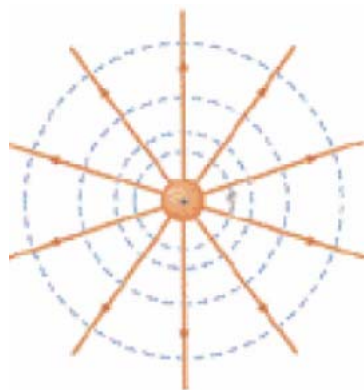


# SPACE CHARGE EFFECTS

Massimo Ferrario

INFN-LNF



# EQUATION OF MOTION

The motion of charged particles is governed by the Lorentz force :

$$\frac{d(m\gamma \mathbf{v})}{dt} = \mathbf{F}_{e.m.}^{ext} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Where  $m$  is the rest mass,  $\gamma$  the relativistic factor and  $\mathbf{v}$  the particle velocity

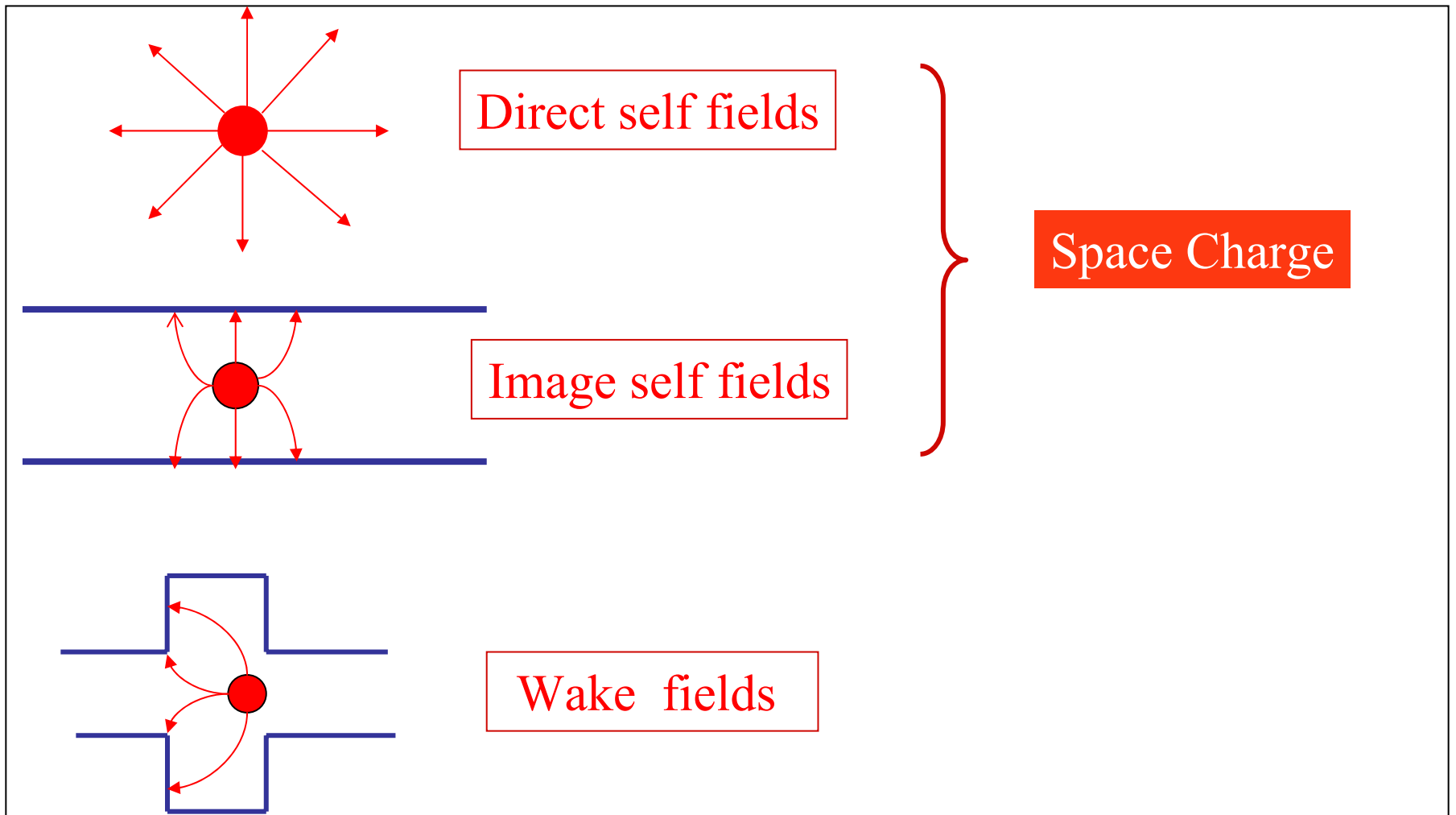
Charged particles are accelerated, guided and confined by external electromagnetic fields.

Acceleration is provided by the electric field of the RF cavity

Magnetic fields are produced in the bending magnets for guiding the charges on the reference trajectory (orbit), in the quadrupoles for the transverse confinement, in the sextupoles for the chromaticity correction.

# SELF FIELDS AND WAKE FIELDS

There is another important source of e.m. fields : **the beam itself**

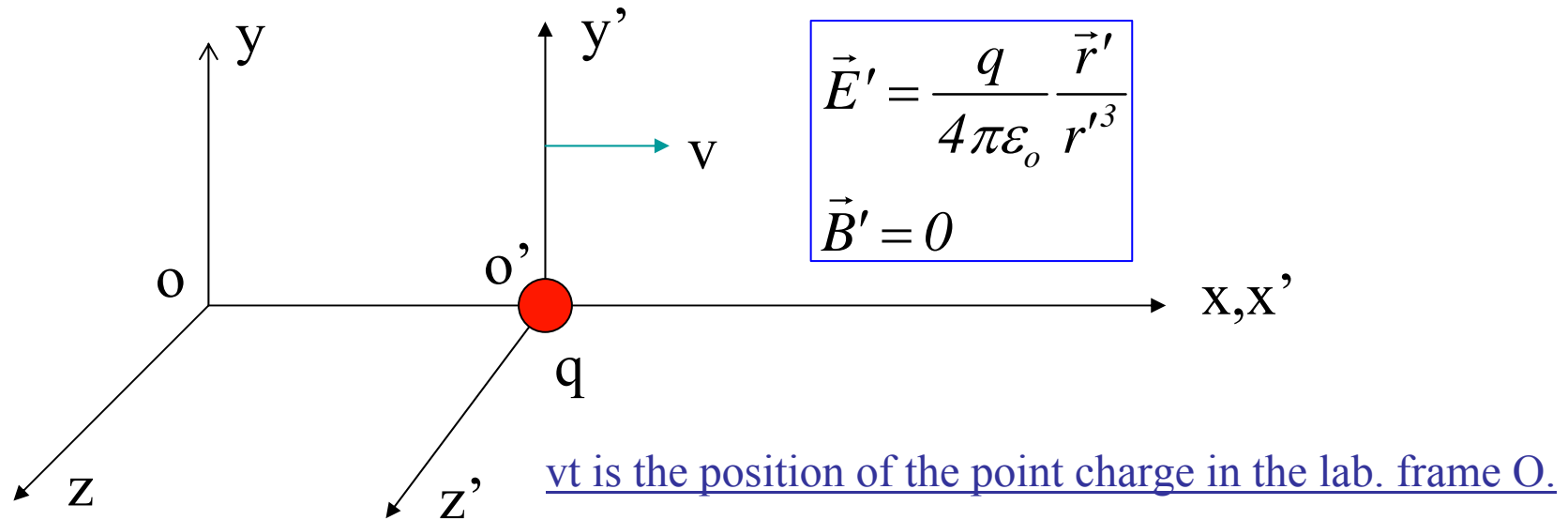


**These fields depend on the current and on the charges velocity.**

**They are responsible of many phenomena of beam dynamics:**

- energy loss (*wake-fields*)
- energy spread and emittance degradation
- shift of the synchronous phase and frequency (tune)
- shift of the betatron frequencies (tunes)
- instabilities.

## Fields of a point charge with uniform motion



- In the moving frame  $O'$  the charge is at rest
- The electric field is radial with spherical symmetry
- The magnetic field is zero

$$E'_x = \frac{q}{4\pi\epsilon_0} \frac{x'}{r'^3}$$

$$E'_y = \frac{q}{4\pi\epsilon_0} \frac{y'}{r'^3}$$

$$E'_z = \frac{q}{4\pi\epsilon_0} \frac{z'}{r'^3}$$

## Relativistic transforms of the fields from O' to O

$$\begin{aligned}E_x &= E'_x & B_x &= B'_x \\E_y &= \gamma(E'_y + vB'_z) & B_y &= \gamma(B'_y - vE'_z / c^2) \\E_z &= \gamma(E'_z - vB'_y) & B_z &= \gamma(B'_z + vE'_y / c^2)\end{aligned}$$

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

$$\begin{aligned}r' &= (x'^2 + y'^2 + z'^2)^{1/2} \\&\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \end{cases} \\r' &= [\gamma^2(x - vt)^2 + y^2 + z^2]^{1/2}\end{aligned}$$

$$\vec{B}' = 0$$

$$E_x = E'_x = \frac{q}{4\pi\epsilon_0} \frac{x'}{r'^3} = \frac{q}{4\pi\epsilon_0} \frac{\gamma(x-vt)}{\left[\gamma^2(x-vt)^2 + y^2 + z^2\right]^{3/2}}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\epsilon_0} \frac{y'}{r'^3} = \frac{q}{4\pi\epsilon_0} \frac{\gamma y}{\left[\gamma^2(x-vt)^2 + y^2 + z^2\right]^{3/2}}$$

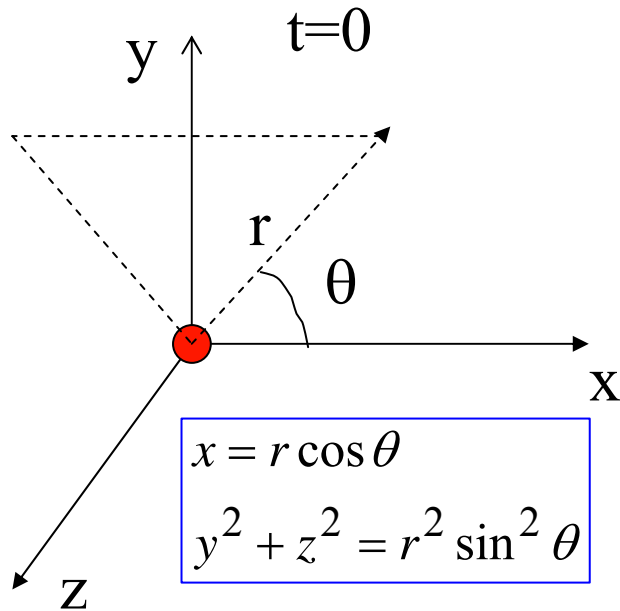
$$E_z = \gamma E'_z = \frac{q}{4\pi\epsilon_0} \frac{z'}{r'^3} = \frac{q}{4\pi\epsilon_0} \frac{\gamma z}{\left[\gamma^2(x-vt)^2 + y^2 + z^2\right]^{3/2}}$$

The field pattern is moving with the charge and it can be observed at  $t=0$ .

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\gamma \vec{r}}{\left[\gamma^2 x^2 + y^2 + z^2\right]^{3/2}}$$

The fields have lost the spherical symmetry

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{\gamma \vec{r}}{[\gamma^2 x^2 + y^2 + z^2]^{3/2}}$$



$$\gamma^2 x^2 + y^2 + z^2 = r^2 \gamma^2 (1 - \beta^2 \sin^2 \theta)$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{(1 - \beta^2)}{r^2 (1 - \beta^2 \sin^2 \theta)^{3/2}} \frac{\vec{r}}{r}$$



$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

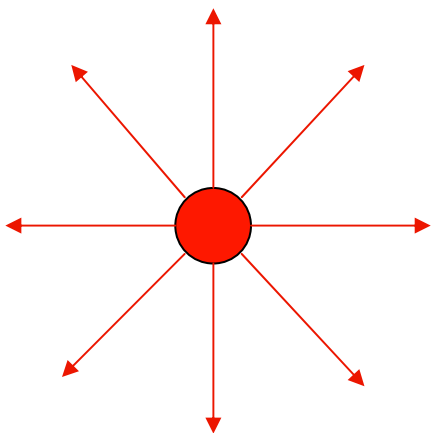
$$\vec{E} = \frac{q}{4\pi\epsilon_0} \frac{(1-\beta^2)}{r^2 (1-\beta^2 \sin^2 \theta)^{3/2}} \frac{\vec{r}}{r}$$

$$\beta = 0 \Rightarrow \vec{E} = \frac{q}{4\pi\epsilon_0} \frac{1}{r^2} \frac{\vec{r}}{r}$$

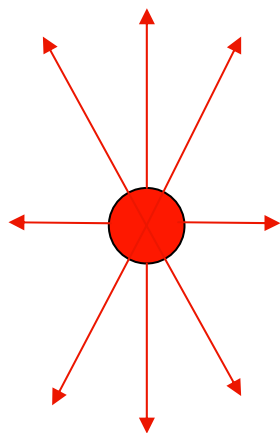
$$\theta = 0 \Rightarrow E_{//} = \frac{q}{4\pi\epsilon_0} \frac{1}{\gamma^2 r^2} \frac{\vec{r}}{r} \xrightarrow{\gamma \rightarrow \infty} 0$$

$$\theta = \frac{\pi}{2} \Rightarrow E_{\perp} = \frac{q}{4\pi\epsilon_0} \frac{\gamma}{r^2} \frac{\vec{r}}{r} \xrightarrow{\gamma \rightarrow \infty} \infty$$

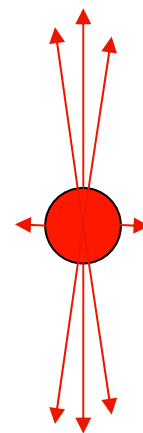
$\gamma=1$



$\gamma \neq 0$



$\gamma \gg 1$



$$\vec{B}' = 0$$

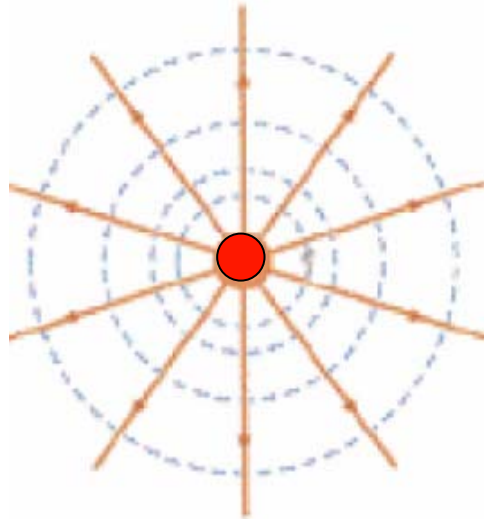
B is transverse to the direction of motion

$$B_x = 0$$

$$B_y = -vE_z / c^2$$

$$B_z = vE_y / c^2$$

$$\vec{B}_\perp = \frac{\vec{v} \times \vec{E}}{c^2}$$



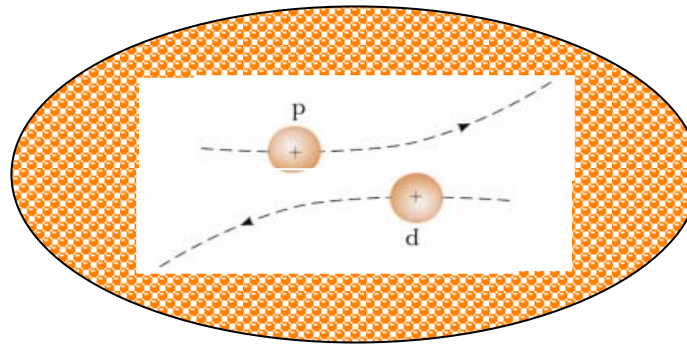
$\gamma \rightarrow \infty$

# **Direct Space Charge Forces**

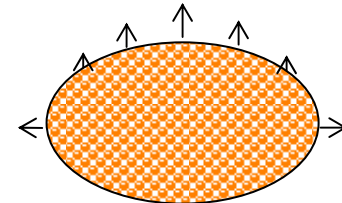
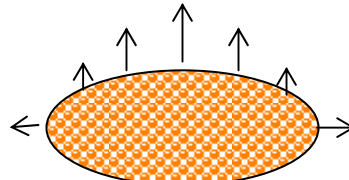
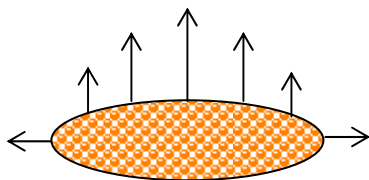
# Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

- 1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



- 2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**

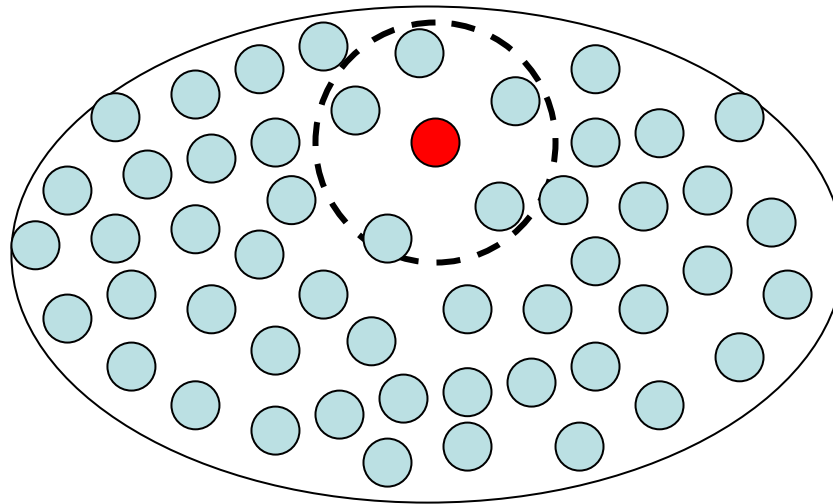


A measure for the relative importance of collisional versus collective effects is the

## Debye Length $\lambda_D$

Let consider a **non-neutralized** system of **identical charged particles**

We wish to calculate the **effective potential** of a fixed **test charged particle** surrounded by other particles that are statistically distributed.



$$\Phi_s(\vec{r}) = ?$$

$$\Phi(\vec{r}) = \frac{C}{r}$$

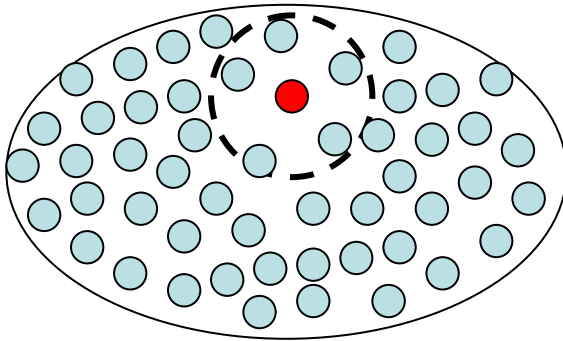
$$C = \frac{e}{4\pi\epsilon_0}$$

**N** total number of particles

**n** average particle density

The effective potential of a test charge can be defined as the sum of the potential of the uniform distribution and a “perturbed” term.

### Poisson Equation



$$\nabla^2 \Phi_s(\vec{r}) = \frac{e}{\epsilon_0} \delta(\vec{r}) + \frac{e}{\epsilon_0} \Delta n(\vec{r})$$

$$\Delta n(\vec{r}) = n e^{-e\Phi_s(\vec{r})/k_B T} - n \approx -\frac{e\Phi_s(\vec{r})}{k_B T}$$

$$\Phi_p(\vec{r}) = \frac{C}{r} e^{-r/\lambda_D}$$

$$\lambda_D = \sqrt{\frac{\epsilon_0 k_B T}{e^2 n}}$$

$k_B$  = Boltzman constant

T = Temperature

$k_B T$  = average kinetic energy of the particles

n = particle density (N/V)

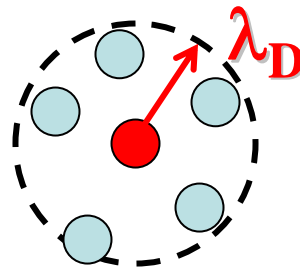
**Conclusion:** the effective interaction range of the test particle is limited to the **Debye length**

The charges surrounding the test particles have a screening effect

$$\Phi_s(\vec{r}) = \frac{C}{r} e^{-r/\lambda_D}$$

$\Phi_s(\vec{r}) \approx \Phi(\vec{r})$  for  $r \ll \lambda_D$

$\Phi_s(\vec{r}) \ll \Phi(\vec{r})$  for  $r \geq \lambda_D$

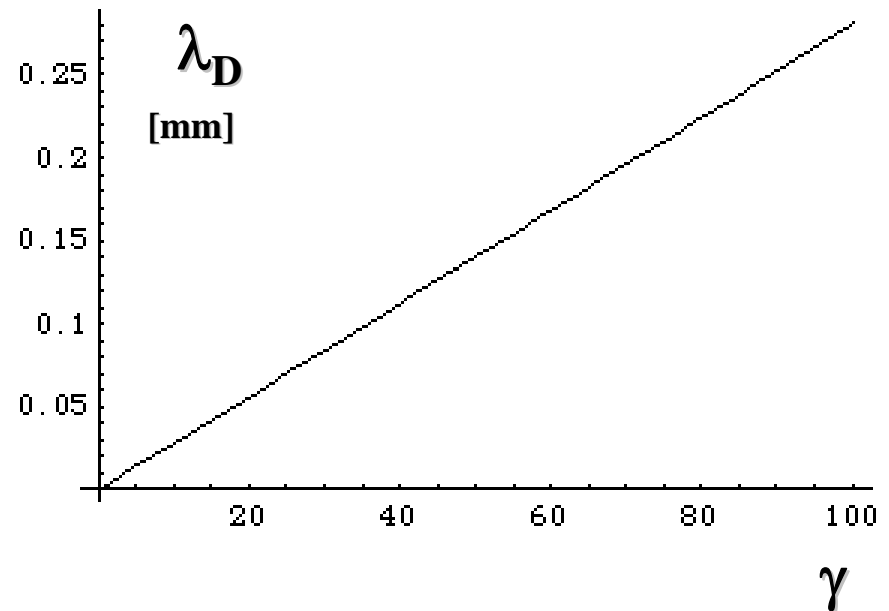
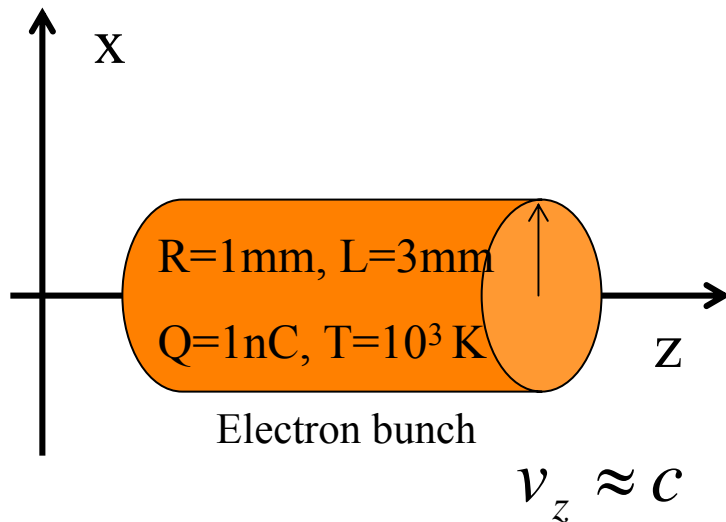


Smooth functions for the charge and field distributions can be used as long as the Debye length remains small compared to the particle bunch size

In a charged particle beam moving at a longitudinal relativistic velocity, assuming that the random transverse motion in the beam is non-relativistic, the Debye length has the following form:

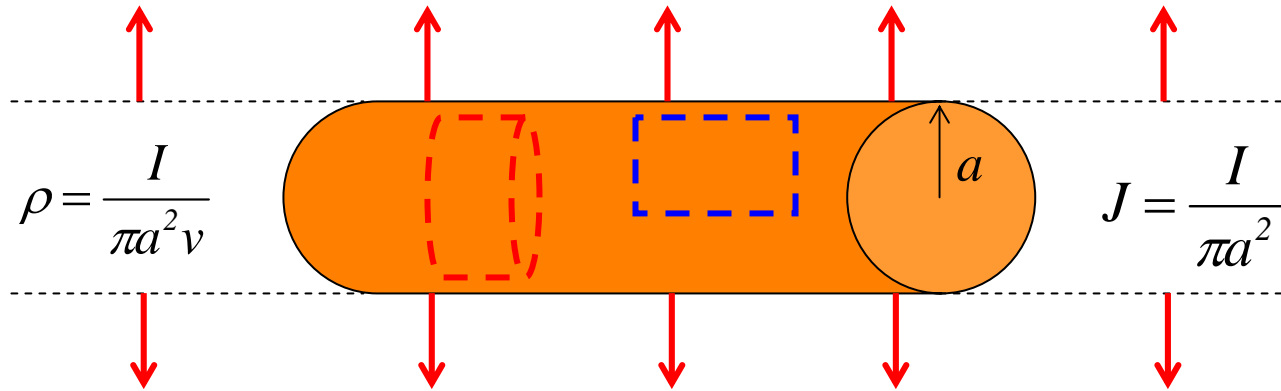
$$\lambda_D = \sqrt{\frac{\epsilon_0 \gamma^2 k_B T}{e^2 n}}$$

$$\langle v_x \rangle = \sqrt{\frac{k_B T}{\gamma m}} \ll c$$





# Continuous Uniform Cylindrical Beam Model



Gauss's law

$$\int \varepsilon_0 E \cdot dS = \int \rho dV$$

$$2\pi r l \varepsilon_0 E_r = \rho \pi r^2 l$$

$$2\pi r l \varepsilon_0 E_r = \rho \pi a^2 l$$

$$E_r = \frac{I}{2\pi \varepsilon_0 a^2 v} r \quad \text{for } r \leq a$$

$$E_r = \frac{I}{2\pi \varepsilon_0 v} \frac{1}{r} \quad \text{for } r > a$$

Ampere's law

$$\int B \cdot dl = \mu_0 \int J \cdot dS$$

$$2l B_g = \mu_0 J l r$$

$$B_g = \frac{\mu_0 J r}{2} = \mu_0 \frac{I r}{2\pi a^2} \quad \text{for } r \leq a$$

$$B_g = \frac{\beta}{c} E_r$$

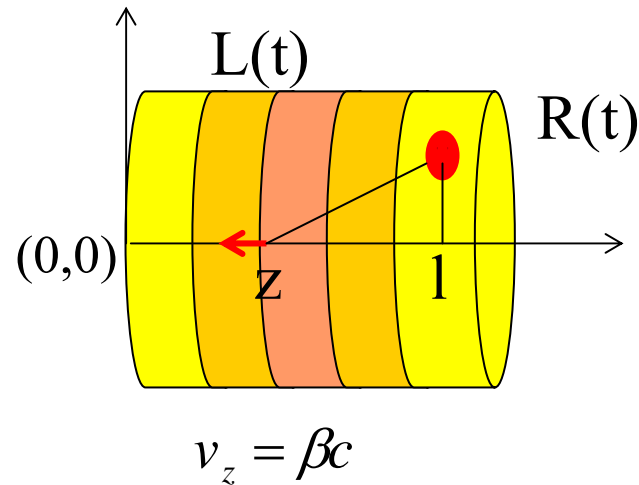
## Lorentz Force

$$F_r = e(E_r - \beta c B_\theta) = e(1 - \beta^2)E_r = \frac{eE_r}{\gamma^2}$$

has only **radial** component and  
is a **linear** function of the transverse coordinate

The attractive magnetic force, which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore, space charge defocusing is primarily a non-relativistic effect

# Bunched Uniform Cylindrical Beam Model



Longitudinal Space Charge field in the bunch moving frame:

$$\tilde{\rho} = \frac{Q}{\pi R^2 \tilde{L}} \quad \tilde{E}_z(\tilde{z}, r=0) = \frac{\tilde{\rho}}{4\pi\epsilon_0} \int_0^R \int_0^{2\pi} \int_0^{\tilde{L}} \frac{(\tilde{l} - \tilde{z})}{\left[ (\tilde{l} - \tilde{z})^2 + r^2 \right]^{3/2}} r dr d\phi d\tilde{l}$$

$$\tilde{E}_z(\tilde{z}, r=0) = \frac{\tilde{\rho}}{2\epsilon_0} \left[ \sqrt{R^2 + (\tilde{L} - \tilde{z})^2} - \sqrt{R^2 + \tilde{z}^2} + (2\tilde{z} - \tilde{L}) \right]$$

## Radial Space Charge field in the bunch moving frame

by series representation of axisymmetric field:

$$\tilde{E}_r(r, \tilde{z}) \cong \left[ \frac{\tilde{\rho}}{\varepsilon_0} - \frac{\partial}{\partial \tilde{z}} \tilde{E}_z(\tilde{z}, 0) \right] \frac{r}{2} + [\dots] \frac{r^3}{16} +$$

$$\tilde{E}_r(r, \tilde{z}) = \frac{\tilde{\rho}}{2\varepsilon_0} \left[ \frac{(\tilde{L} - \tilde{z})}{\sqrt{R^2 + (\tilde{L} - \tilde{z})^2}} + \frac{\tilde{z}}{\sqrt{R^2 + \tilde{z}^2}} \right] \frac{r}{2}$$

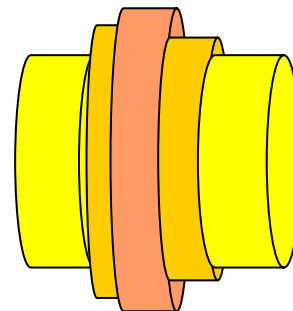
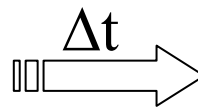
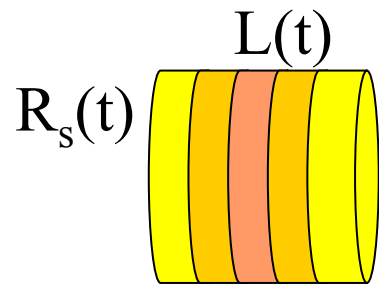
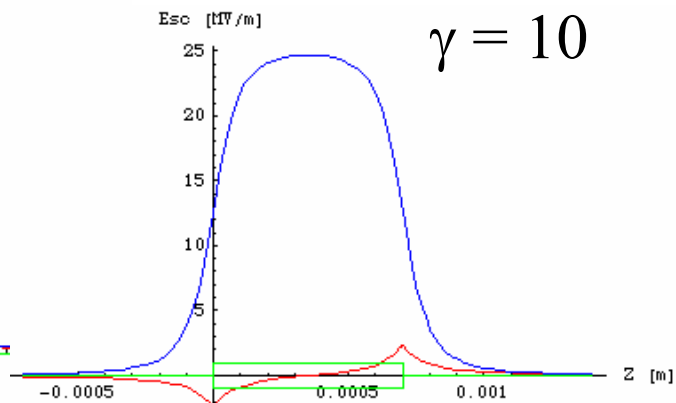
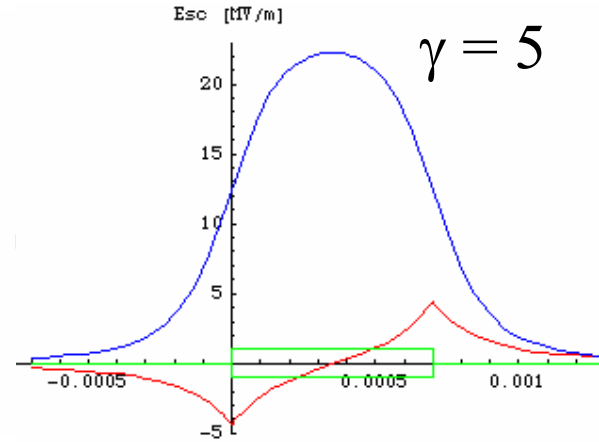
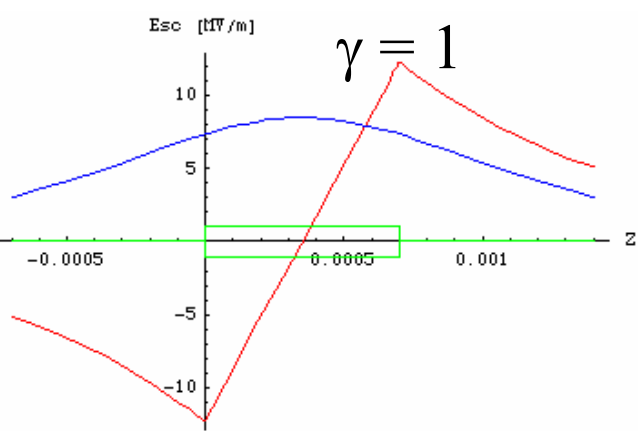
## Lorentz Transformation to the Lab frame

$$\begin{aligned} E_z &= \tilde{E}_z & \tilde{L} = \gamma L &\Rightarrow \tilde{\rho} = \frac{\rho}{\gamma} \\ E_r &= \gamma \tilde{E}_r & \tilde{z} &= \gamma z \end{aligned}$$

$$E_z(0, z) = \frac{\rho}{\gamma 2\epsilon_0} \left[ \sqrt{R^2 + \gamma^2 (L - z)^2} - \sqrt{R^2 + \gamma^2 z^2} + \gamma(2z - L) \right]$$

$$E_r(r, z) = \frac{\rho}{2\epsilon_0} \left[ \frac{\gamma(L - z)}{\sqrt{R^2 + \gamma^2 (L - z)^2}} + \frac{\gamma z}{\sqrt{R^2 + \gamma^2 z^2}} \right] \frac{r}{2}$$

**It is still a linear field with r but with a longitudinal correlation z**



# **Beam motion in a linear channel**

## Equation of motion in a drift space:

$$\gamma m \frac{d^2 r}{dt^2} = \frac{eE_r}{\gamma^2} = \frac{eI}{2\pi\gamma^2 \varepsilon_0 a^2 v} r$$

$$\frac{d^2 r}{dt^2} = \beta^2 c^2 \frac{d^2 r}{dz^2}$$

$$\frac{d^2 r}{dz^2} = \frac{eI}{2\pi m \gamma^3 \varepsilon_0 a^2 v^3} r = \frac{K}{a^2} r$$

$$K = \frac{eI}{2\pi m \gamma^3 \varepsilon_0 v^3} = \frac{2I}{I_0 \beta^3 \gamma^3}$$

Generalized perveance

$$I_0 = \frac{4\pi\varepsilon_0 mc^3}{e}$$

Alfven current



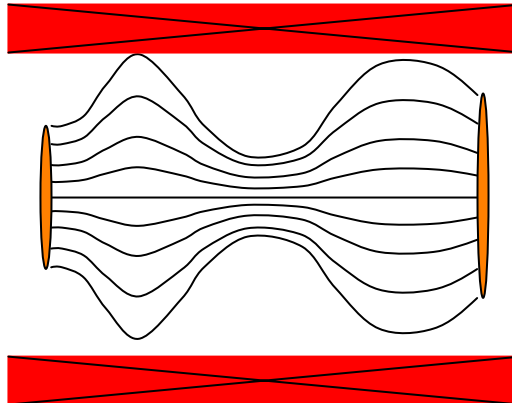
# Laminar Beam

If the initial particle velocities depend linearly on the initial coordinates

$$\frac{dr(0)}{dt} = Ar(0)$$

then the **linear dependence will be conserved during the motion**, because of the linearity of the equation of motion.

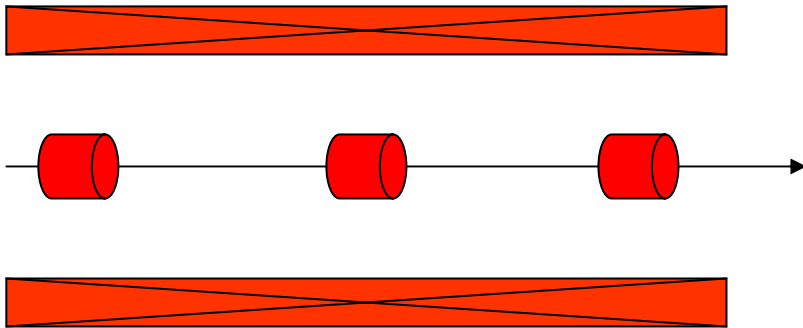
This means that particle trajectories do not cross ==> **Laminar Beam**



# Transport in a Long Solenoid

$$k_s = \frac{qB}{2mc\beta\gamma}$$

$$K(\zeta) = \frac{2I_g(\zeta)}{I_o(\beta\gamma)^3} \quad \zeta = \frac{z}{L}$$



$$R'' + k_s^2 R = \frac{K(\zeta)}{R}$$

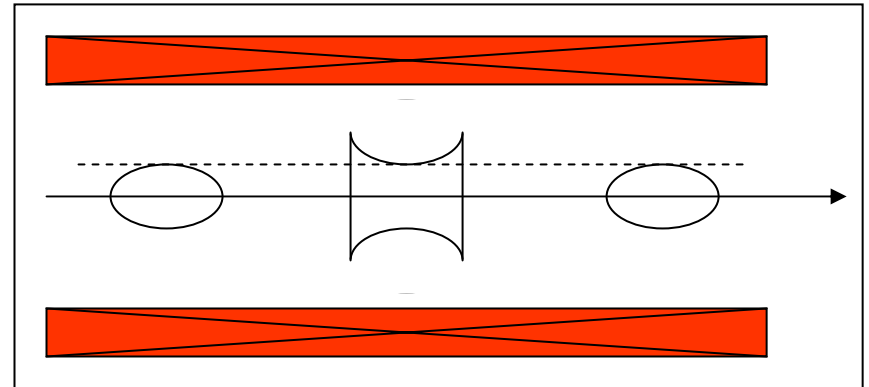
$$R'' = 0 \implies \text{Envelope equilibrium solution} \implies R_{eq}(\zeta) = \frac{\sqrt{K(\zeta)}}{k_s}$$

$$R(\zeta) = R_{eq}(\zeta) + \delta r(\zeta)$$

## Small perturbations around the equilibrium solution

$$k_s = \frac{qB}{2mc\beta\gamma}$$

$$R(\zeta) = R_{eq}(\zeta) + \delta r(\zeta) \cos(\sqrt{2}k_s z)$$
$$R'(\zeta) = -\delta r(\zeta) \sin(\sqrt{2}k_s z)$$

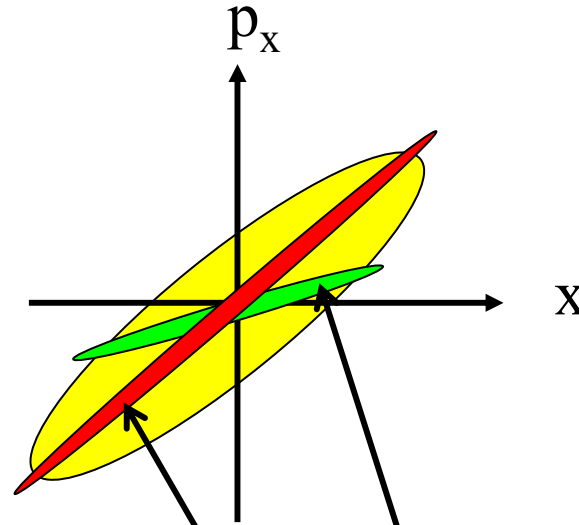


QuickTime™ and a  
Animation decompressor  
are needed to see this picture.

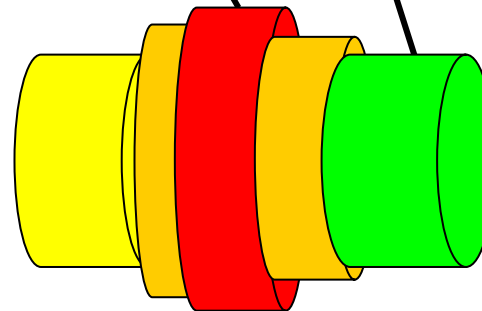
QuickTime™ and a  
Animation decompressor  
are needed to see this picture.

# Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

Projected Phase Space



Slice Phase Spaces



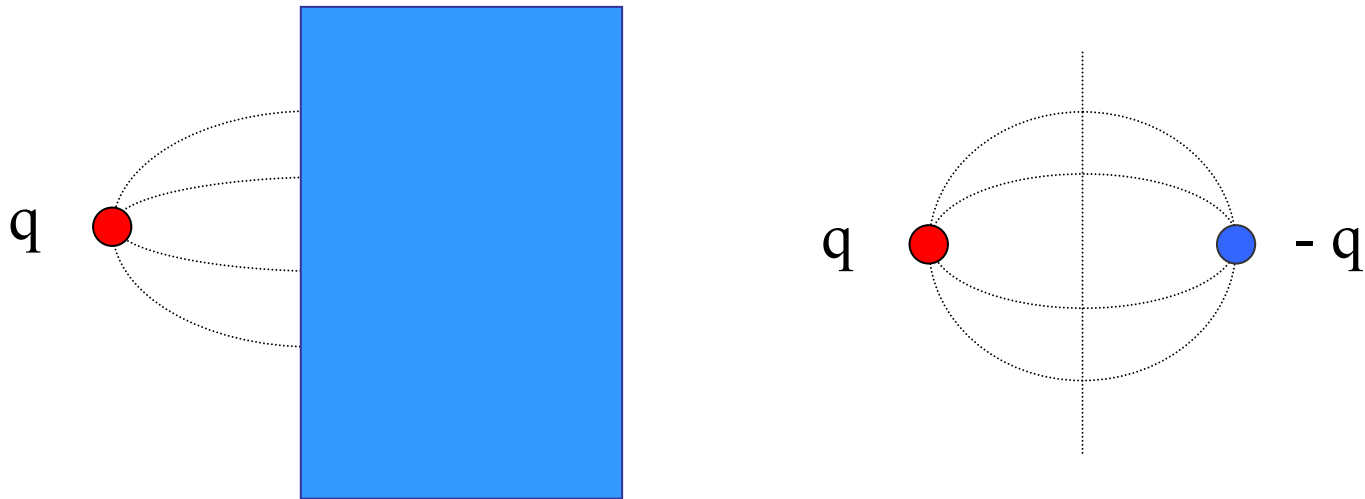
QuickTime™ and a  
Animation decompressor  
are needed to see this picture.

# Space charge with image currents

## Effects of conducting or magnetic screens

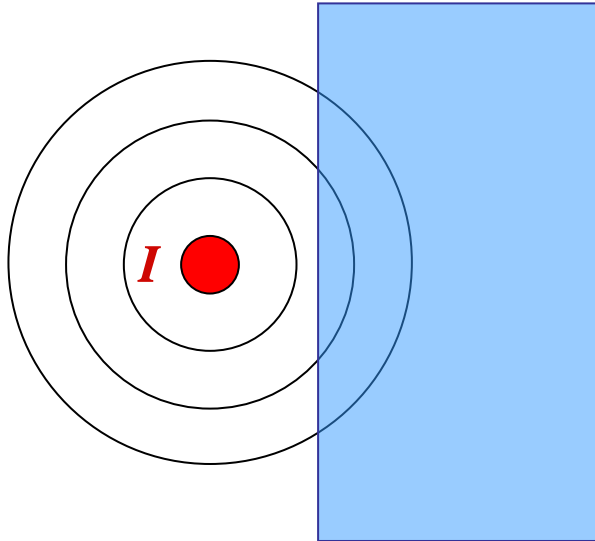
Let us consider a point charge close to a **conducting screen**.

The electrostatic field can be derived through the "image method". Since the metallic screen is an **equi-potential plane**, it can be removed provided that a "virtual" charge is introduced such that the potential is constant at the position of the screen

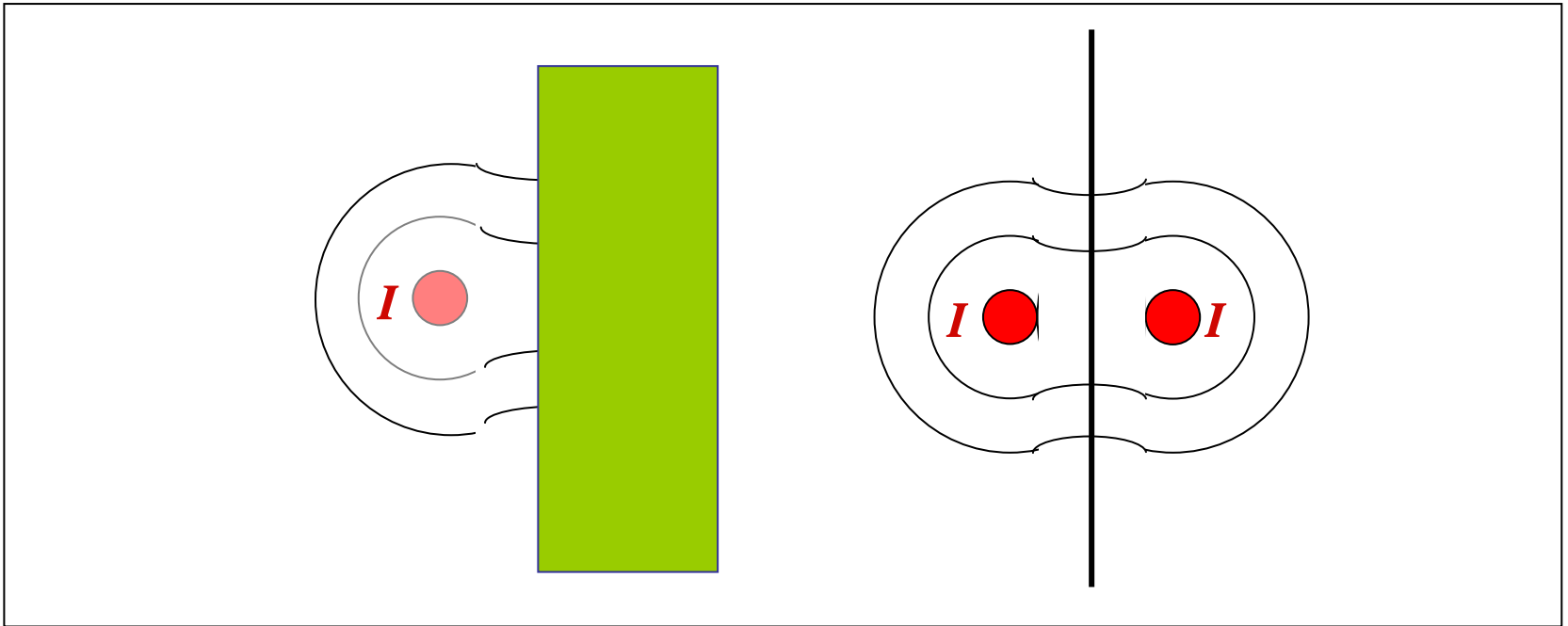


A constant current in the free space produces circular magnetic field.

If  $\mu_r \approx 1$ , the material, even in the case of a good conductor, does not affect the field lines.

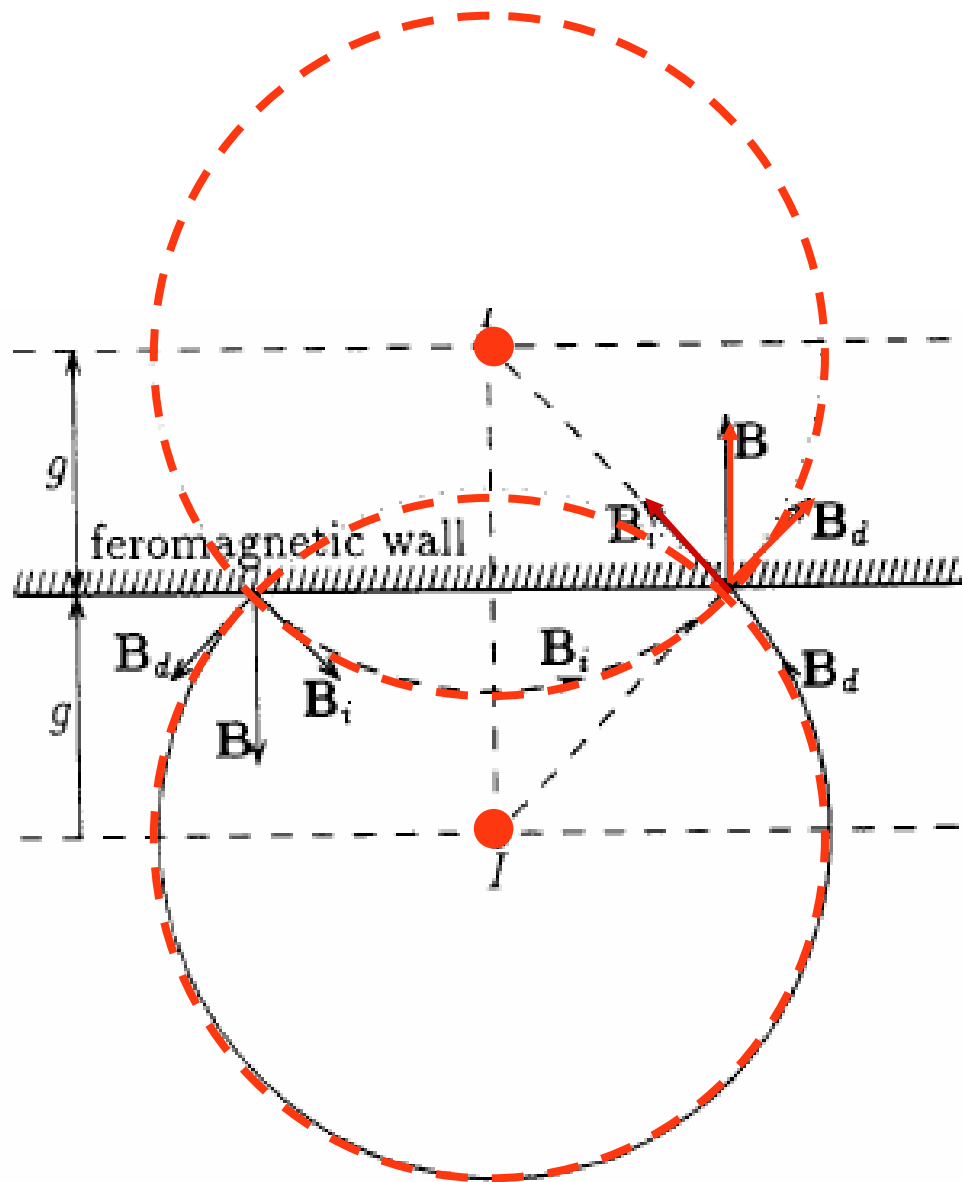


However, if the material is of **ferromagnetic type**, with  $\mu_r \gg 1$ , due to its magnetisation, the magnetic field lines are strongly affected, inside and outside the material. In particular **a very high magnetic permeability makes the tangential magnetic field zero at the boundary** so that the magnetic field is perpendicular to the surface, just like the electric field lines close to a conductor.



In analogy with the image method for charges close to conducting screens, we get the magnetic field, in the region outside the material, as superposition of the fields due to two symmetric equal currents flowing in the same direction.





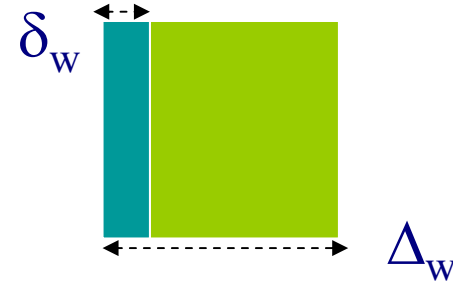
Satisfying a magnetic boundary condition by an image current

*A. Hofmann*

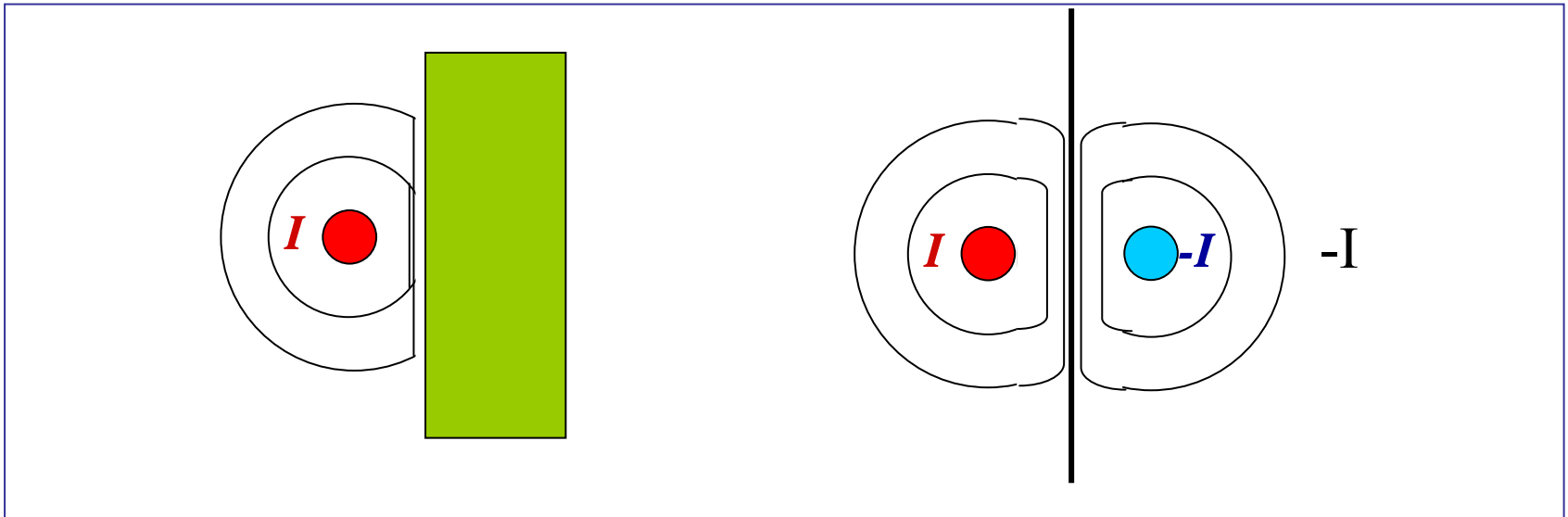
# Time-varying fields

It is necessary to compare the **wall thickness** and the **skin depth** (region of penetration of the e.m. fields) in the conductor.

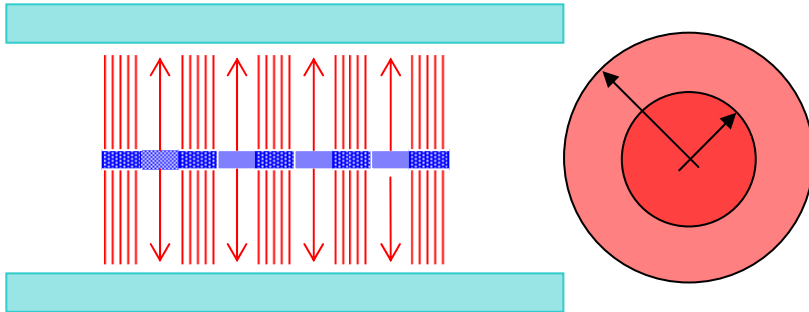
$$\delta_w \cong \sqrt{\frac{2}{\omega\sigma\mu}}$$



If the **fields penetrate** and pass through the material, we are practically in the **static boundary conditions case**. Conversely, if the **skin depth is very small**, fields do not penetrate, the electric field lines are perpendicular to the wall, as in the static case, while **the magnetic field lines are tangent to the surface**.



## Circular Perfectly Conducting Pipe (Beam at Center)



In the case of charge distribution, and  $\gamma \rightarrow \infty$ , the electric field lines are perpendicular to the direction of motion. The transverse fields intensity can be computed like in the static case, applying the Gauss and Ampere laws.

$$\lambda_o = \rho \pi a^2$$

$$\lambda(r) = \lambda_o (r / a)^2$$

$$J = \beta c \rho$$

$$I = J \pi a^2 = \beta c \lambda_o (A)$$

for  $r < a$

$$E_r(r) = \frac{\lambda_o r}{2\pi\epsilon_o a^2}$$

$$B_\theta(r) = \frac{\beta}{c} E_r(r) = \frac{\lambda_o \beta}{2\pi\epsilon_o c} \frac{r}{a^2}$$

$$F_r = e(E_r - \beta c B_\theta) = e(1 - \beta^2) E_r = \frac{e E_r}{\gamma^2} = \frac{e \lambda_o r}{2\pi\epsilon_o \gamma^2 a^2}$$

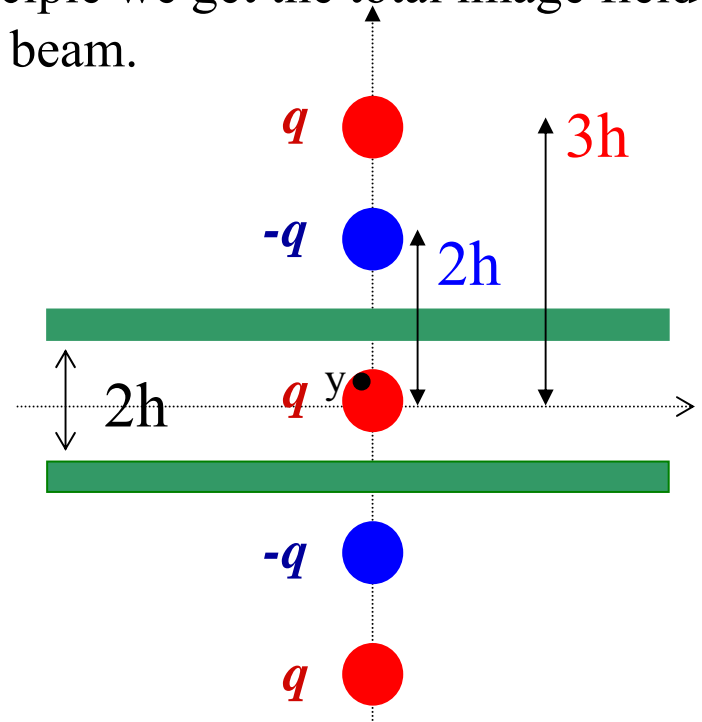
there is a cancellation of the electric and magnetic forces

## Parallel Plates (Beam at Center)

In some cases, the beam pipe cross section is such that we can consider only the surfaces closer to the beam, which behave like two parallel plates. In this case, we use the image method to a charge distribution of radius  $a$  between two conducting plates  $2h$  apart. By applying the superposition principle we get the total image field at a position  $y$  inside the beam.

$$E_y^{im}(z, y) = \frac{\lambda(z)}{2\pi\epsilon_0} \sum_{n=1}^{\infty} (-1)^n \left[ \frac{1}{2nh+y} - \frac{1}{2nh-y} \right]$$

$$E_y^{im}(z, y) = \frac{\lambda(z)}{2\pi\epsilon_0} \sum_{n=1}^{\infty} (-1)^n \frac{-2y}{(2nh)^2 - y^2} \cong \frac{\lambda(z)}{4\pi\epsilon_0 h^2} \frac{\pi^2}{12} y$$



Where we have assumed  $h \gg a > y$ .

For d.c. or slowly varying currents, the boundary condition imposed by the conducting plates does not affect the magnetic field. As a consequence there is no cancellation effect for the fields produced by the "image" charges.

From the divergence equation we derive also the other transverse component, notice the opposite sign:

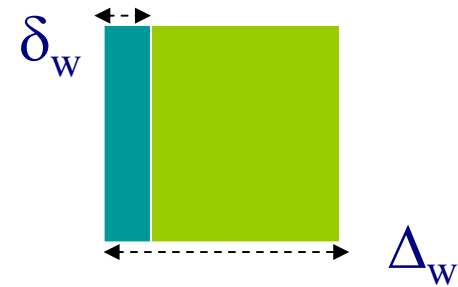
$$\frac{\partial}{\partial x} E_x^{im} = -\frac{\partial}{\partial y} E_y^{im} \Rightarrow E_x^{im}(z, x) = \frac{-\lambda(z)}{4\pi\epsilon_0 h^2} \frac{\pi^2}{12} x$$

Including also the direct space charge force, we get:

$$F_x(z, x) = \frac{e\lambda(z)x}{\pi\epsilon_0} \left( \frac{1}{2a^2\gamma^2} - \frac{\pi^2}{48h^2} \right)$$
$$F_y(z, x) = \frac{e\lambda(z)y}{\pi\epsilon_0} \left( \frac{1}{2a^2\gamma^2} + \frac{\pi^2}{48h^2} \right)$$

Therefore, for  $\gamma \gg 1$ , and for d.c. or slowly varying currents the cancellation effect applies only for the direct space charge forces. There is no cancellation of the electric and magnetic forces due to the "image" charges.

## Parallel Plates (Beam at Center) a.c. currents

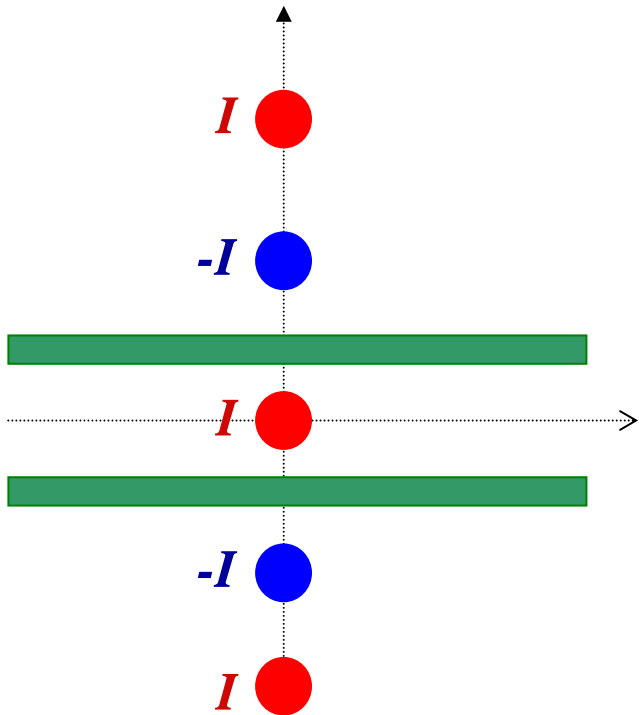


Usually, the frequency beam spectrum is quite rich of harmonics, especially for bunched beams.

It is convenient to decompose the current into a d.c. component,  $I$ , for which  $\delta_w \gg \Delta_w$ , and an a.c. component,  $\hat{I}$ , for which  $\delta_w \ll \Delta_w$ .

While the d.c. component of the magnetic field does not perceive the presence of the material, **its a.c. component is obliged to be tangent at the wall.** For a charge density  $\lambda$  we have  $I = \lambda v$ .

We can see that this current produces a magnetic field able to cancel the effect of the electrostatic force.



$$\tilde{E}_y(z, x) = \frac{\tilde{\lambda}(z)y}{\pi \epsilon_0} \frac{\pi^2}{48h^2}; \quad \tilde{B}_x(z, x) = \frac{\beta}{c} \tilde{E}_y(z, x)$$

$$\tilde{F}_y(z, x) = \frac{e \tilde{\lambda}(z)y}{\pi \epsilon_0 \gamma^2} \frac{\pi^2}{48h^2}$$

$$\tilde{F}_x(z, x) = \frac{e \tilde{\lambda}(z)x}{2\pi \epsilon_0 \gamma^2} \left( \frac{1}{a^2} - \frac{\pi^2}{24h^2} \right)$$

$$\tilde{F}_y(z, x) = \frac{e \tilde{\lambda}(z)y}{2\pi \epsilon_0 \gamma^2} \left( \frac{1}{a^2} + \frac{\pi^2}{24h^2} \right)$$

*There is cancellation of the electric and magnetic forces !!*

## Parallel Plates - General expression of the force

Taking into account all the boundary conditions for d.c. and a.c. currents, we can write the expression of the force as:

$$F_u = \frac{e}{2\pi \varepsilon_0} \left[ \frac{1}{\gamma^2} \left( \frac{1}{a^2} \mp \frac{\pi^2}{24h^2} \right) \lambda \mp \beta^2 \left( \frac{\pi^2}{24h^2} + \frac{\pi^2}{12g^2} \right) \bar{\lambda} \right] u$$

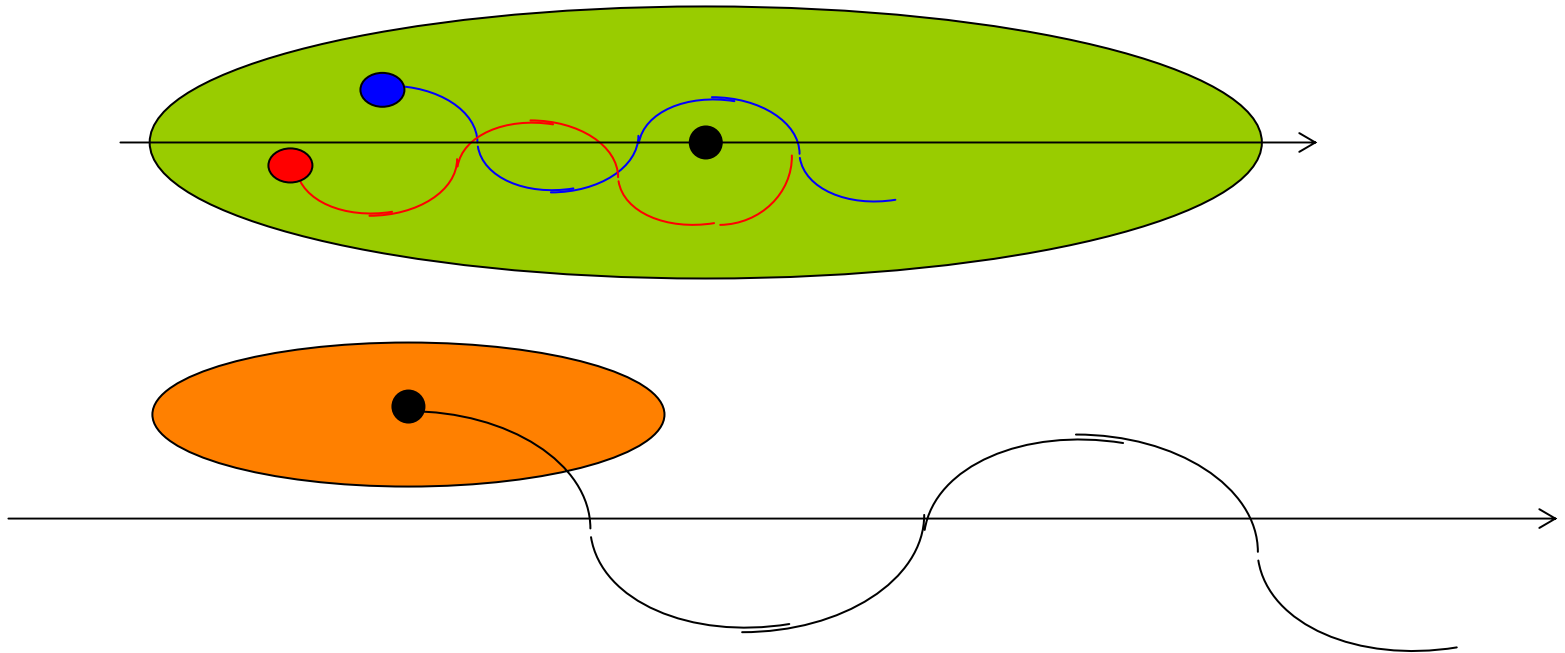
where  $\lambda$  is the total current, and  $\bar{\lambda}$  its d.c. part. We take the sign (+) if  $u=y$ , and the sign (−) if  $u=x$ .



# Space charge effects in storage rings

## Incoherent and Coherent Transverse Effects

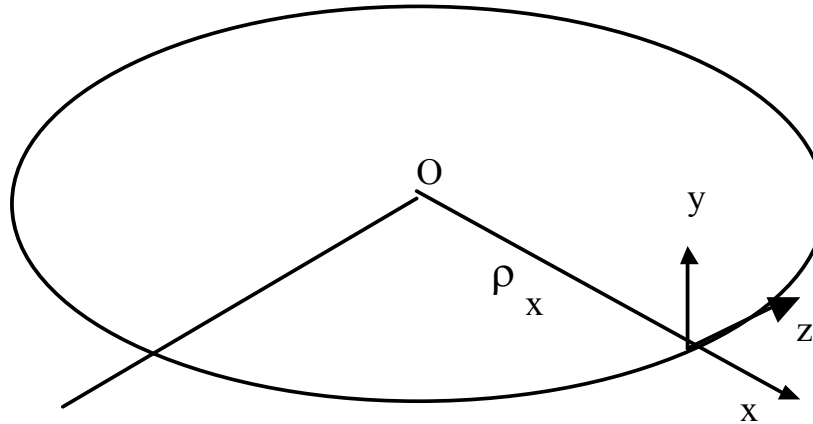
When the beam is located at the centre of symmetry of the pipe, the e.m. forces due to space charge and images cannot affect the motion of the centre of mass (**coherent**), but change the trajectory of individual charges in the beam (**incoherent**).



These force may have a complicate dependence on the charge position. A simple analysis is done considering only the linear expansion of the self-fields forces around the equilibrium trajectory.

## Self Fields and betatron motion

Consider a perfectly circular accelerator with radius  $\rho_x$ . The beam circulates inside the beam pipe. The transverse single particle motion in the linear regime, is derived from the equation of motion. Including the self field forces in the motion equation, we have



$$\frac{d(m\gamma\mathbf{v})}{dt} = \mathbf{F}^{ext}(\vec{r}) + \mathbf{F}^{self}(\vec{r})$$

$$\frac{d\mathbf{v}}{dt} = \frac{\mathbf{F}^{ext}(\vec{r}) + \mathbf{F}^{self}(\vec{r})}{m\gamma}$$

Following the same steps already seen in the "transverse dynamics" lectures, we write:

$$\vec{r} = (\rho_x + x)\hat{e}_x + y\hat{e}_y$$

$$\vec{v} = \dot{X}\hat{e}_x + \dot{Y}\hat{e}_y + \omega_o(\rho_x + x)\hat{e}_z$$

$$\vec{a} = [\ddot{X} - \omega_o^2(\rho_x + x)]\hat{e}_x + \ddot{Y}\hat{e}_y + [\dot{\omega}_o(\rho_x + x) + 2\omega_o\dot{X}]\hat{e}_z$$

For the motion along x:

$$\ddot{X} - \omega_o^2(\rho_x + x) = \frac{1}{m\gamma} (F_x^{ext} + F_x^{self})$$

which is rewritten with respect to the azimuthal position  $s = v_z t$ :

$$\ddot{X} = v_z^2 x'' = \omega_o^2 (\rho_x + x)^2 x''$$

$$x'' - \frac{1}{\rho_x + x} = \frac{1}{mv_z^2 \gamma} (F_x^{ext} + F_x^{self})$$

We assume small transverse displacements  $x$  with respect to the closed orbit, and only dipoles for bending and quadrupole to keep the beam around the closed orbit:

$$F_x^{ext} = F_o^{ext} + \left( \frac{\partial F_x^{ext}}{\partial x} \right)_{x=0} x \quad x \ll \rho_x$$

Around the closed orbit, putting  $v_z = \beta c$ , we get

$$x'' + \left[ \frac{1}{\rho_x^2} - \frac{1}{\beta^2 E_o} \left( \frac{\partial F_x^{ext}}{\partial x} \right)_{x=0} \right] x = \frac{1}{\beta^2 E_o} F_x^{self}(x)$$

where  $E_o$  is the particle energy. This equation expressed as function of “s” reads:

$$x''(s) + \left[ \frac{1}{\rho_x^2(s)} + K_x(s) \right] x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x, s)$$

In the analysis of the motion of the particles in presence of the self field, we will adopt a simplified model where particles execute simple harmonic oscillations around the reference orbit.

This is the case where the focussing term is constant. Although this condition is never fulfilled in a real accelerator, it provides a reliable model for the description of the beam instabilities

$$x''(s) + K_x x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x)$$

$Q_x$  Betatron tune: n. of betatron oscillations per turn  $K_x = \left( \frac{Q_x}{\rho_x} \right)^2$

$$x''(s) + \left( \frac{Q_x}{\rho_x} \right)^2 x(s) = \frac{1}{\beta^2 E_o} F_x^{self}(x, s)$$

## Transverse Incoherent Effects

We take the linear term of the transverse force in the betatron equation:

$$F_x^{s.c.}(x, z) \cong \left( \frac{\partial F_x^{s.c.}}{\partial x} \right)_{x=0} x$$

$$x'' + \left( \frac{Q_x}{\rho_x} \right)^2 x = \frac{1}{\beta^2 E_o} \left( \frac{\partial F_x^{s.c.}}{\partial x} \right)_{x=0} x$$

$$x'' + \left( \left( \frac{Q_x}{\rho_x} \right)^2 - \frac{1}{\beta^2 E_o} \left( \frac{\partial F_x^{s.c.}}{\partial x} \right)_{x=0} \right) x = 0$$

$$(Q_x + \Delta Q_x)^2 \cong Q_x^2 + 2Q_x \Delta Q_x \Rightarrow \Delta Q_x = -\frac{\rho_x^2}{2\beta^2 E_o Q_x} \left( \frac{\partial F_x^{s.c.}}{\partial x} \right)$$

The betatron shift is negative since the space charge forces are defocusing on both planes. Notice that the tune shift is in general function of “z”, therefore there is a tune spread inside the beam.

**Example: Incoherent betatron tune shift for an uniform electron beam of radius  $a$ , length  $l_0$ , inside circular perfectly conducting Pipe**

$$\left( \frac{\partial F_x^{s.c.}}{\partial x} \right) = \frac{\partial}{\partial x} \frac{e\lambda_0 x}{2\pi\epsilon_0\gamma^2 a^2} = \frac{e\lambda_0}{2\pi\epsilon_0\gamma^2 a^2}$$

$$\Delta Q_x = - \frac{\rho_x^2 N e^2}{4\pi\epsilon_0 a^2 \beta^2 \gamma^2 E_0 Q_{xo} l_0}$$

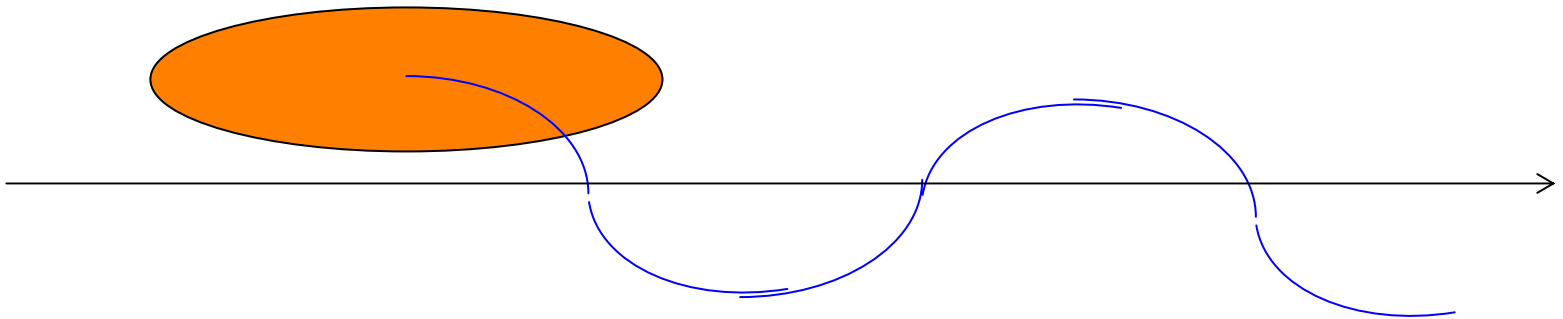
$$r_{e,p} = \frac{e^2}{4\pi\epsilon_0 m_0 c^2} \text{ (electrons : } 2.82 \cdot 10^{-15} \text{ m, protons : } 1.53 \cdot 10^{-18} \text{ m)}$$

$$\Delta Q_x = - \frac{\rho_x^2 N r_{e,p}}{a^2 \beta^2 \gamma^3 Q_{xo} l_0}$$

**For a real bunched beams the space charge forces, and the tune shift depend on the longitudinal and radial position of the charge.**



## Transverse Coherent Effects



The whole beam can execute betatron oscillations.

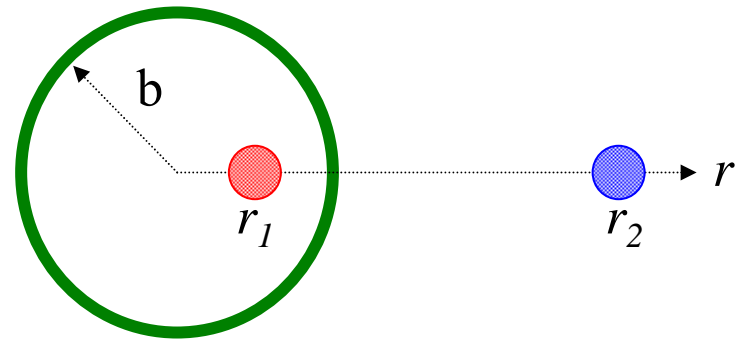
The environment affects this motion, causing a shift of the coherent tune which is generally different from the incoherent one.

For instance, in the case of circular pipe with an off-centre beam, there is a transverse force which tends to further deflect the beam, producing a defocusing coherent effect.

## Circular Perfectly Conducting Pipe (Beam Off-Center)

Only in the case of pure cylindrical symmetry, the external pipe does not affect the transverse space charge fields. When the symmetry is broken, the image charges give a significant contribution.

If the beam is displaced by  $r_1$  from the axis, by applying the image theorem we find that the image charge is located at the position  $r_2$  such that  $r_1 r_2 = b^2$ . Assuming that  $r_1 \ll b$  and that the beam radius  $a \ll b$ , we find the electric field:

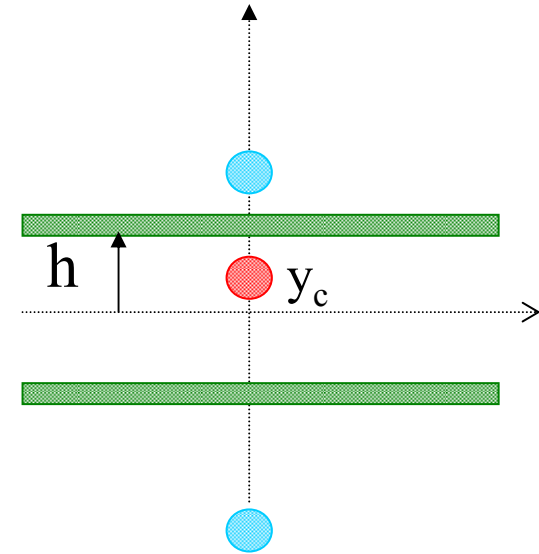


$$E_r(\mathbf{r}, z) = \frac{\lambda(z)}{2\pi\epsilon_0} \frac{r_1}{b^2}$$

$$\Delta Q_x^{c.} = \frac{-\rho_x^2}{2Q_u\beta^2 E_o} \left( \frac{\partial F_u^{s.c.}}{\partial u} \right)^{c.} = -\frac{e\rho_x^2 \lambda_o}{4\pi\epsilon_o\beta^2 E_o Q_x b^2}$$

## Parallel Plates (Beam Off-Center)

If the whole beam is displaced from the axis by  $y_c$ , the image charges produce a transverse field which leads to coherent effects:



$$E_y^{im-c}(z, y_c) = \frac{\lambda(z)}{4\pi\epsilon_0 h^2} \frac{\pi^2}{4} y_c$$

$$\Delta Q_y^c = \frac{-\rho_x^2}{2Q_u \beta^2 E_o} \left( \frac{\partial F_u^{s.c.}}{\partial u} \right)^c = \frac{-\rho_x^2}{2\pi Q_y \beta^2 E_o} \frac{\lambda_0}{\epsilon_0 h^2} \frac{\pi^2}{16}$$

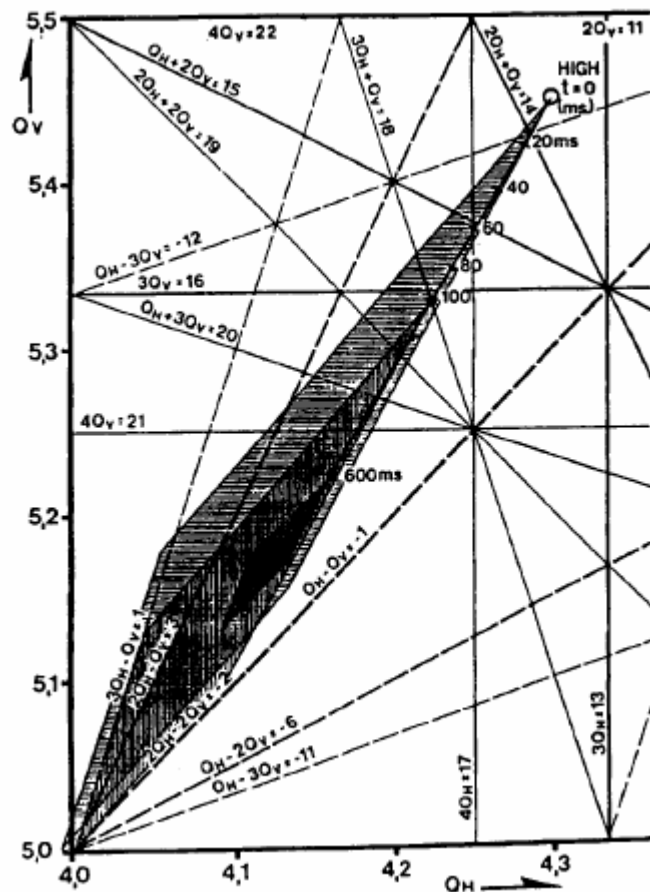
## Consequences of the space charge tune shifts

In circular accelerators the values of the betatron tunes should not be close to rational numbers in order to avoid the crossing of linear and non-linear resonances where the beam becomes unstable.

The tune spread induced by the space charge force can make hard to satisfy this basic requirement. Typically, in order to avoid major resonances the stability requires

$$|\Delta Q_u| < 0.3$$

**END**



PS Booster, accelerate proton bunches From 50 to 800 MeV in about 0.6 s. The tunes occupied by the particle are indicated in the diagram by the shaded area. As time goes on, the energy increase and the space charge tune spread gets smaller covering at  $t=100$  ms the tune area shown by the darker area. The point of highest tune correspond to the particles which are least affected by the space charge. This point moves in the Q diagram since the external focusing is adjusted such that the reduced tune spread lies in a region free of harmful resonances.

Finally, the small dark area shows the situation at  $t=600$  ms when the beam has Reached the energy of 800 MeV. The tune spread reduction is lower than expected with the energy increase ( $1/\gamma^3$ ) dependence since the bunch dimensions also decrease during the acceleration.