l}{\rho} \]
Dispersion is visible

\[ x_D = D(s) \times \frac{\Delta p}{p} \]

\[ \rightarrow \text{closed orbit is moved to a dispersions trajectory} \]

Attention: at the Interaction Points we require \( D = D' = 0 \)
16.) Momentum Compaction Factor:

The dispersion function relates the momentum error of a particle to the horizontal orbit coordinate.

**inhomogeneous differential equation**

\[ x'' + K(s) \ast x = \frac{1}{\rho} \frac{\Delta p}{p} \]

**general solution**

\[ x(s) = x_\beta(s) + D(s) \frac{\Delta p}{p} \]

*But it does much more:*

*it changes the length of the off-energy-orbit!!*
particle with a displacement \( x \) to the design orbit

\[ \Rightarrow \text{path length } dl \ldots \]

\[
\frac{dl}{ds} = \frac{\rho + x}{\rho}
\]

\[ \rightarrow dl = \left( 1 + \frac{x}{\rho(s)} \right) ds \]

**circumference of an off-energy closed orbit**

\[ l_{\Delta E} = \int dl = \int \left( 1 + \frac{x_{\Delta E}}{\rho(s)} \right) ds \]

**remember:**

\[ x_{\Delta E}(s) = D(s) \frac{\Delta p}{p} \]

\[ \Delta l_{\Delta E} = \frac{\Delta p}{p} \int \left( \frac{D(s)}{\rho(s)} \right) ds \]

*The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.*
Definition: \[ \frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p} \]

\[ \rightarrow \alpha_{cp} = \frac{1}{L} \int \left( \frac{D(s)}{\rho(s)} \right) ds \]

For first estimates assume: \( \frac{1}{\rho} = \text{const} \)

\[ \int D(s) ds = \sum_d (l_{\text{dipoles}}^d)^* \langle D \rangle_{\text{dipole}} \]

\[ \alpha_{cp} = \frac{1}{L} l_{\text{dipoles}} \langle D \rangle \frac{1}{\rho} = \frac{1}{L} 2\pi \rho \langle D \rangle \frac{1}{\rho} \]

\[ \rightarrow \alpha_{cp} \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R} \]

Assume: \( v \approx c \)

\[ \frac{\delta T}{T} = \frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p} \]

\( \alpha_{cp} \) combines via the dispersion function the momentum spread with the longitudinal motion of the particle.
17.) Tune and Quadrupoles

**Question:** what will happen, if you do not make too many mistakes and your particle performs one complete turn?

Transfer Matrix from point „0“ in the lattice to point „s“:

\[
M = \begin{pmatrix}
\sqrt{\frac{\beta_s}{\beta_0}} \cos \psi_s + \alpha_0 \sin \psi_s & \sqrt{\beta_s \beta_0} \sin \psi_s \\
(\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s & \sqrt{\frac{\beta_0}{\beta_s}} \cos \psi_s - \alpha_s \sin \psi_s
\end{pmatrix}
\]
Matrix for one complete turn

the Twiss parameters are periodic in $L$:

$$\beta(s + L) = \beta(s)$$

$$\alpha(s + L) = \alpha(s)$$

$$\gamma(s + L) = \gamma(s)$$

\[
M_{\text{turn}} = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} = \begin{pmatrix} \cos \psi_{\text{turn}} + \alpha \sin \psi_{\text{turn}} & \beta \sin \psi_{\text{turn}} \\ -\gamma \sin \psi_{\text{turn}} & \cos \psi_{\text{turn}} - \alpha \sin \psi_{\text{turn}} \end{pmatrix}
\]

Definition: phase advance of the particle oscillation per revolution in units of $2\pi$ is called tune

\[
Q = \frac{\Delta \psi_{\text{turn}}}{2\pi} = \frac{\mu}{2\pi}
\]
Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

\[ M_{\text{dist}} = M_{\Delta k} \cdot M_0 = \begin{pmatrix} 1 & 0 \\ \Delta Kds & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos \psi_{\text{turn}} + \alpha \sin \psi_{\text{turn}} & \beta \sin \psi_{\text{turn}} \\ -\gamma \sin \psi_{\text{turn}} & \cos \psi_{\text{turn}} - \alpha \sin \psi_{\text{turn}} \end{pmatrix} \]

\[ M_{\text{dist}} = \begin{pmatrix} \cos \psi_{\text{turn}} + \alpha \sin \psi_{\text{turn}} & \beta \sin \psi_{\text{turn}} \\ \Delta Kds \left( \cos \psi_{\text{turn}} + \alpha \sin \psi_{\text{turn}} \right) - \gamma \sin \psi_{\text{turn}} & \Delta Kds \cdot \beta \sin \psi_{\text{turn}} + \cos \psi_{\text{turn}} - \alpha \sin \psi_{\text{turn}} \end{pmatrix} \]

rule for getting the tune

\[ \text{Trace}(M) = 2 \cos \psi = 2 \cos \psi_0 + \Delta Kds \beta \sin \psi_0 \]

\[ \psi = \psi_0 + \Delta \psi \]  

Quadrupole error \rightarrow Tune Shift
\[ \cos(\psi_0 + \Delta \psi) = \cos \psi_0 + \frac{\Delta K ds \beta \sin \psi_0}{2} \]

remember the old fashioned trigonometric stuff and assume that the error is small !!!

\[ \cos \psi_0 \cos \Delta \psi - \sin \psi_0 \sin \Delta \psi = \cos \psi_0 + \frac{\Delta K ds \beta \sin \psi_0}{2} \]

\[
\approx 1 \\
\approx \Delta \psi
\]

\[ \Delta \psi = \frac{\Delta K ds \beta}{2} \]

and referring to \( Q \) instead of \( \psi \): \[ \psi = 2\pi Q \]

\[ \Delta Q = \int_{s_0}^{s_0 + L} \frac{\Delta K(s) \beta(s) ds}{4\pi} \]
a quadrupol error leads to a shift of the tune:

\[
\Delta Q = \int_{s_0}^{s_0+L} \frac{\Delta K(s) \beta(s) ds}{4\pi} \approx \frac{\Delta K\beta}{4\pi}
\]

the tune shift is proportional to the $\beta$-function at the quadrupole

field quality, power supply tolerances etc are much tighter at places where $\beta$ is large

mini beta quads: $\beta \approx 1900$

arc quads: $\beta \approx 80$

$\beta$ is a measure for the sensitivity of the beam

Example: measurement of $\beta$ in a storage ring:

tune spectrum ...

tune shift as a function of a gradient change
18.) Chromaticity:  
A Quadrupole Error for $\Delta p/p \neq 0$

Influence of external fields on the beam: *prop. to magn. field & prop. zu 1/p*

**Dipole Magnet**  
$$\alpha = \frac{\int B \, dl}{p/e}$$  
$$x_D(s) = D(s) \frac{\Delta p}{p}$$

**Focusing Lens**  
$$k = \frac{g}{p/e}$$

-- Figure 29: FODO cell

-- Particle having ...
  - to high energy
  - to low energy
  - ideal energy
**Chromaticity: \( Q' \)**

\[
k = \frac{g}{p/e} \quad p = p_0 + \Delta p
\]

in case of a momentum spread:

\[
k = \frac{eg}{p_0 + \Delta p} \approx \frac{e}{p_0} \left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k
\]

\[
\Delta k = -\frac{\Delta p}{p_0} k_0
\]

... which acts like a quadrupole error in the machine and leads to a tune spread:

\[
\Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds
\]

definition of chromaticity:

\[
\Delta Q = Q' \frac{\Delta p}{p}
\]
Problem: chromaticity is generated by the lattice itself!!

\( \xi \) is a number indicating the size of the tune spot in the working diagram,
\( \xi \) is always created if the beam is focussed
→ it is determined by the focusing strength \( k \) of all quadrupoles

\[
Q' = \frac{-1}{4\pi} \int k(s) \beta(s) \, ds
\]

\( k = \text{quadrupole strength} \)
\( \beta = \text{betafunction} \) indicates the beam size … and even more the sensitivity of the beam to external fields

Example: HERA

HERA-p: \( Q' = -70 \ldots -80 \)
\( \Delta p/p = 0.5 \times 10^{-3} \)
\( Q = 0.257 \ldots 0.337 \)

→ Some particles get very close to resonances and are lost
Correction of \( Q' \):

1.) sort the particles according to their momentum

\[ x_D(s) = D(s) \frac{\Delta p}{p} \]

2.) apply a magnetic field that rises quadratically with \( x \) (sextupole field)

\[
\begin{align*}
B_x &= \tilde{g}xz \\
B_z &= \frac{1}{2} \tilde{g}(x^2 - z^2)
\end{align*}
\]

Sextupole Magnets:

\[
\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \tilde{g}x
\]

linear rising „gradient“:

\[
k_1 \text{ normalised quadrupole strength} \\
k_2 \text{ normalised sextupole strength}
\]

\[
k_1(\text{sext}) = \frac{\tilde{g}x}{p / e} = k_2 \ast x
\]

\[
k_1(\text{sext}) = k_2 \ast D \ast \frac{\Delta p}{p}
\]

corrected chromaticity:

\[
Q'_x = \frac{-1}{4\pi} \int k_1(s) \beta(s) \, ds + \frac{1}{4\pi} \sum_{F, \text{sext}} k_2^F l_{\text{sext}}^F D_x^F \beta_x^F - \frac{1}{4\pi} \sum_{D, \text{sext}} k_2^D l_{\text{sext}}^D D_x^D \beta_x^D
\]
Chromaticity in the FoDo Lattice

\[ Q' = \frac{-1}{4\pi} \int k(s) \beta(s) \, ds \]

\[ \beta = \frac{(1 + \sin \frac{\mu}{2})L}{\sin \mu} \]

\[ \beta = \frac{(1 - \sin \frac{\mu}{2})L}{\sin \mu} \]

\[ Q' = -\frac{1}{4\pi} N \frac{\hat{\beta} - \bar{\beta}}{f_Q} \]

\[ Q' = -\frac{1}{4\pi} N \frac{1}{f_Q} \left\{ \frac{L(1 + \sin \frac{\mu}{2}) - L(1 - \sin \frac{\mu}{2})}{\sin \mu} \right\} \]
using some TLC transformations ...

\[ Q' = \frac{-1}{4\pi} N \frac{1}{f_Q} \left( \frac{2L \sin \frac{\mu}{2}}{\sin \mu} \right) \]

\[ Q' = \frac{-1}{4\pi} N \frac{1}{f_Q} \left( \frac{L \sin \frac{\mu}{2}}{\sin \frac{\mu}{2} \cos \frac{\mu}{2}} \right) \]

\[ Q'_{\text{cell}} = \frac{-1}{4\pi f_Q} \left( \frac{L \tan \frac{\mu}{2}}{\sin \frac{\mu}{2}} \right) \]

\[ Q'_{\text{cell}} = \frac{-1}{\pi} \tan \frac{\mu}{2} \]

remember ...

\[ \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \]

putting ...

\[ \sin \frac{\mu}{2} = \frac{L}{4f_Q} \]

contribution of one FoDo Cell to the chromaticity of the ring:
question: main contribution to $\xi$ in a lattice … ?

Chromaticity

$$Q' = -\frac{1}{4\pi} \int K(s) \beta(s) ds$$
19.) Resume:\textsuperscript{\textdegree}:

beam emittance: \[ \varepsilon \propto \frac{1}{\beta \gamma} \]

beta function in a drift: \[
\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2
\]

\[ \beta(s) = \beta_0 + \frac{s^2}{\beta_0} \]

… and for \( \alpha = 0 \)

particle trajectory for \( \Delta p/p \neq 0 \)

inhomogenous equation:

\[
x'' + x\left(\frac{1}{\rho^2} - k\right) = \Delta\frac{p}{p_0} \frac{1}{\rho}
\]

… and its solution:

\[
x(s) = x_\beta(s) + D(s) \cdot \frac{\Delta p}{p}
\]

momentum compaction:

\[
\frac{\delta l_\varepsilon}{L} = \alpha_{cp} \frac{\Delta p}{p} \quad \alpha_{cp} \approx \frac{2\pi}{L} \langle D \rangle \approx \frac{\langle D \rangle}{R}
\]

quadrupole error:

\[
\Delta Q = \int_{s_0}^{s_0+l} \frac{\Delta K(s) \beta(s) ds}{4\pi}
\]

chromaticity:

\[
Q' = -\frac{1}{4\pi} \int K(s) \beta(s) ds
\]