Kinematics of Particle Beams - Relativity

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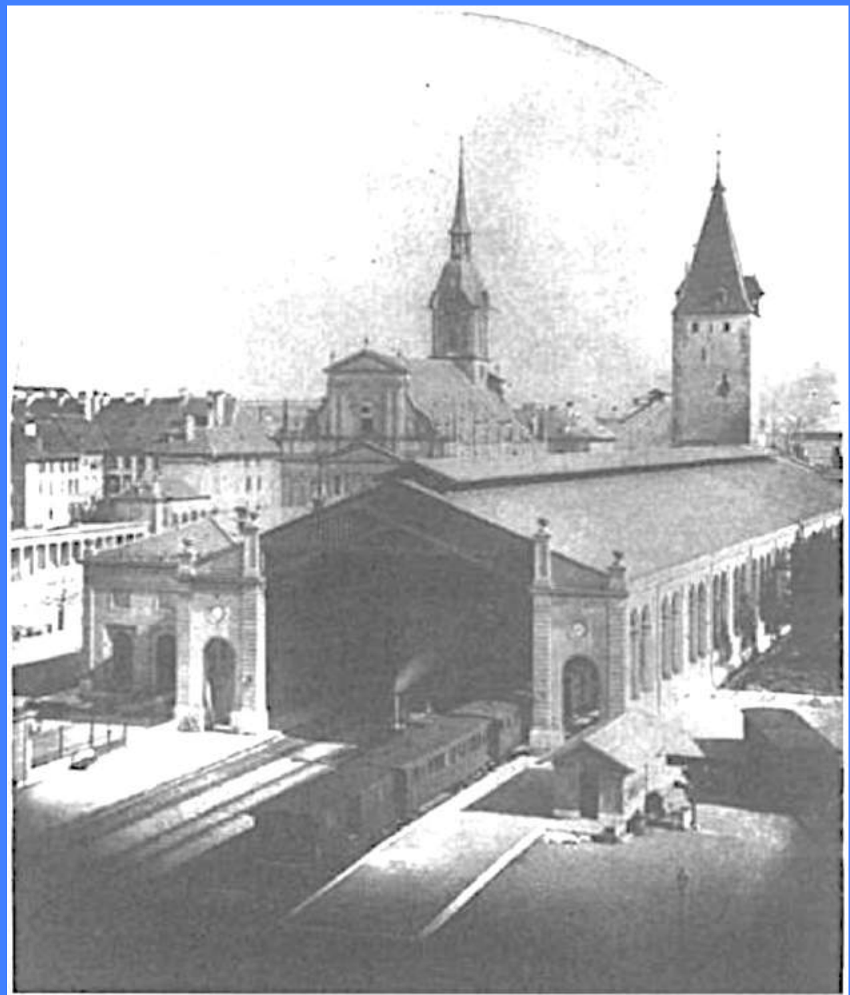


Introduction to Accelerator Physics - Constanta – 17 September 2018

FIG. 16.—Map of electrical clock network in Bern. Harvard Map Collection, using data from Jakob Messerli, *Gleichmässig pünktlich schnell*.

Master Clocks

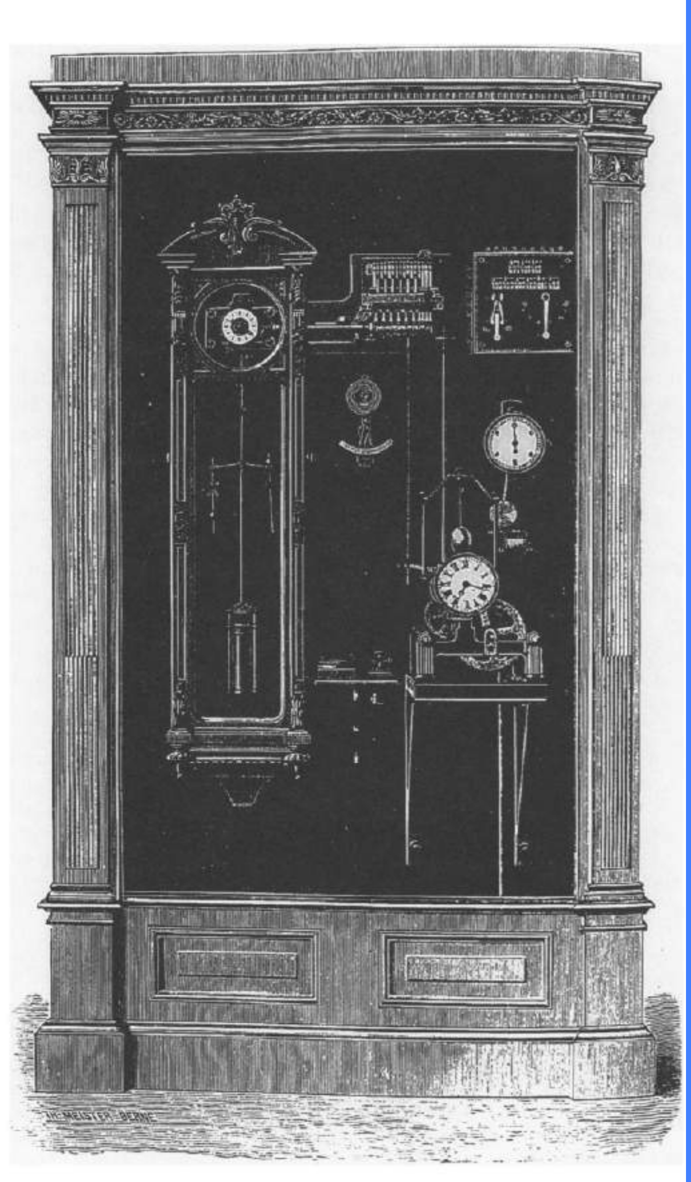


FIG. 5.—L'Horloge-mère, Neuchâtel. From A. Favarger, *L'Électricité et ses applications à la chronométrie* (Neuchâtel, 1924), p. 414.

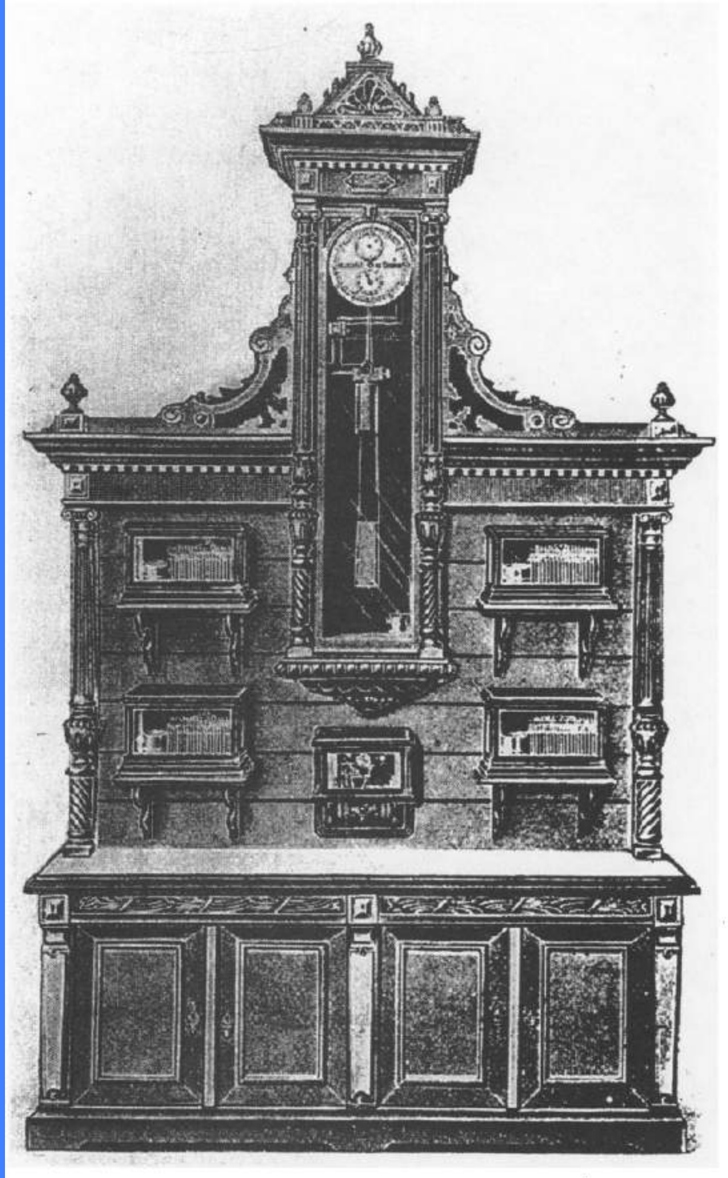
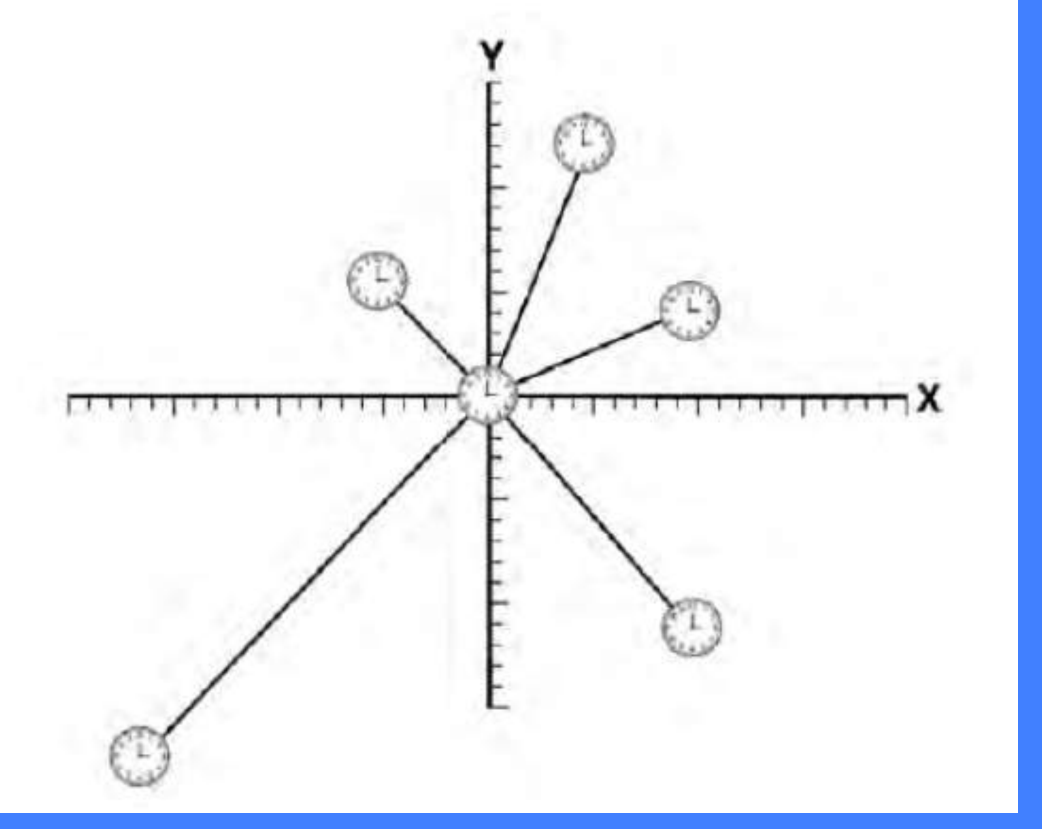


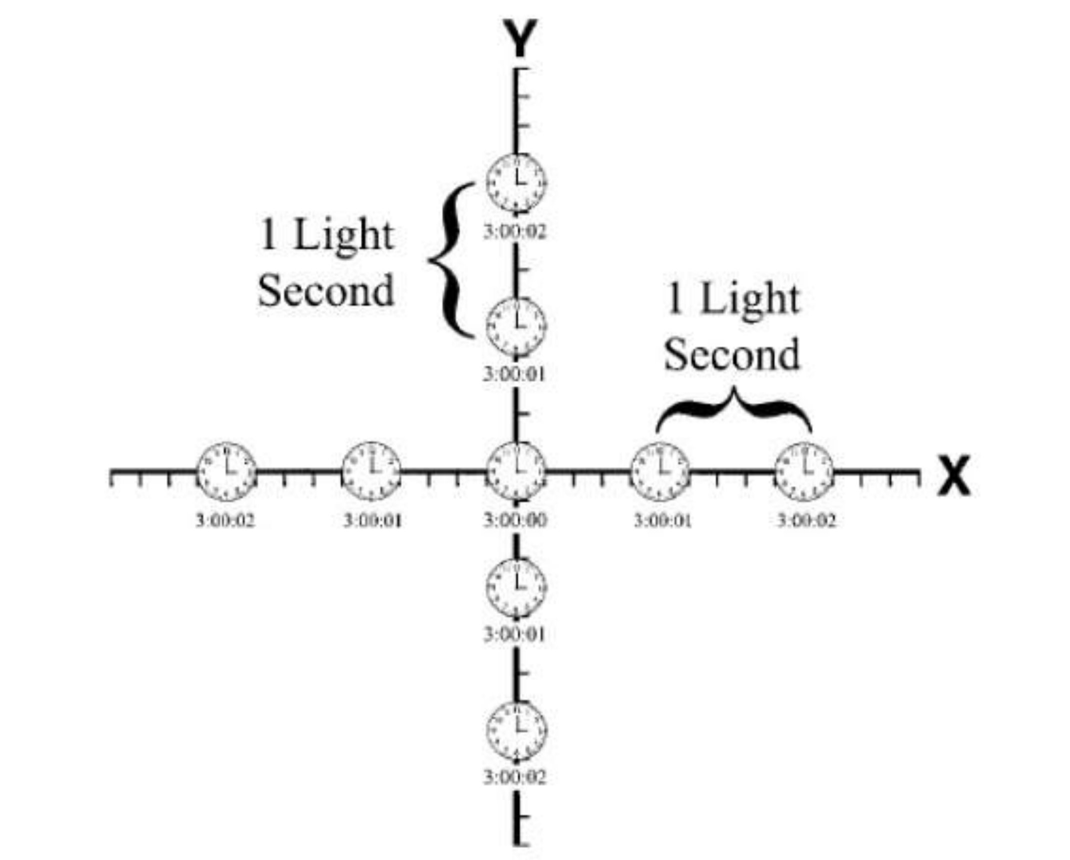
FIG. 6.—Normaluhr, Silesischer Bahnhof. From A. Favarger, *L'Électricité et ses applications à la chronométrie*, p. 470.

Master Clock Synchronization



In his 1905 paper on special relativity, Einstein introduced—and rejected—a scheme of clock coordination in which the central clock sent a signal to all other clocks; these secondary clocks set their times when the signal arrived.
Einstein's objection: the secondary clocks were at different distances from the center so close clockswould be set by the arriving signal before distant ones. This made the simultaneity of two clocks depend (unacceptably to Einstein) on the arbitrary circumstance of where the time-setting "central" clock happened to be.

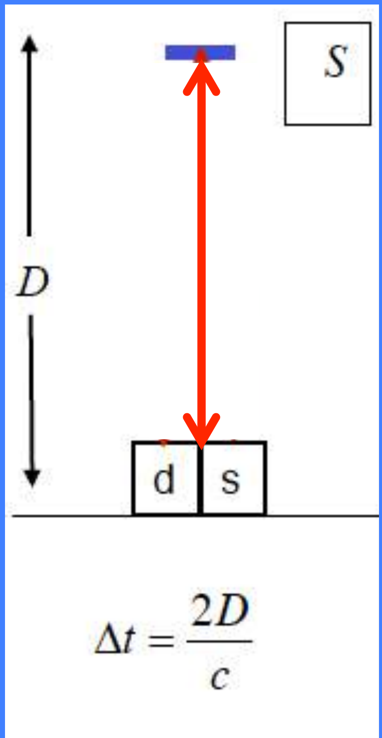
Einstein’s Clock Synchronization

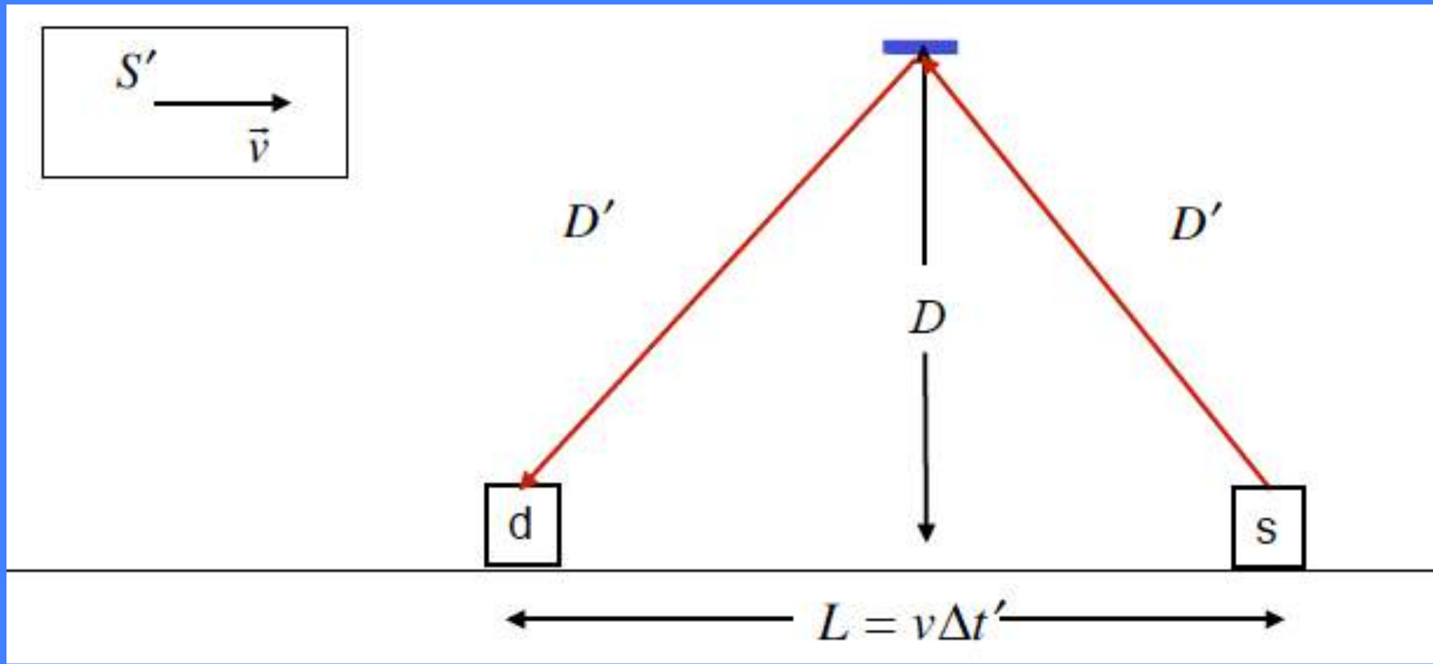


Set clocks not to the time that the signal was launched, but to the time of the initial clock plus the time it took for the signal to travel the distance from the initial clock to the clock being synchronized.

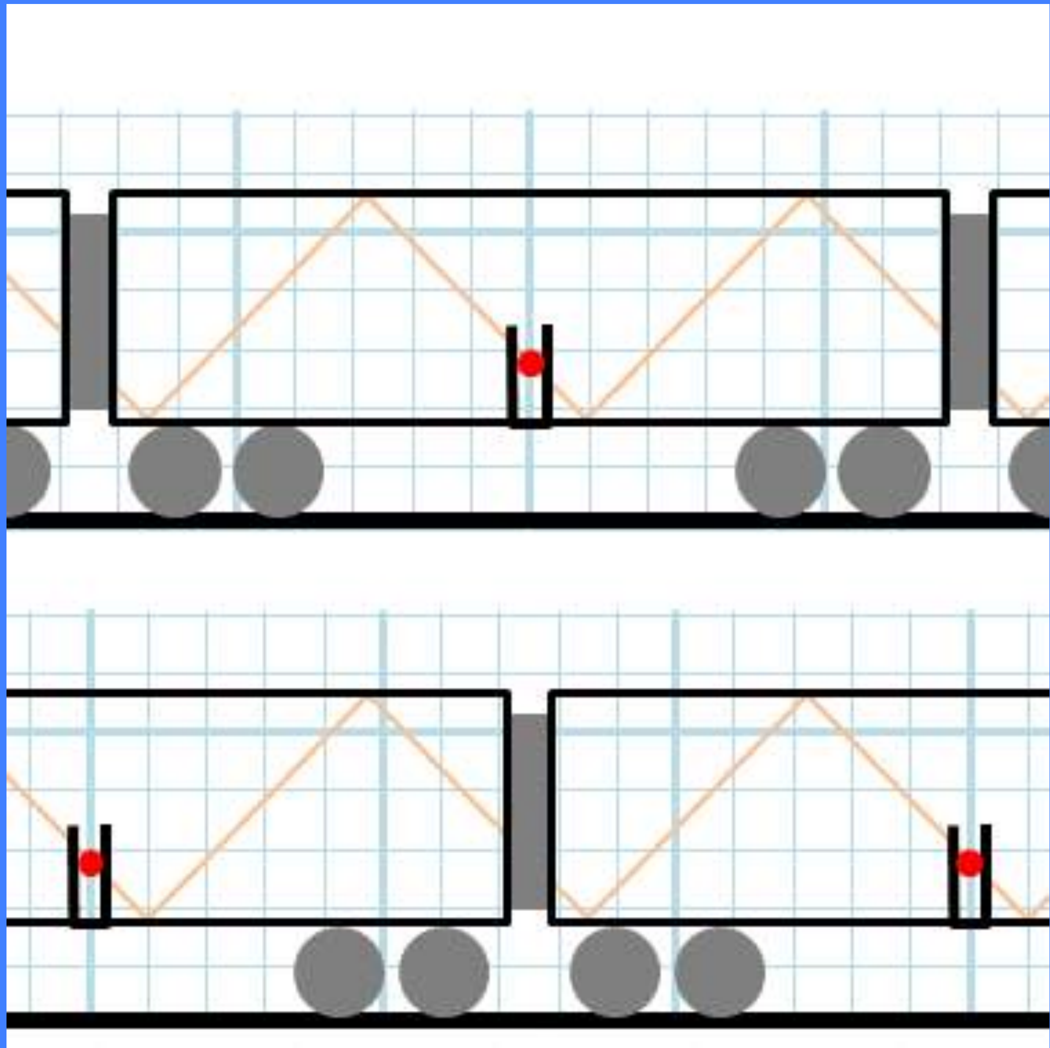
Specifically, he advocated sending a **round-trip signal** from the initial clock to the distant clock and then setting the distant clock to the initial clock’s time plus half the round-trip time. In this way the location of the “central” clock made no difference–one could start the procedure at any point and unambiguously fix simultaneity.

Light Clock and moving systems

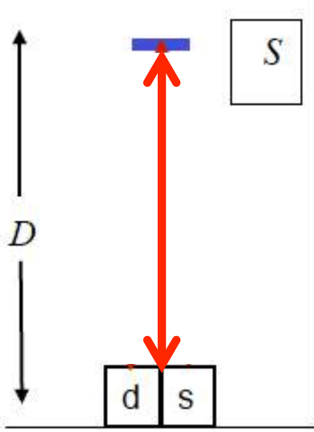




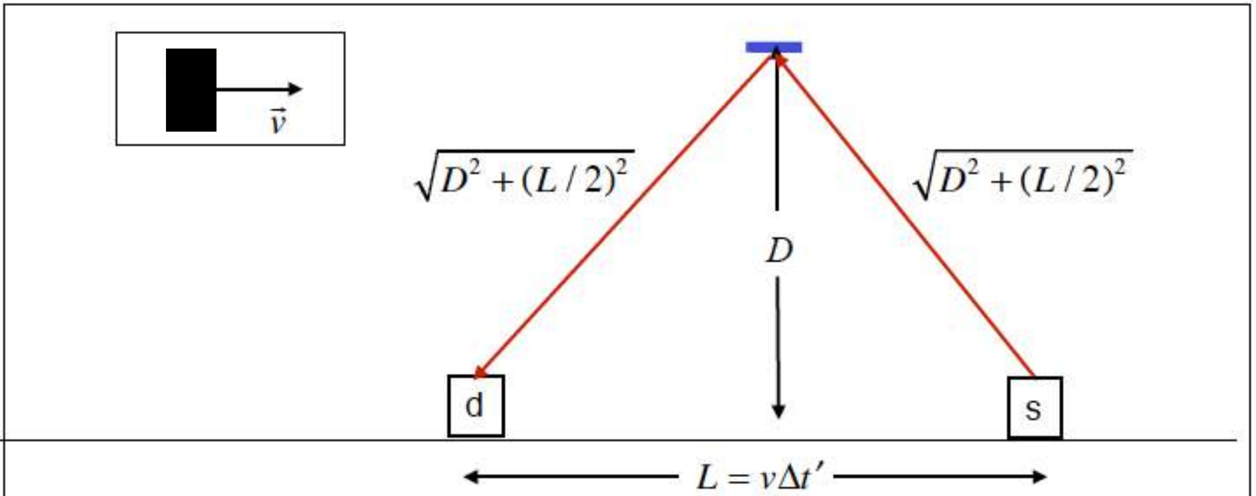
Light Clock and moving systems



Light Clock and moving systems



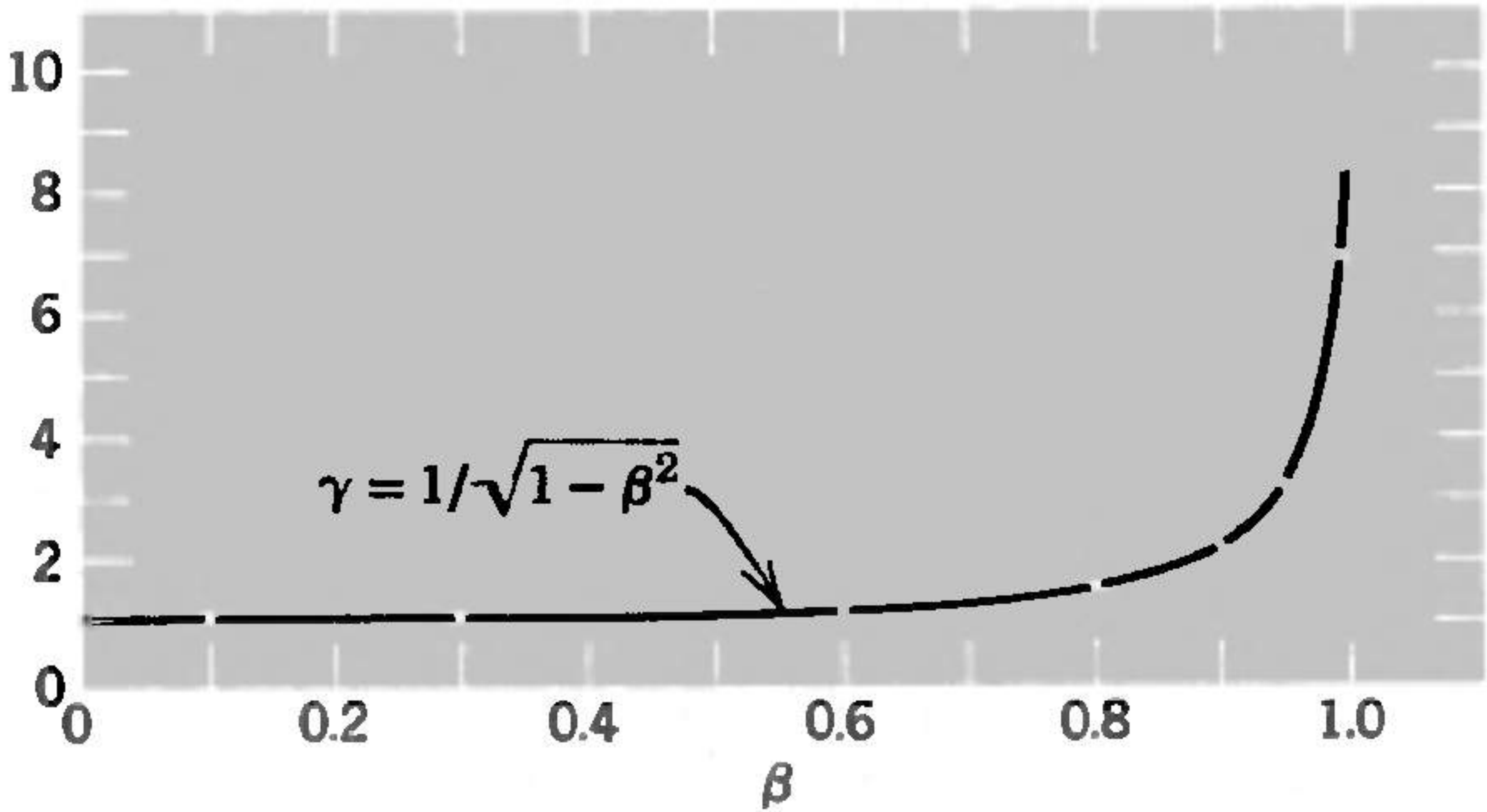
$$\Delta t = \frac{2D}{c}$$



$$\Delta t' = \frac{2\sqrt{\frac{L^2}{4} + D^2}}{c} = \frac{\sqrt{L^2 + 4D^2}}{c} = \frac{\sqrt{v^2 \Delta t'^2 + 4D^2}}{c}$$

$$c^2 \Delta t'^2 = v^2 \Delta t'^2 + 4D^2, \qquad \Delta t'^2 c^2 \left(1 - \frac{v^2}{c^2}\right) = 4D^2, \qquad \Delta t'^2 \left(1 - \frac{v^2}{c^2}\right) = \frac{4D^2}{c^2} = \Delta t^2$$

$$\Delta t' = \frac{\Delta t}{\sqrt{1 - \frac{v^2}{c^2}}}, \qquad \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}}, \qquad \beta \equiv \frac{v}{c}, \qquad \boxed{\Delta t' = \frac{\Delta t}{\sqrt{1 - \beta^2}} = \gamma \Delta t}$$



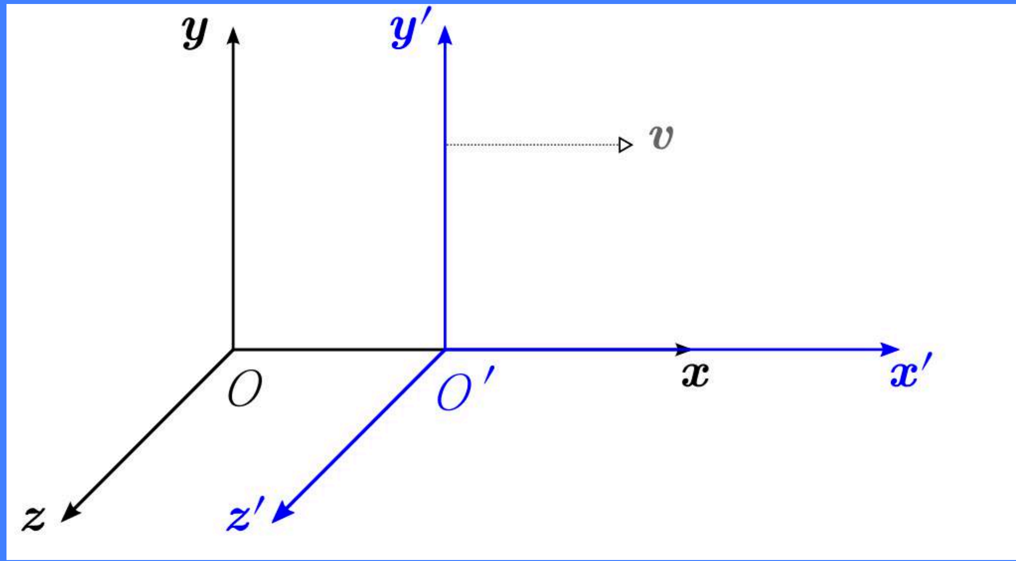
10
8
6
4
2
0
0
0.2
0.4
0.6
0.8
1.0
$\gamma = 1/\sqrt{1-\beta^2}$
β

Relativity - Basic principles

- **The Principle of Relativity** – The laws of physics are invariant (i.e. identical) in all inertial systems (non-accelerating frames of reference) =>
  - All experiments run the same in all inertial frames of reference and are described by the same equations

- **The Principle of Invariant Light Speed** – The speed of light in a vacuum is the same for all observers, regardless of the motion of the light source =>
  - Experimental result: c = 299792458 m/s in vacuum

Inertial Systems

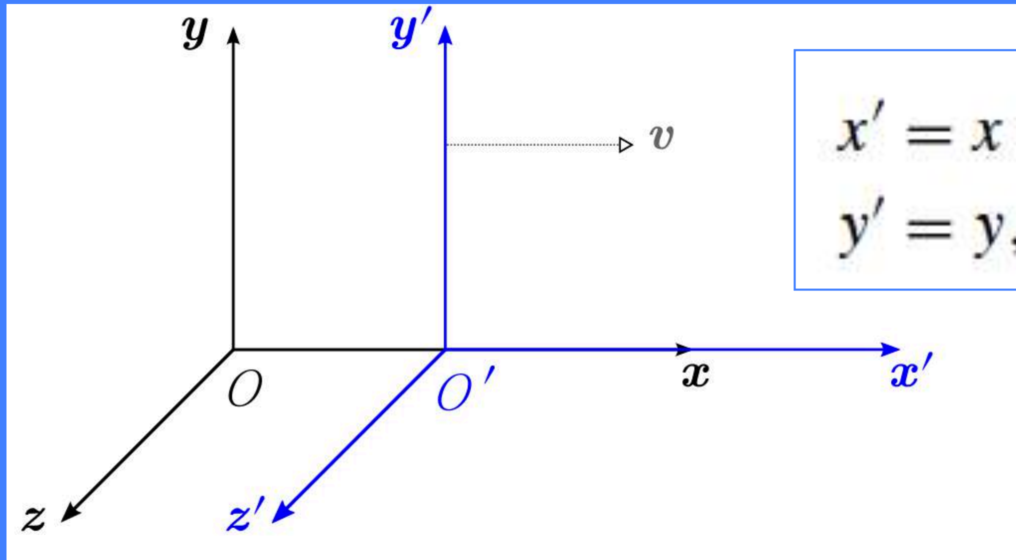


- For the mathematical description of a mass point one specifies its relative motion with respect to an **arbitrary** coordinate frame.

- It is convenient for this purpose to adopt a **non accelerated reference** frame (inertial system).There are, however, arbitrarily many alternative inertial system that are moving uniformly against the first one.

- If one changes from an inertial system (K) into another one (K') , then the laws of **Newtonian mechanics** remain unchanged. As a consequence, one cannot decide from mechanical experiments whether an inertial system at absolute rest exists.

Galileo Transformations



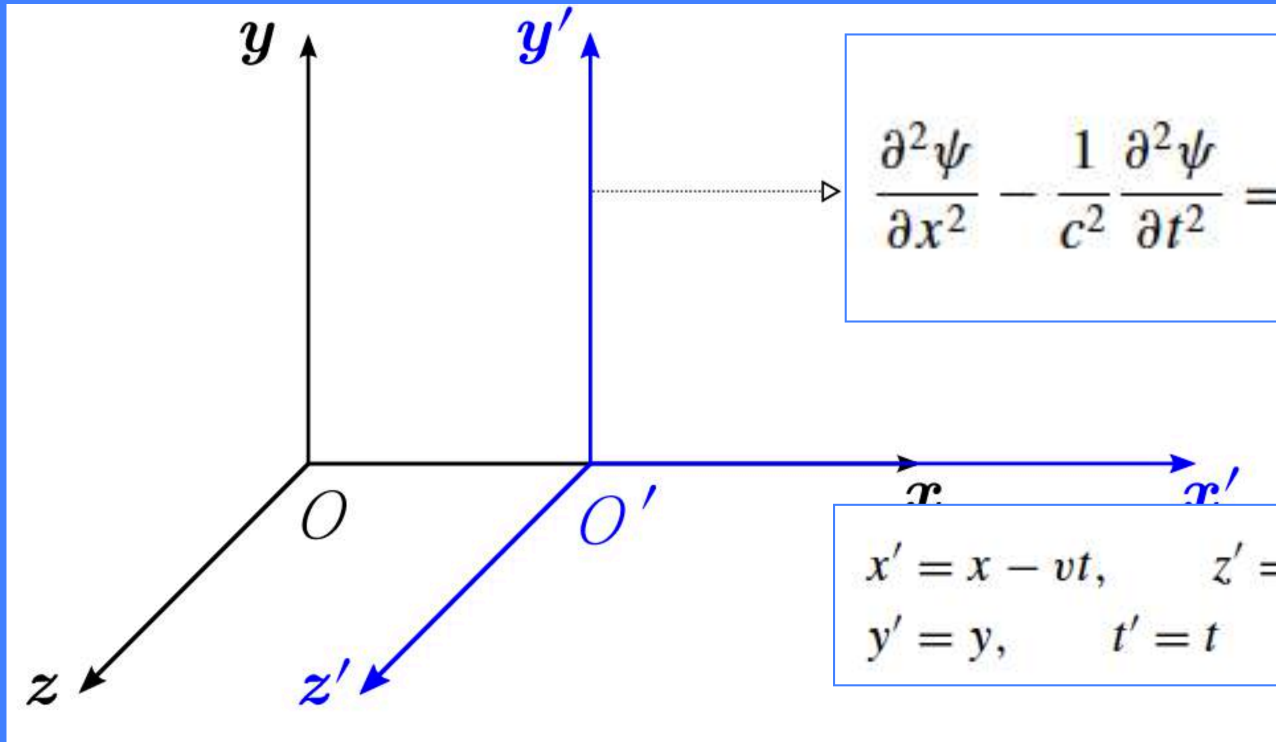
$$x' = x - vt, \qquad z' = z,$$
$$y' = y, \qquad t' = t$$

The transformations specifying the transition from one reference frame to another frame moving with constant velocity v against the former one are the Galileo transformations:

$$F = m\ddot{x} = m\ddot{x}' = F'$$

Newton's law, if it holds in one inertial system, also holds in any other inertial system, that is, the Newtonian mechanics remains unchanged.

Invariance of the Wave Equation ?



$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} = 0.$$

$$x' = x - vt, \qquad z' = z,$$
$$y' = y, \qquad t' = t$$

The partial derivatives with respect to the non primed coordinates must be replaced by derivatives with respect to the primed coordinates.
Use the chain rule:

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j}$$

The partial derivatives with respect to the non primed coordinates must be replaced by derivatives with respect to the primed coordinates.

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial x'_j}{\partial x_i} \frac{\partial}{\partial x'_j}$$

$$x' = x - vt, \qquad t' = t.$$

The partial derivatives are related by

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}, \qquad \frac{\partial}{\partial t} = \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'},$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - v\frac{\partial}{\partial x'},$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}, \qquad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t'^2} + v^2\frac{\partial^2}{\partial x'^2} - 2v\frac{\partial}{\partial t'}\frac{\partial}{\partial x'}.$$

Galileo Transformations Fail !

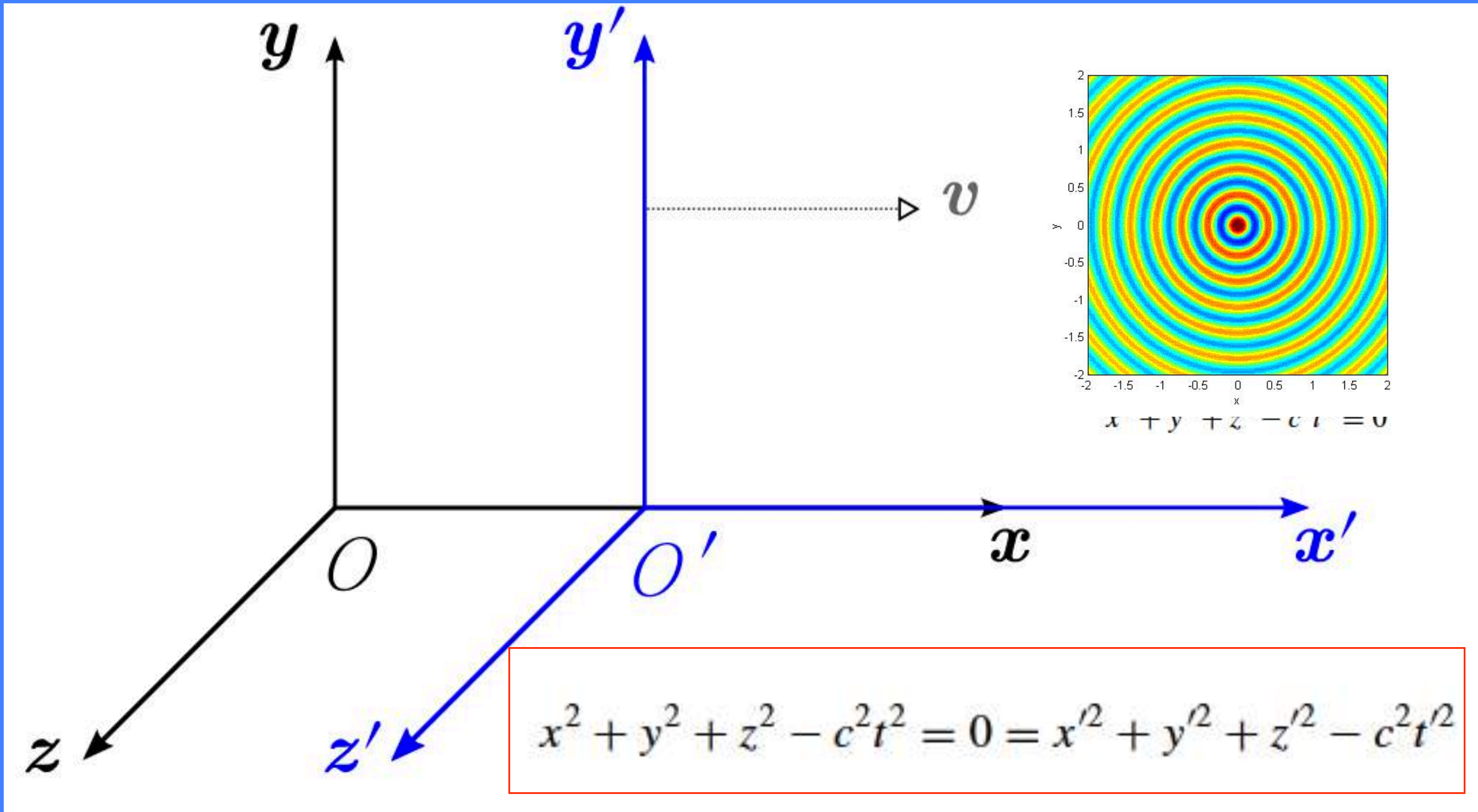
$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial x'^2}, \qquad \frac{\partial^2}{\partial t^2} = \frac{\partial^2}{\partial t'^2} + v^2\frac{\partial^2}{\partial x'^2} - 2v\frac{\partial}{\partial t'}\frac{\partial}{\partial x'}.$$

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = 0.$$

Insertion into the equation yields

$$\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} - \frac{v^2}{c^2}\frac{\partial^2\psi}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial^2\psi}{\partial t'\partial x'} = 0$$

In both reference frames a spherical wave propagates with velocity c and must remain spherical



$$x^2 + y^2 + z^2 - c^2 t^2 = 0 = x'^2 + y'^2 + z'^2 - c^2 t'^2$$

An ellipsoidally deformed wave in the moving system or another propagation velocity would allow one to establish the state of motion, and thus would violate the relativity principle.

Derivation of Lorentz Transformations

The principle of inertia in both frames a = 0 ⇔ a' = 0 implies that the transformations are linear => homogeneity and isotropy of space

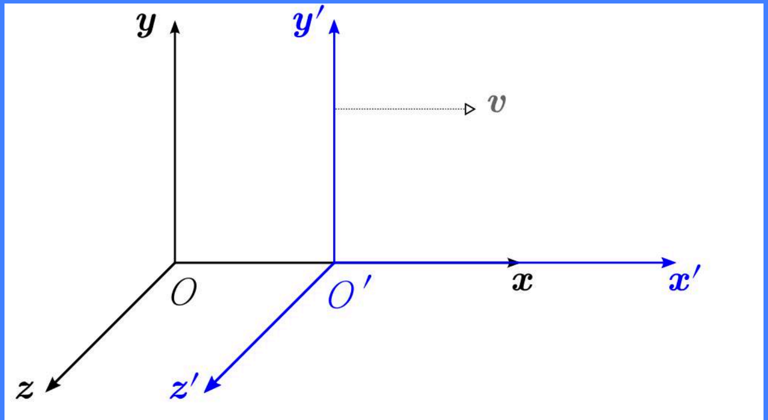
$$\begin{cases} x' = a_{11}x + a_{12}y + a_{13}z + a_{14}t \\ y' = a_{21}x + a_{22}y + a_{23}z + a_{24}t \\ z' = a_{31}x + a_{32}y + a_{33}z + a_{34}t \\ t' = a_{41}x + a_{42}y + a_{43}z + a_{44}t \ . \end{cases}$$

+

$$\boxed{x^2 + y^2 + z^2 - c^2t^2 = 0 = x'^2 + y'^2 + z'^2 - c^2t'^2}$$

Derivation of Lorentz Transformations

$$\begin{cases} x' = a_{11}x + a_{14}t \\ y' = y \\ z' = z \\ t' = a_{41}x + a_{44}t \end{cases}$$



A point having x'=0 appears to move in the direction of positive x-axis with speed v so that when x'=0 => x=vt

$$\begin{cases} x' = a_{11}(x - Vt) \\ y' = y \\ z' = z \\ t' = a_{41}x + a_{44}t\ . \end{cases}$$

Derivation of Lorentz Transformations

$$x^2 + y^2 + z^2 - c^2t^2 = 0 = x'^2 + y'^2 + z'^2 - c^2t'^2$$

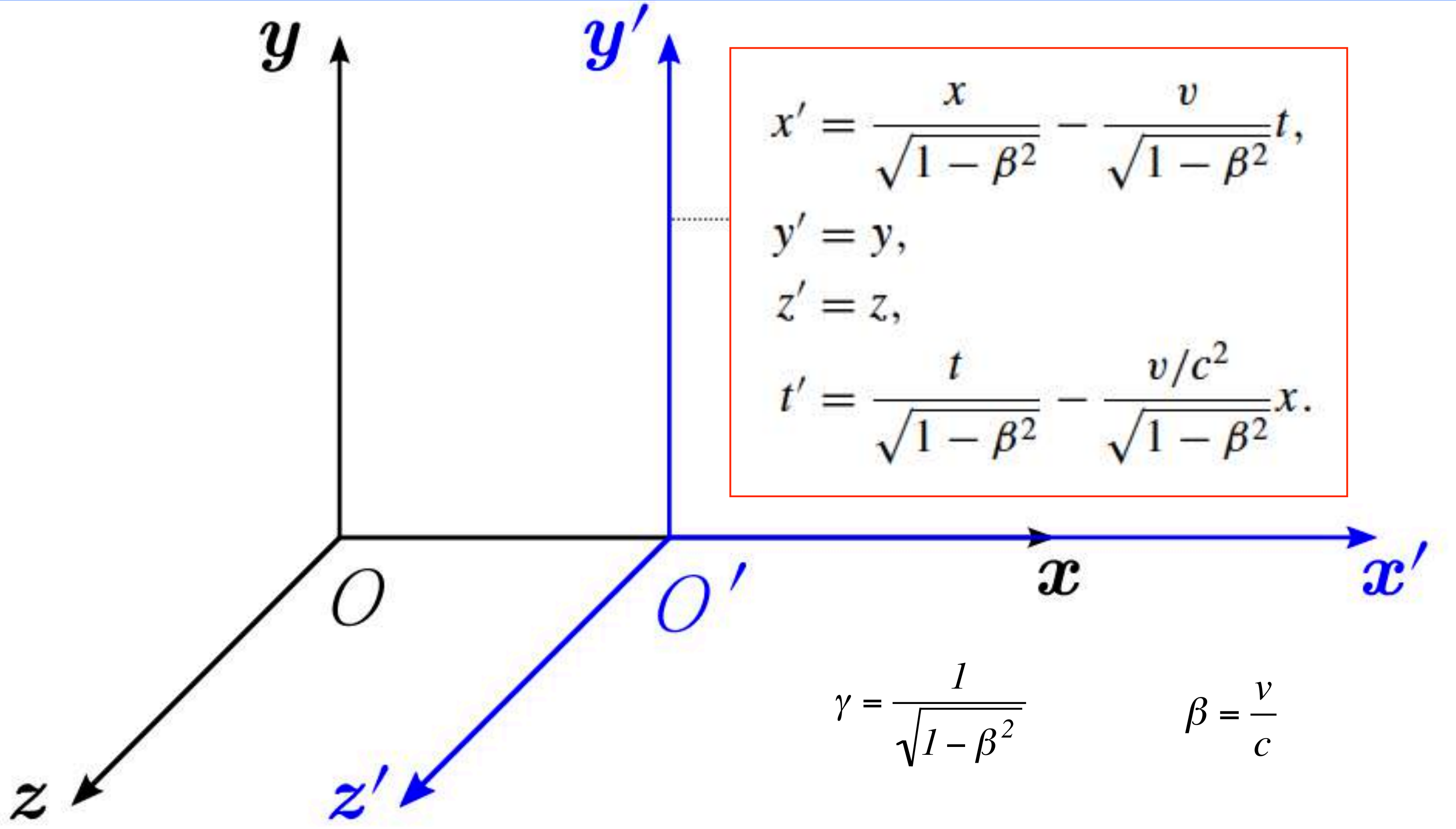
$$a_{11}^2(x - Vt)^2 + y^2 + z^2 - c^2(a_{41}x + a_{44}t)^2 = 0$$

$$(a_{11}^2 - c^2a_{41}^2)x^2 + y^2 + z^2 - 2(Va_{11}^2 + c^2a_{41}a_{44})xt - (c^2a_{44}^2 - V^2a_{11}^2)t^2 = 0$$

$$\begin{cases} a_{11}^2 - c^2a_{41}^2 = 1 \\ Va_{11}^2 + c^2a_{41}a_{44} = 0 \\ c^2a_{44}^2 - V^2a_{11}^2 = c^2 \ . \end{cases}$$

$$\begin{cases} a_{11} = \dfrac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \gamma \\ a_{41} = -\dfrac{V}{c^2\sqrt{1 - \frac{V^2}{c^2}}} = -\gamma\dfrac{V}{c^2} \\ a_{44} = \dfrac{1}{\sqrt{1 - \frac{V^2}{c^2}}} = \gamma \ , \end{cases}$$

Lorentz Transformations



$$x' = \frac{x}{\sqrt{1-\beta^2}} - \frac{v}{\sqrt{1-\beta^2}}t,$$
$$y' = y,$$
$$z' = z,$$
$$t' = \frac{t}{\sqrt{1-\beta^2}} - \frac{v/c^2}{\sqrt{1-\beta^2}}x.$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{v}{c}$$

They represent the correct transformation laws connecting two inertial frame in agreament with the principle of relativity

Lorentz Transformations

$$\begin{cases} x' = \gamma(x - Vt) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \frac{V}{c^2}x\right) \ . \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = \frac{V}{c}$$

$$\begin{cases} x = \gamma(x' + Vt') \\ y = y' \\ z = z' \\ t = \gamma\left(t' + \frac{V}{c^2}x'\right) \ . \end{cases}$$

Reciprocity: according to the principle of relativity, it is equivalent to say that S is moving at velocity V with respect to S', or that S' is moving at velocity –V with respect to S.

Lorentz Transformations and the invariance of light vector

$$
\begin{aligned}
& x'^2 + y'^2 + z'^2 - c^2 t'^2 \\
&= \frac{1}{1-\beta^2}(x - vt)^2 + y^2 + z^2 - \frac{c^2}{1-\beta^2}\left(t - \frac{v}{c^2}x\right)^2 \\
&= \left[\frac{1}{1-\beta^2} - \frac{v^2/c^2}{1-\beta^2}\right] x^2 + y^2 + z^2 - c^2 t^2 \left[\frac{1}{1-\beta^2} - \frac{v^2/c^2}{1-\beta^2}\right] \\
&\quad - tx\left[\frac{2v}{1-\beta^2} - \frac{2v}{1-\beta^2}\right] \\
&= x^2 + y^2 + z^2 - c^2 t^2
\end{aligned}
$$

Lorentz Transformations and the invariance of wave equation

$$x' = \frac{x}{\sqrt{1-\beta^2}} - \frac{v}{\sqrt{1-\beta^2}}t,$$
$$y' = y,$$
$$z' = z,$$
$$t' = \frac{t}{\sqrt{1-\beta^2}} - \frac{v/c^2}{\sqrt{1-\beta^2}}x.$$

$$\frac{\partial}{\partial x_i} = \sum_j \frac{\partial x'_j}{\partial x_i}\frac{\partial}{\partial x'_j}$$

- First derivative

$$\frac{\partial}{\partial x} = \frac{1}{\sqrt{1-\beta^2}}\frac{\partial}{\partial x'} - \frac{v/c^2}{\sqrt{1-\beta^2}}\frac{\partial}{\partial t'}$$

$$\frac{\partial}{\partial t} = \frac{1}{\sqrt{1-\beta^2}}\frac{\partial}{\partial t'} - \frac{v}{\sqrt{1-\beta^2}}\frac{\partial}{\partial x'}$$

- Second derivative

$$\frac{\partial^2}{\partial x^2} = \frac{1}{1-\beta^2}\frac{\partial^2}{\partial x'^2} - \frac{2v}{c^2}\frac{1}{1-\beta^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'} + \frac{v^2}{c^4(1-\beta^2)}\frac{\partial^2}{\partial t'^2}$$

$$\frac{\partial^2}{\partial t^2} = \frac{1}{1-\beta^2}\frac{\partial^2}{\partial t'^2} - \frac{2v}{1-\beta^2}\frac{\partial}{\partial x'}\frac{\partial}{\partial t'} + \frac{v^2}{1-\beta^2}\frac{\partial^2}{\partial x'^2}$$

Lorentz Transformations and the invariance of wave equation

By insertion into the wave equation, we obtain:

$$
\begin{aligned}
\frac{\partial^2\psi}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} &= \frac{1}{1-\beta^2}\left(\frac{\partial^2\psi}{\partial x'^2} - \frac{2v}{c^2}\frac{\partial^2\psi}{\partial x'\partial t'} + \frac{v^2\partial^2\psi}{c^4\partial t'^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} + \frac{2v}{c^2}\frac{\partial^2\psi}{\partial x'\partial t'} - \frac{v^2}{c^2}\frac{\partial^2\psi}{\partial x'^2}\right) \\
&= \frac{1}{1-\beta^2}\left[\frac{\partial^2\psi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}\right] - \frac{v^2/c^2}{1-\beta^2}\left[\frac{\partial^2\psi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2}\right] \\
&= \frac{\partial^2\psi}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2\psi}{\partial t'^2} = 0.
\end{aligned}
$$

Thewave equation is invariant under Lorentz tranformations!

Consequences of Lorentz Transformations

Let (x_A, y_A, z_A), (x_B, y_B, z_B) be the coordinates of A and B, respectively, and t_A, t_B the times of the corresponding events, as measured in S, and let the primed symbols refer to the same quantities relative to S'.

$$\Delta x = x_B - x_A; \quad \Delta x' = x'_B - x'_A,$$
$$\Delta t = t_B - t_A; \quad \Delta t' = t'_B - t'_A.$$

$$\Delta x' = \gamma(V)(\Delta x - V\Delta t),$$

$$\Delta y' = \Delta y,$$

$$\Delta z' = \Delta z,$$

$$\Delta t' = \gamma(V)\left(\Delta t - \frac{V}{c^2}\Delta x\right),$$

in contrast to Galilean transformations, the time lapse and the space interval between two events is no longer invariant

Consequences of Lorentz Transformations

- **Time Dilation:**

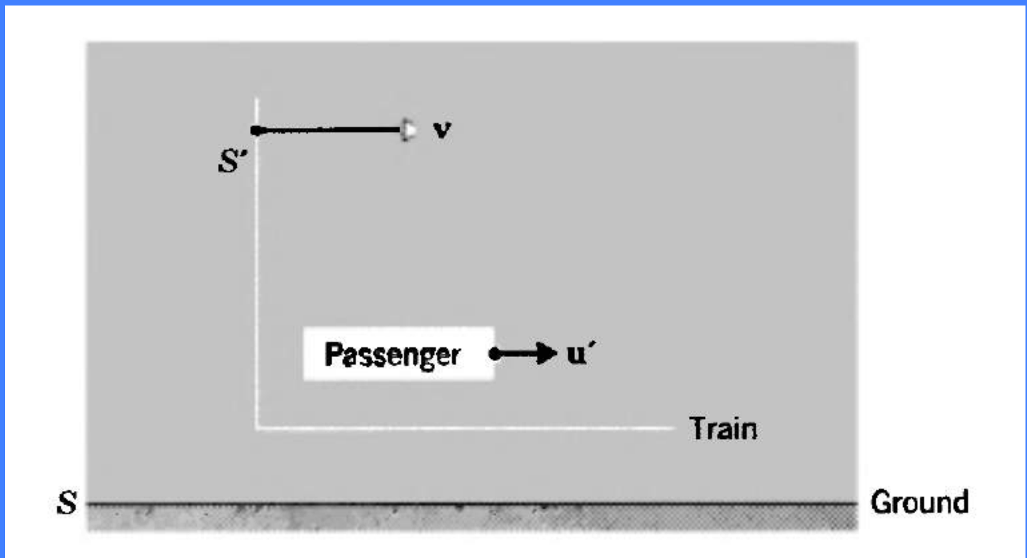
$$\Delta t = \gamma \Delta t' \qquad \Delta t' = \gamma \Delta t$$

- **Length Contraction:**

$$\Delta x = \frac{\Delta x'}{\gamma} \qquad \Delta x' = \frac{\Delta x}{\gamma}$$

The classical addition of velocities

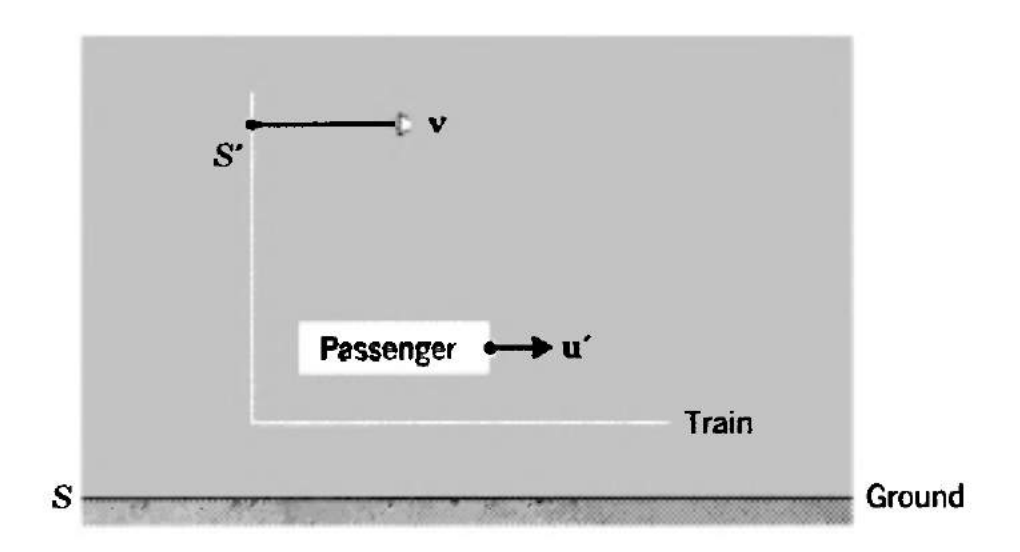
In classical physics, if we have a train moving with velocity v with respect to ground and a passenger on the train moves with velocity u’ with respect to train, then the passangers’s velocity relative to ground u is just:



$$\mathbf{u} = \mathbf{u}' + \mathbf{v}$$

The Relativistic addition of velocities

The passanger's speed in the S' frame is u' and his position on the train as time goes on can be described by **x'=u't'**. We need the Lorentz Tr. to evaluate the speed of the passanger observed from the ground.



$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} = u't'$$

$$t' = \frac{t - (v/c^2)x}{\sqrt{1 - v^2/c^2}}.$$

Combining the Lorentz Tr. and solving for x:

$$x - vt = u'(t - \frac{v}{c^2}x),$$

$$x = \frac{(u' + v)}{(1 + u'v/c^2)} t = ut.$$

The Relativistic addition of velocities

$$u = \frac{u' + v}{1 + u'v/c^2}$$

The resulting velocity is reduced w.r.t Galileo tr.

$$u = \frac{c + v}{1 + cv/c^2} = \frac{c + v}{c(c + v)} c^2 = c.$$

The velocity of light is independent of the velocity of the source!

It always holds, including u'=c or u'=-c or u'=c and v=c

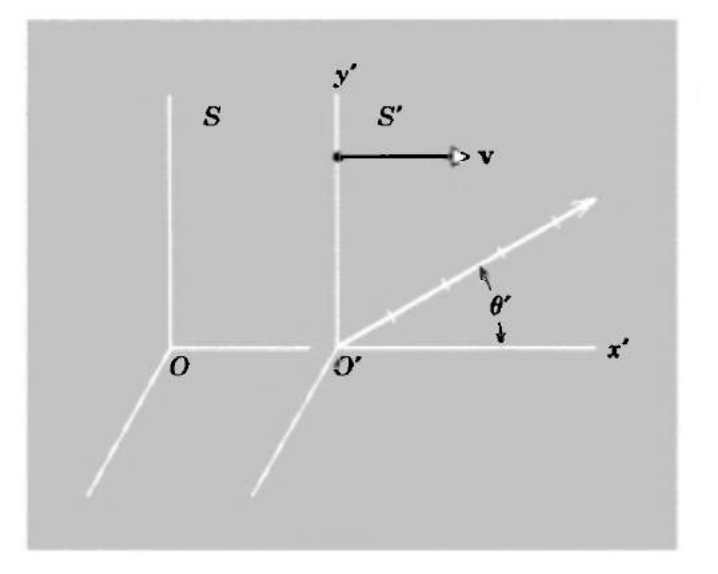
The Relativistic addition of velocities

For velocities that are perpendicular to the direction of relative motion the result is more involved. Using similar argument we can obtain:

$u_x' = \dfrac{u_x - v}{1 - u_x v/c^2}$	$u_x = \dfrac{u_x' + v}{1 + u_x' v/c^2}$
$u_y' = \dfrac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$	$u_y = \dfrac{u_y'\sqrt{1 - v^2/c^2}}{1 + u_x' v/c^2}$
$u_z' = \dfrac{u_z\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2}$	$u_z = \dfrac{u_z'\sqrt{1 - v^2/c^2}}{1 + u_x' v/c^2}$

The transverse components in S are related both to transverse and longitudinal components in S'

The Relativistic Doppler Effect



Consider a plane monochromatic light wave emotted from a source at the origin of the moving S' frame wave:

$$\cos 2\pi\left[\frac{x' \cos\theta' + y' \sin\theta'}{\lambda'} - \nu' t'\right],$$

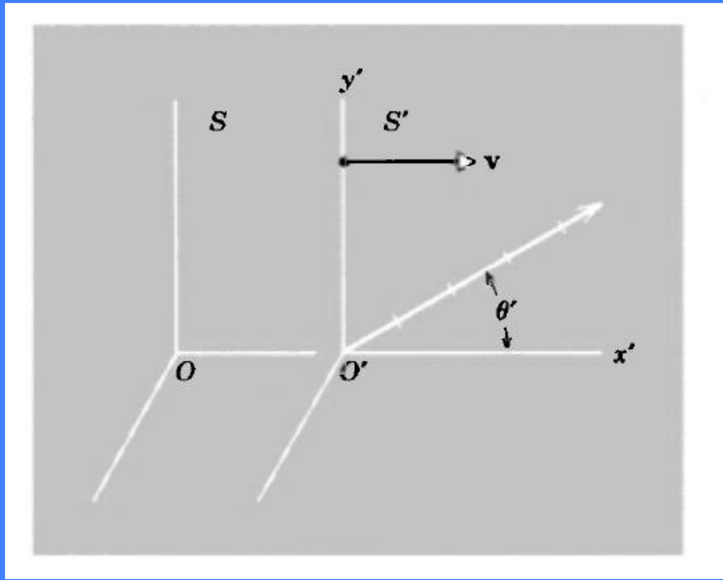
In the lab frame S frame the equation describing the wave propagation will be:

$$\cos 2\pi\left[\frac{x \cos\theta + y \sin\theta}{\lambda} - \nu t\right]$$

Using L. Tr. the result must be the same

$$\cos 2\pi\left[\frac{1}{\lambda'}\frac{(x - vt)}{\sqrt{1-\beta^2}}\cos\theta' + \frac{y \sin\theta'}{\lambda'} - \nu'\frac{[t - (v/c^2)x]}{\sqrt{1-\beta^2}}\right]$$

The Relativistic Doppler Effect



$$\cos 2\pi\left[\frac{x\cos\theta + y\sin\theta}{\lambda} - \nu t\right]$$

$$\cos 2\pi\left[\frac{\cos\theta' + \beta}{\lambda'\sqrt{1-\beta^2}}x + \frac{\sin\theta'}{\lambda'}y - \frac{(\beta\cos\theta' + 1)\nu'}{\sqrt{1-\beta^2}}t\right]$$

the result is the same if:

$$\frac{\cos\theta}{\lambda} = \frac{\cos\theta' + \beta}{\lambda'\sqrt{1-\beta^2}}$$

$$\frac{\sin\theta}{\lambda} = \frac{\sin\theta'}{\lambda'}$$

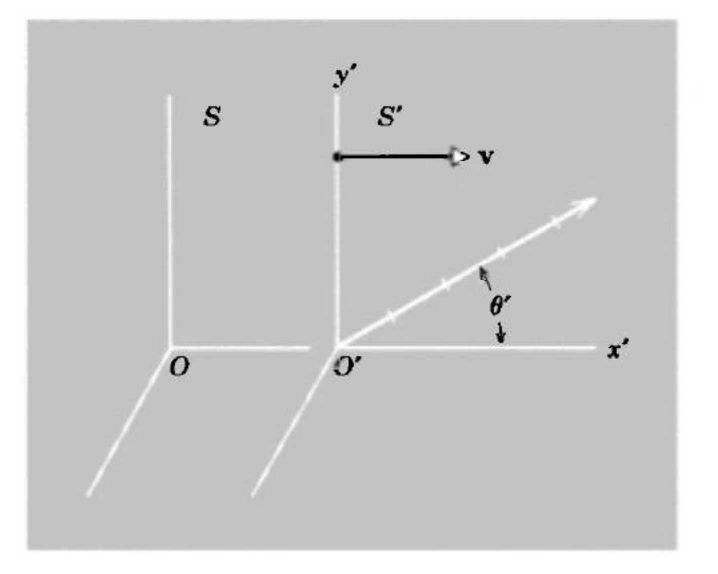
$$\tan\theta = \frac{\sin\theta'\sqrt{1-\beta^2}}{\cos\theta' + \beta}$$

the relativistic equation of Doppler effect

$$\nu = \frac{\nu'(1 + \beta\cos\theta')}{\sqrt{1-\beta^2}}$$

$$\lambda\nu = \lambda'\nu' = c,$$

The Relativistic Doppler Effect



$$\nu = \frac{\nu'(1+\beta\cos\theta')}{\sqrt{1-\beta^2}}$$

$\theta = 0$ Longitudinal effect $\nu = \nu' \sqrt{\frac{1+\beta}{1-\beta}} = \nu' \sqrt{\frac{c+v}{c-v}};$

$\theta = 90°$ Transverse effect $\nu = \nu'\sqrt{1-\beta^2}.$

The Relativistic transformations of acceleration

We can obtain the relativistic transformation equations by time differentiation of the velocity transformations:

$$a_x{}' = a_x \frac{(1 - v^2/c^2)^{3/2}}{(1 - u_x v/c^2)^3},$$

The acceleration of a particle, unlike the classical result, depends on the inertial reference frame in wich it is measured

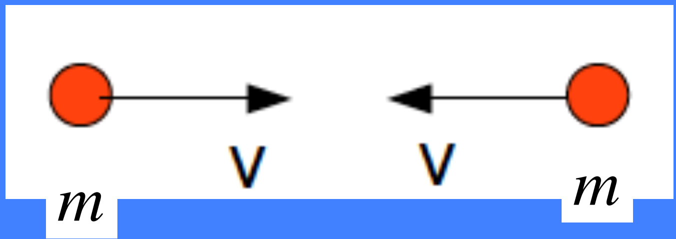
Conservation of conservation laws

- the laws of conservation of momentum and energy are valid classically, ex. during collisions .
- if we require that these conservation laws be invariant under a Lorentz transformation we might need to modify them.

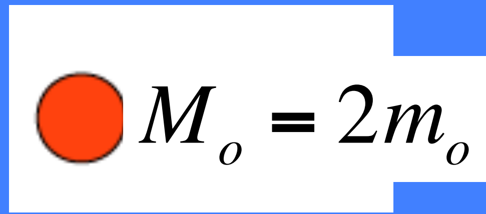
Inelastic collision between 2 identical particles

In the Laboratory frame S:

Before collision

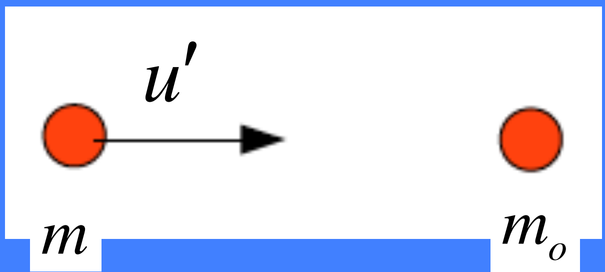


After collision

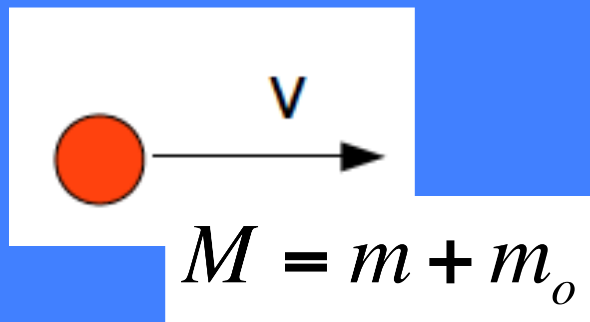


In the Moving frame S' with **–v** where left particle is at rest:

Before collision



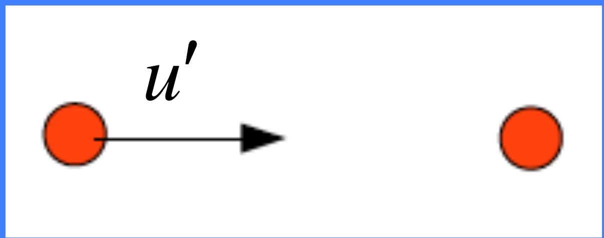
After collision



Inelastic collision between 2 identical particles

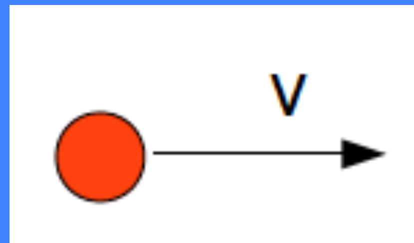
In the moving frame S' (with **–v)** the left particle is at rest before collision, but after collision the composite particle M will move with **+v** :

Before collision



m $\qquad$ m_o

After collision



$$M = m + m_o$$

Conservation of momentum

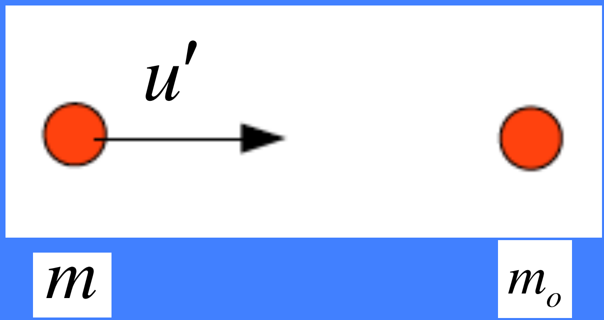
$$mu' = Mv$$

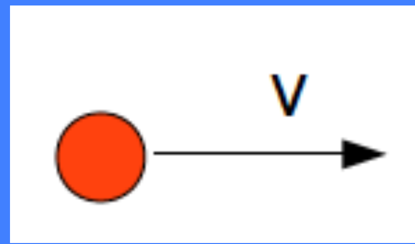
$$mu' = (m + m_o)v$$

$$u' = \frac{2v}{1 + \left(\frac{v}{c}\right)^2}$$

$$v = \frac{c^2}{u'}\left(1 - \sqrt{1 - \frac{u'^2}{c^2}}\right)$$

Inelastic collision between 2 identical particles





$$M = m + m_o$$

$$mu' = (m + m_o)\frac{c^2}{u'}\left(1 - \sqrt{1 - \frac{u'^2}{c^2}}\right)$$

Solving for m:

$$m = \frac{m_o}{\sqrt{1-\beta^2}} = \gamma m_o$$

$$\beta = \frac{u'}{c}$$

New definition of momentum

if we insist in defining the mass as independent of the velocity, then the conservation of momentum can not hold in any reference frame, thus violating the principle of relativity

The relativistic definition of momentum becomes:

$$\boxed{p = mv = \gamma m_o v}$$

$$\boxed{m = \frac{m_o}{\sqrt{1-\beta^2}} = \gamma m_o}$$

$$\beta = \frac{v}{c}$$

Relativistic equation of motion

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \qquad \mathbf{p} = \gamma m_o \mathbf{v} \qquad \boldsymbol{\beta} = \frac{\mathbf{v}}{c} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad 1 + \beta^2\gamma^2 \equiv \gamma^2$$

$$\gamma m_o \frac{d\mathbf{v}}{dt} + m_o \mathbf{v} \frac{d\gamma}{dt} = \mathbf{F} \qquad \frac{d\gamma}{dt} = \frac{d}{dt}\left(1 - \frac{|\mathbf{v}|^2}{c^2}\right)^{-1/2} = \left(1 - \frac{|\mathbf{v}|^2}{c^2}\right)^{-3/2} \frac{\mathbf{v}}{c^2} \cdot \frac{d\mathbf{v}}{dt} = \gamma^3 \frac{\mathbf{v}}{c^2} \cdot \mathbf{a}$$

$$\boxed{\frac{d\mathbf{p}}{dt} = m_o \gamma \mathbf{a} + m_o \gamma^3 \frac{\mathbf{a} \cdot \mathbf{v}}{c^2} \mathbf{v} = \mathbf{F}}$$

$$\mathbf{a} = \frac{\mathbf{F} - (\mathbf{F} \cdot \boldsymbol{\beta})\boldsymbol{\beta}}{\gamma m_o}$$

Acceleration does not generally point in the direction of the applied force

$$\mathbf{F} \perp \mathbf{v} \qquad \mathbf{a}_\perp = \frac{\mathbf{F}_\perp}{\gamma m_o}$$

$$\mathbf{F} // \mathbf{v} \qquad \mathbf{a}_{//} = \frac{\mathbf{F}_{//}(1-\beta^2)}{\gamma m_o} = \frac{\mathbf{F}_{//}}{\gamma^3 m_o}$$

A moving body is more inert in the longitudinal direction than in the transverse direction

Relativistic equation of motion are coupled:

$$\frac{d\boldsymbol{\beta}}{dt} = \frac{\mathbf{F} - \left(\mathbf{F}\cdot\boldsymbol{\beta}\right)\boldsymbol{\beta}}{\gamma m_o c}$$

$$\begin{cases} \dfrac{d\beta_x}{dt} = \dfrac{1}{\gamma m_o c}\left[F_x - \left(F_x\beta_x + F_y\beta_y + F_z\beta_z\right)\beta_x\right] \\ \dfrac{d\beta_y}{dt} = \dfrac{1}{\gamma m_o c}\left[F_y - \left(F_x\beta_x + F_y\beta_y + F_z\beta_z\right)\beta_y\right] \\ \dfrac{d\beta_z}{dt} = \dfrac{1}{\gamma m_o c}\left[F_z - \left(F_x\beta_x + F_y\beta_y + F_z\beta_z\right)\beta_z\right] \end{cases}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\boxed{\beta = \sqrt{\beta_x^2 + \beta_y^2 + \beta_z^2}}$$

We are now ready to determine the relativistic expression for the kinetic energy of a particle by computing the work done by the total force F acting on it.

Let consider a particle moving along x subject to a collinear force

$$dT = F_x dx = \gamma^3 m_o a_x dx = \gamma^3 m_o \frac{dv}{dt} dx = \gamma^3 m_o v dv$$

$$T = m_o c^2 \int_o^{\beta} \frac{\beta d\beta}{\left(1-\beta^2\right)^{3/2}} = \gamma m_o c^2 - m_o c^2 = E - E_o$$

The kinetic energy is then expressed as the difference between the total energy and the rest energy, which is the amount of energy a mass possesses when it is at rest:

$$E = T + m_o c^2 = \gamma m_o c^2$$

This result implies the equivalence between mass ed energy:

$$\Delta E = \Delta m c^2$$

Energy-momentum relation

$$p = \gamma \cdot m_0 \, v = \gamma \cdot \beta \cdot c \cdot m_0$$

we re-write:

$$E^2 = m^2 c^4 = \gamma^2 m_0^2 c^4 = (1 + \gamma^2 \beta^2) m_0^2 c^4$$

and finally get:

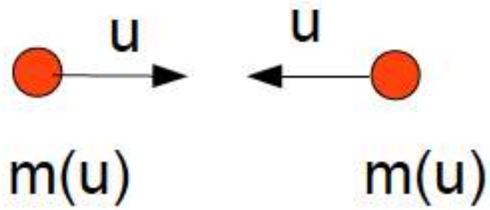
$$E^2 = (m_0 c^2)^2 + (pc)^2 \rightarrow \boxed{\frac{E}{c} = \sqrt{(m_0 c)^2 + p^2}}$$

Mass of a proton: $m_p = 1.672 \cdot 10^{-27}$ **Kg**
Energy(at rest): $m_p c^2 = 938$ **MeV** $= 0.15$ **nJ**

Collider versus Fixed target

1) 3.5 TeV head-on p-p collider

before collision

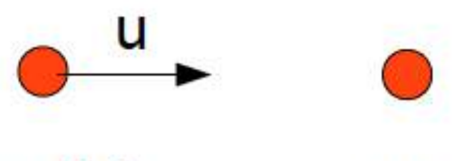


composite particle at rest after collision

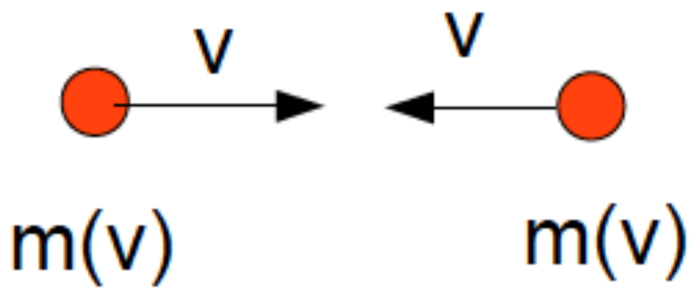
$$M_0 = 2\gamma(u)m_o \quad \Rightarrow \quad E_{CM} = M_o c^2 = 2E = 7TeV$$

2) 3.5 TeV fixed target p-machine

before collision



In the CM reference frame moving to the right with u'=v the energy is given by E_{cm} but now gamma is different:



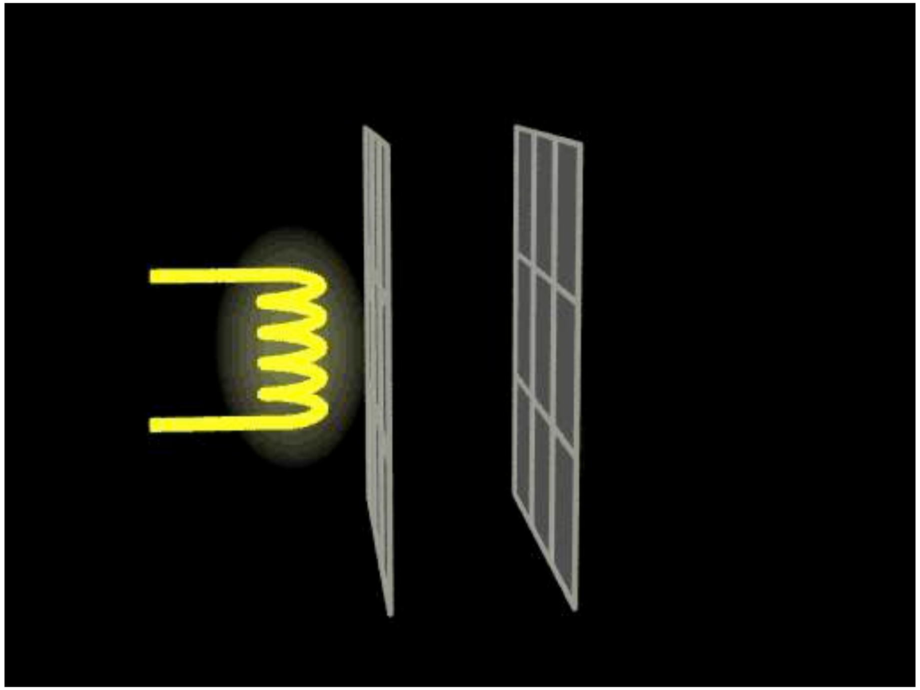
$$v = \frac{c^2}{u}\left(1-\sqrt{1-\frac{u^2}{c^2}}\right) \qquad \gamma(v) = \sqrt{\frac{1}{2}\left(1+\gamma(u)\right)}$$

$$E_{CM} = 2\gamma(v)E_o = \sqrt{2\left(1+\gamma(u)\right)}E_o$$

$$= \sqrt{2\left(E_o + E\right)E_o} = 81GeV$$

$$E_o = 938MeV$$

Longitudinal motion in the laoratory frame
==> ex: beam dynamics in a relativistic capacitor



Consider longitudinal motion only :

$$\boxed{\gamma^3 \frac{d\beta}{dt} = \frac{a_o}{c}} \qquad a_o = \frac{eE_z}{m_o}$$

$$\int_{\beta_o}^{\beta} \frac{d\beta}{\left(1-\beta^2\right)^{3/2}} = \frac{a_o}{c}\int_{t_o}^{t} dt$$

$$\frac{\beta}{\sqrt{1-\beta^2}} - \beta_o\gamma_o = \frac{a_o}{c}\left(t - t_o\right)$$

Solving explicitly for β one can find:

$$\boxed{\beta(t) = \frac{a_o(t - t_o) + c\beta_o\gamma_o}{\sqrt{c^2 + \left(c\beta_o\gamma_o + a_o(t - t_o)\right)^2}}}$$

After separating the variables one can integrate once more to obtain the position as a function of time :

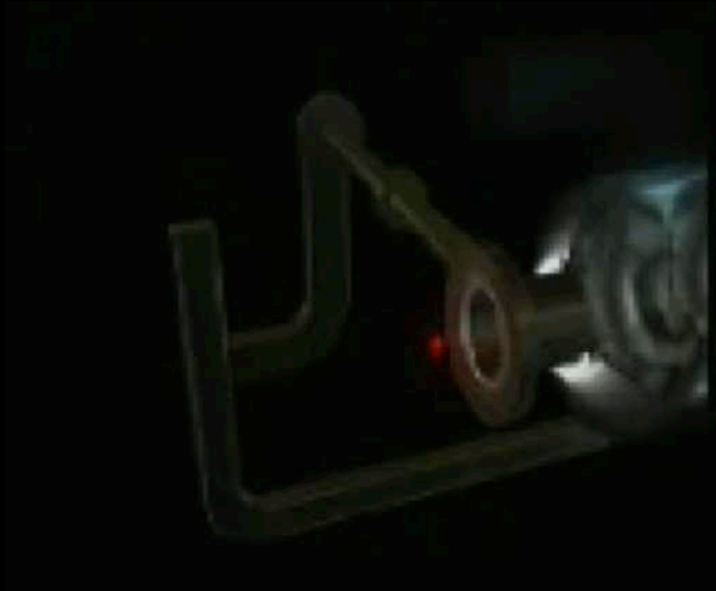
$$\boxed{z(t) - z_o = \frac{c^2}{a_o}\left(\sqrt{1 + \left(\beta_o\gamma_o + \frac{a_o}{c}(t - t_o)\right)^2} - \gamma_o\right) = h(t)}$$

In the non relativistic limit: $z(t) - z_o = \beta_o c(t - t_o) + \frac{1}{2}a_o(t - t_o)^2$

The paradox of relativistic bunch compression

Low energy electron bunch injected in a linac:

$$\gamma \approx 1$$
$$L_b = 3mm \approx L_b'$$

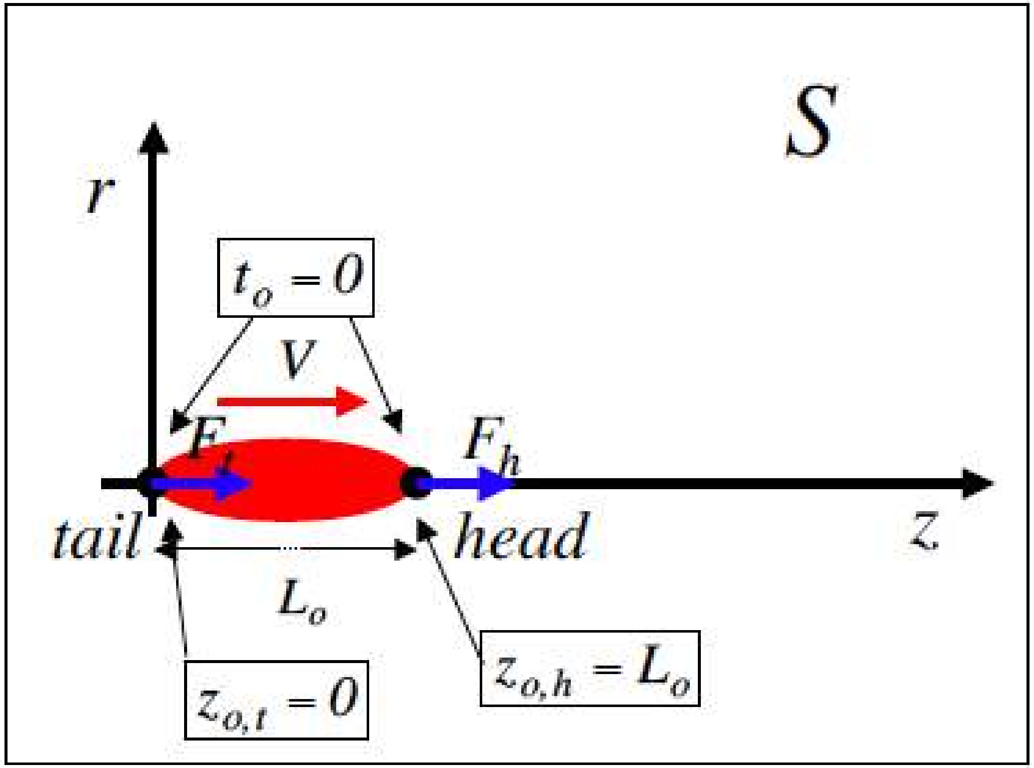


Length contraction?

$$\cancel{\gamma = 1000}$$
$$\cancel{L_b = \frac{L_b'}{\gamma} = 3\mu m}$$

Bunch length in the laboratory frame S

Let consider an electron bunch of initial length L_o inside a capacitor when the field is suddenly switched on at the time t_o.



$$L(t) = z_h(t) - z_t(t)$$

$$L(t) = \left(L_o + h(t)\right) - h(t) = L_o$$

Thus a simple computation show that no observable contraction occurs in the laboratory frame, as should be expected since both ends are subject to the same acceleration at the same time.

Bunch length in the moving frame S'

More interesting is the bunch dynamics as seen by a moving reference frame S', that we assume it has a relative velocity V with respect to S such that at the end of the process the accelerated bunch will be at rest in the moving frame S'. It is actually a deceleration process as seen by S'

Inverse Lorentz transformations:

$$\begin{cases} ct' = \gamma\left(ct - \dfrac{V}{c}z\right) \\ z' = \gamma(z - Vt) \end{cases}$$

leading for the tail particle to:

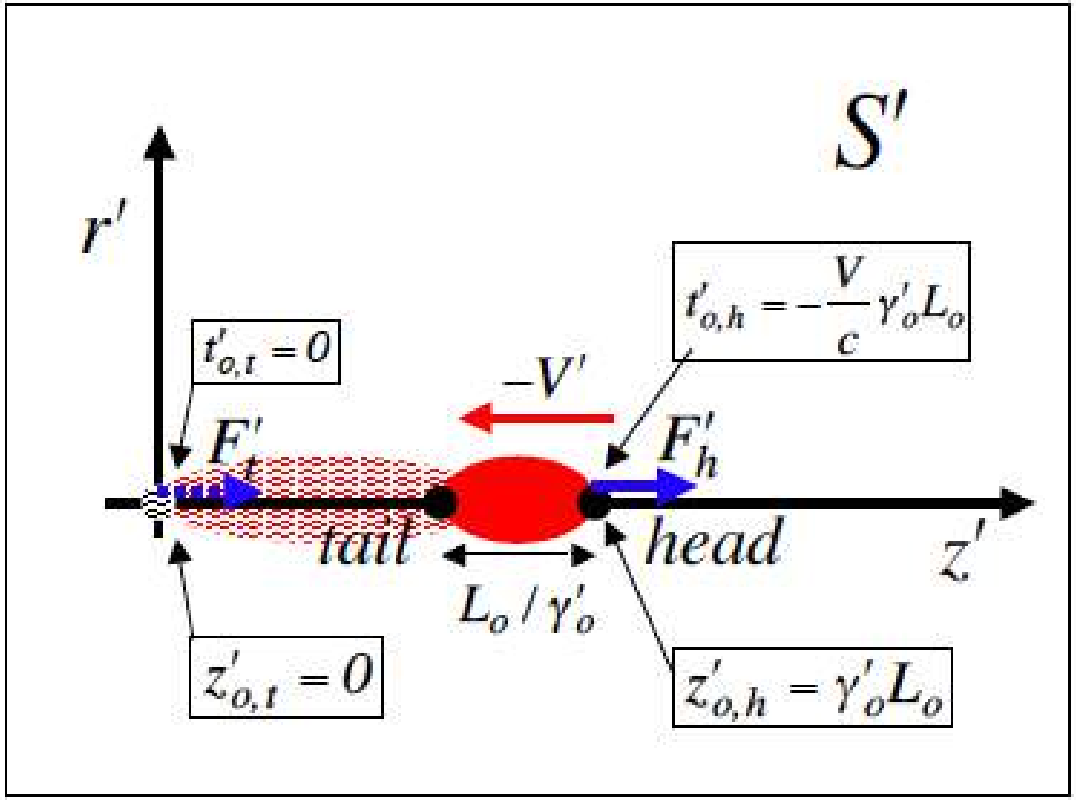
$$\begin{cases} t'_{o,t} = t_o = 0 \\ z'_{o,t} = z_{o,t} = 0 \end{cases}$$

and for the head particle to:

$$\begin{cases} t'_{o,h} = -\dfrac{V}{c}\gamma'_o L_o < t_o \\ z'_{o,h} = \gamma'_o L_o > z_{o,h} \end{cases}$$

The key point is that as seen from S' the decelerating force is not applied *simultaneously* along the bunch but with a *delay* given by:

$$\Delta t'_o = t'_{o,h} - t'_{o,t} = -\frac{V}{c}\gamma'_o L_o < 0$$



At the end of the process when both particle have been subject to the same decelerating field for the same amount of time the bunch length results to be:

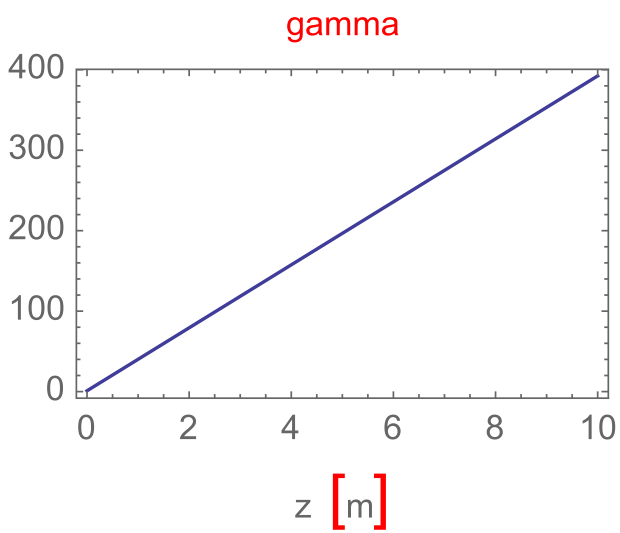
$$L'(t') = \left(\gamma' L_o + h'(t')\right) - h'(t') = \gamma' L_o$$

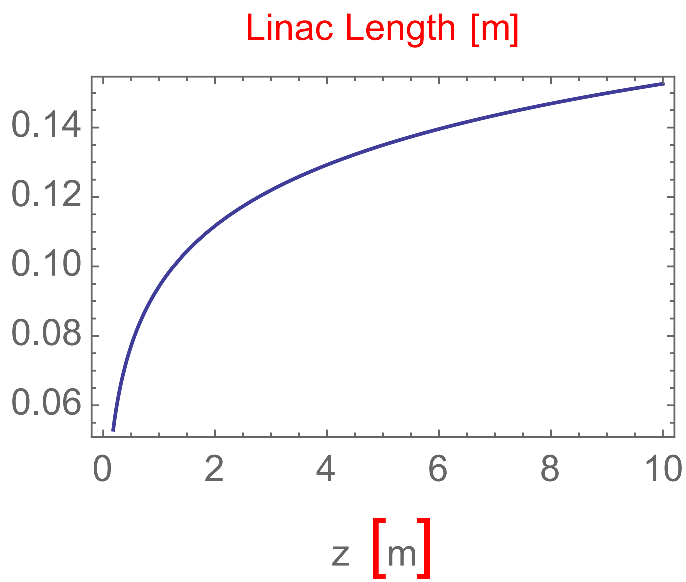
$$z'(t') - z'_o = \frac{c^2}{a_o}\left(\sqrt{1 + \left(\beta'_o\gamma'_o + \frac{a_o}{c}\left(t' - t'_{o,h}\right)\right)^2} - \gamma'_o\right) = h'(t')$$

Accelerator length in the moving frame $\tilde{\Sigma}$

$$\tilde{L}_{linac} = \int_0^{L_{linac}} d\tilde{z} = \int_0^{L_{linac}} \frac{dz}{\gamma} = \int_0^{L_{linac}} \frac{dz}{\gamma' z + \gamma_o} = \left[\frac{1}{\gamma'} \ln\left(\gamma' z + \gamma_o\right)\right]_o^{L_{linac}} = \frac{1}{\gamma'} \ln\left(\frac{\gamma_f}{\gamma_o}\right)$$

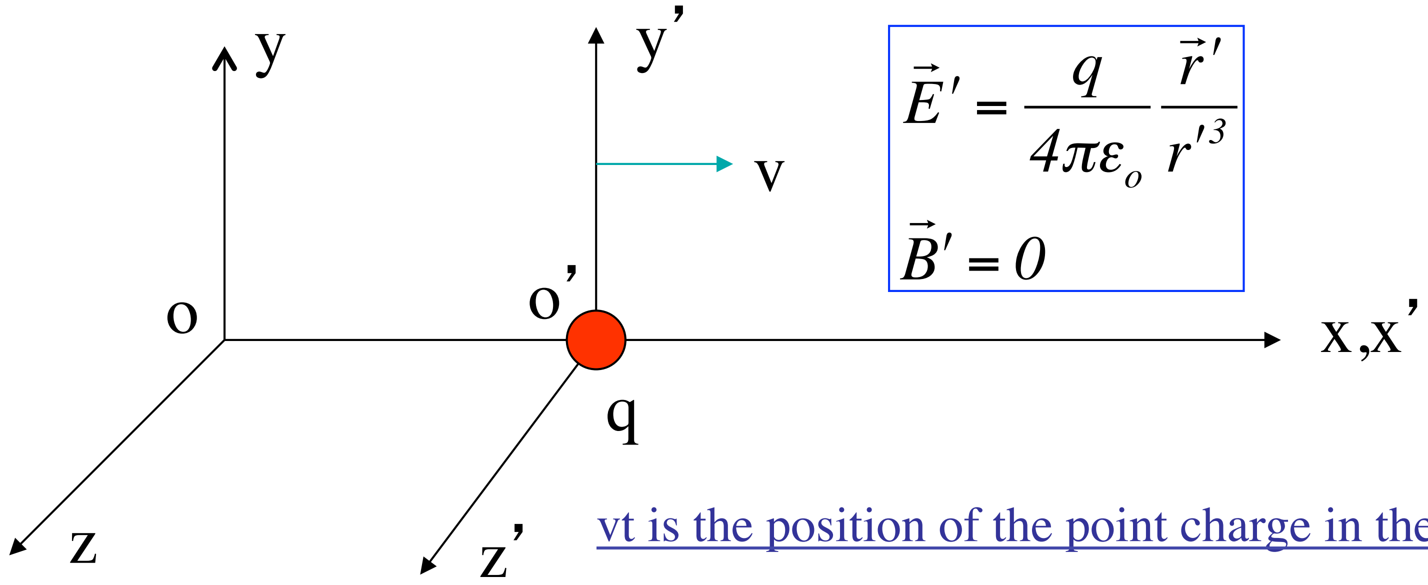
$$\gamma = \frac{d\gamma}{dz} z + \gamma_o \qquad \frac{d\gamma}{dz} = \gamma' = \frac{eE_{acc}}{mc^2}$$





Electromagnetic Fields of a moving charge

Fields of a point charge with uniform motion



$$\vec{E}' = \frac{q}{4\pi\varepsilon_o}\frac{\vec{r}'}{r'^3}$$

$$\vec{B}' = 0$$

vt is the position of the point charge in the lab. frame O.

- In the moving frame O' the charge is at rest
- The electric field is radial with spherical symmetry
- The magnetic field is zero

$$E'_x = \frac{q}{4\pi\varepsilon_o}\frac{x'}{r'^3} \qquad E'_y = \frac{q}{4\pi\varepsilon_o}\frac{y'}{r'^3} \qquad E'_z = \frac{q}{4\pi\varepsilon_o}\frac{z'}{r'^3}$$

Relativistic transforms of the fields from O' to O

$$\begin{cases} E_x = E'_x \\ E_y = \gamma(E'_y + vB'_z) \\ E_z = \gamma(E'_z - vB'_y) \end{cases} \qquad \begin{cases} B_x = B'_x \\ B_y = \gamma(B'_y - vE'_z / c^2) \\ B_z = \gamma(B'_z + vE'_y / c^2) \end{cases}$$

$$\begin{cases} x' = \gamma(x - vt) \\ y' = y \\ z' = z \\ ct' = \gamma\left(ct - \frac{v}{c}x\right) \end{cases}$$

$$r' = \left(x'^2 + y'^2 + z'^2\right)^{1/2}$$

$$r' = \left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{1/2}$$

$$E_x = E'_x = \frac{q}{4\pi\varepsilon_o}\frac{x'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma(x-vt)}{\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}}$$

$$E_y = \gamma E'_y = \frac{q}{4\pi\varepsilon_o}\frac{y'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma y}{\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}}$$

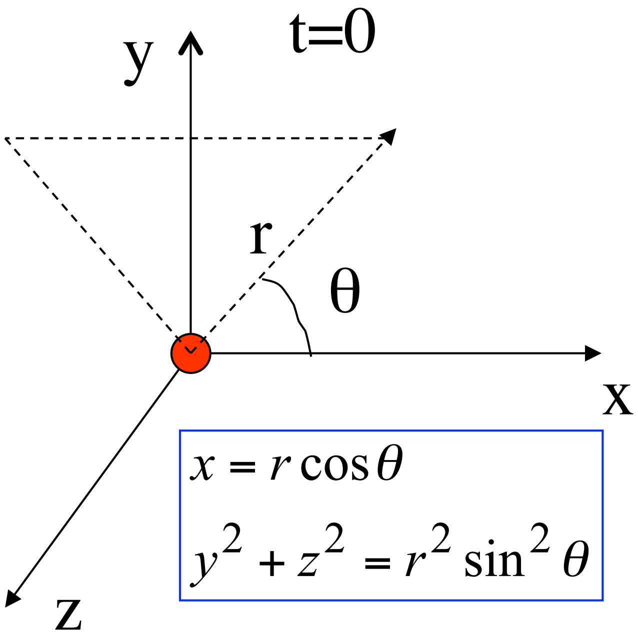
$$E_z = \gamma E'_z = \frac{q}{4\pi\varepsilon_o}\frac{z'}{r'^3} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma z}{\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}}$$

The field pattern is moving with the charge and it can be observed at t=0:

$$\vec{E} = \frac{q}{4\pi\varepsilon_o}\frac{\gamma\,\vec{r}}{\left[\gamma^2 x^2+y^2+z^2\right]^{3/2}}$$

The fields have lost the spherical symmetry

$$\vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{\gamma \vec{r}}{\left[\gamma^2 x^2 + y^2 + z^2\right]^{3/2}}$$



$$x = r\cos\theta$$

$$y^2 + z^2 = r^2 \sin^2\theta$$

$$\gamma^2 x^2 + y^2 + z^2 = r^2\gamma^2\left(1-\beta^2 \sin^2\theta\right)$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{\left(1-\beta^2\right)}{r^2\left(1-\beta^2 \sin^2\theta\right)^{3/2}} \frac{\vec{r}}{r}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{\left(1-\beta^2\right)}{r^2\left(1-\beta^2 \sin^2\theta\right)^{3/2}} \frac{\vec{r}}{r}$$

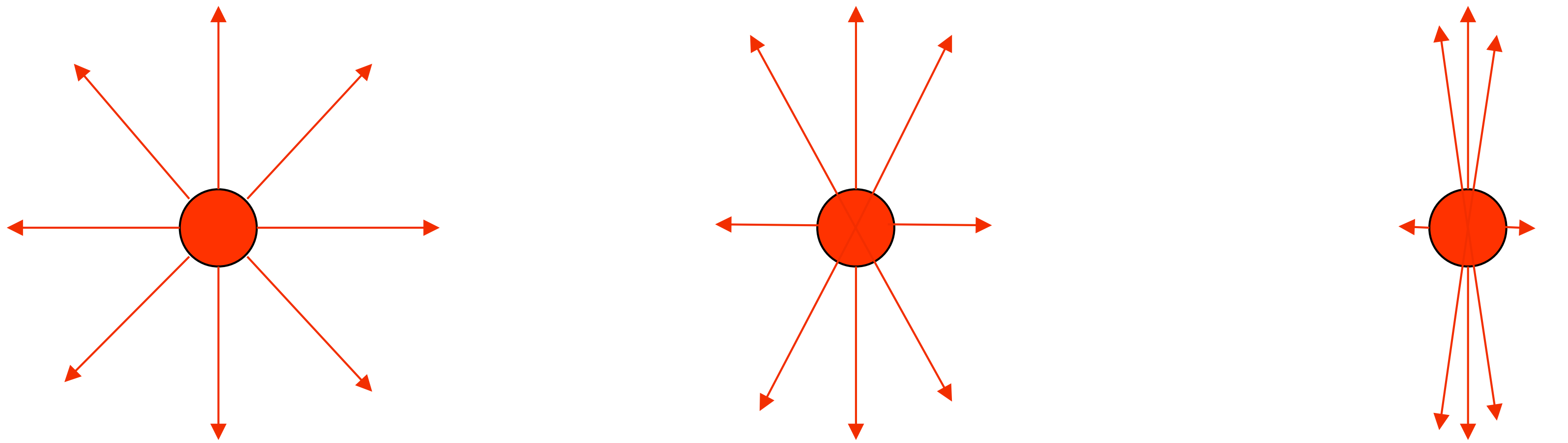
$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\beta = 0 \Rightarrow \vec{E} = \frac{q}{4\pi\varepsilon_o} \frac{1}{r^2} \frac{\vec{r}}{r}$$

$$\theta = 0 \Rightarrow E_{//} = \frac{q}{4\pi\varepsilon_o} \frac{1}{\gamma^2 r^2} \frac{\vec{r}}{r} \xrightarrow{\gamma \to \infty} 0$$

$$\theta = \frac{\pi}{2} \Rightarrow E_{\perp} = \frac{q}{4\pi\varepsilon_o} \frac{\gamma}{r^2} \frac{\vec{r}}{r} \xrightarrow{\gamma \to \infty} \infty$$

γ =1 γ >1 γ >>1

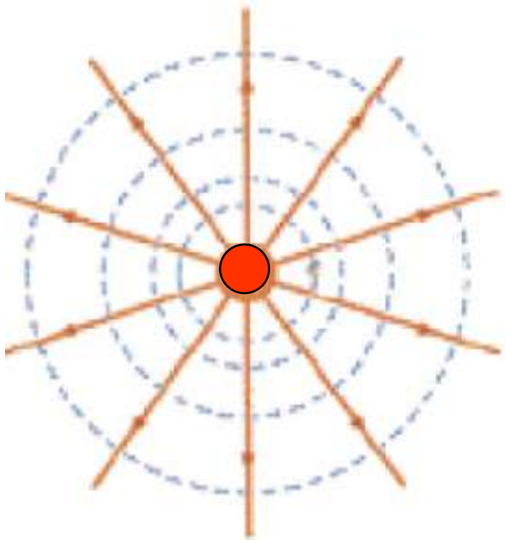


$$\boxed{\vec{B}' = 0}$$

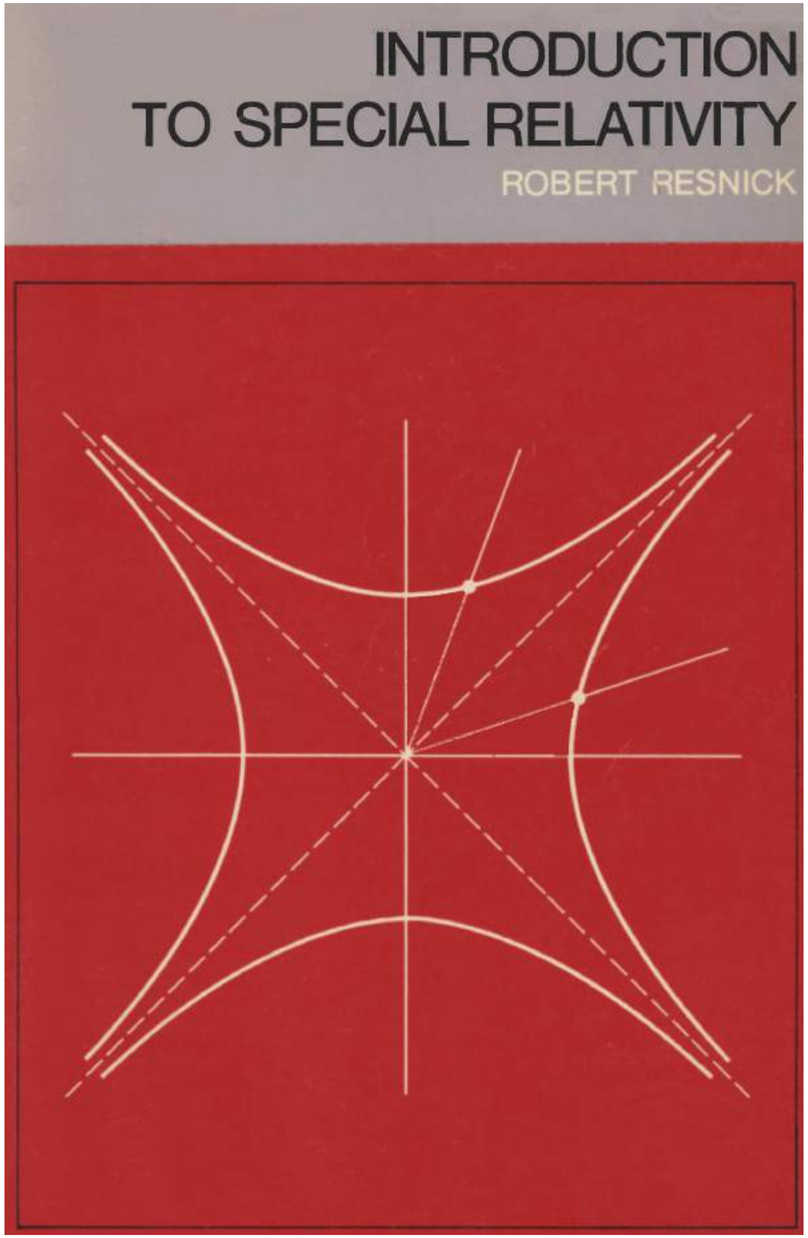
B is transverse to the direction of motion

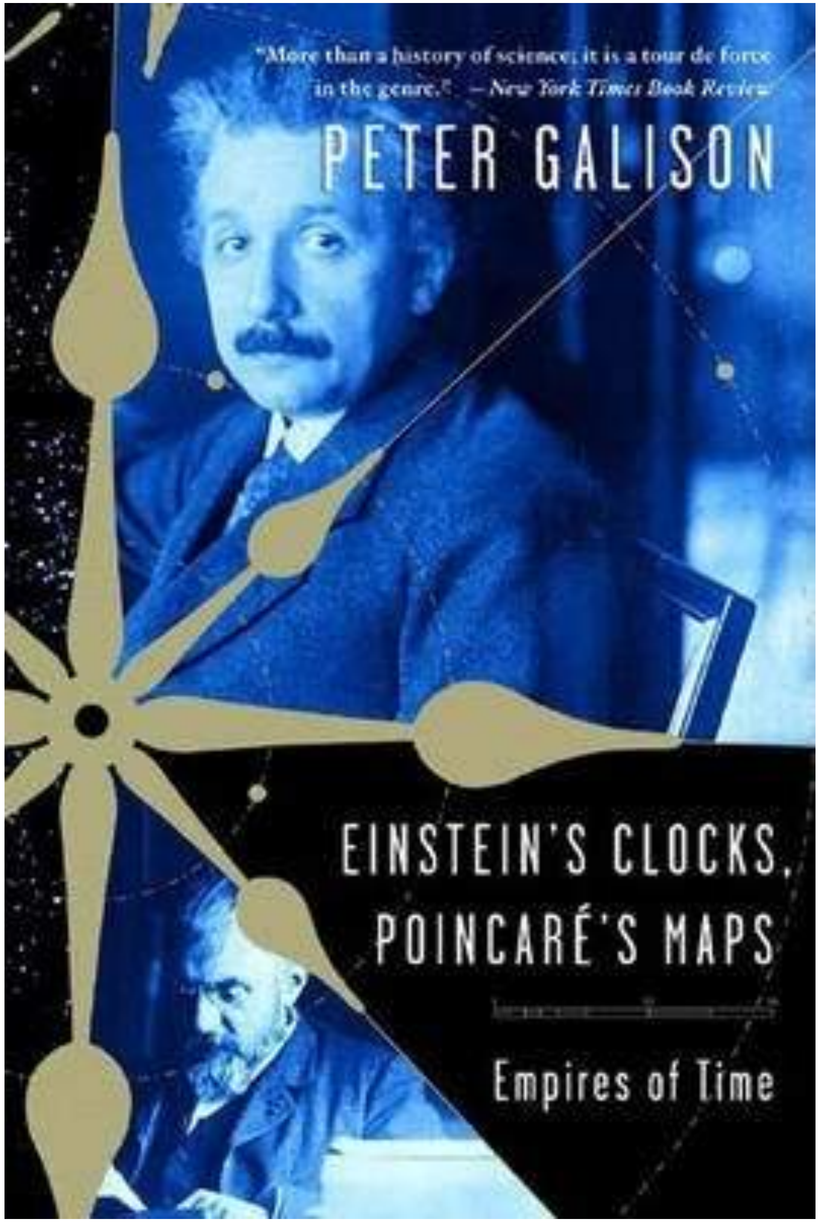
$$B_x = 0$$
$$B_y = -vE_z / c^2$$
$$B_z = vE_y / c^2$$

$$\vec{B}_\perp = \frac{\vec{v} \times \vec{E}}{c^2}$$



$\gamma \longrightarrow \infty$


INTRODUCTION
TO SPECIAL RELATIVITY
ROBERT RESNICK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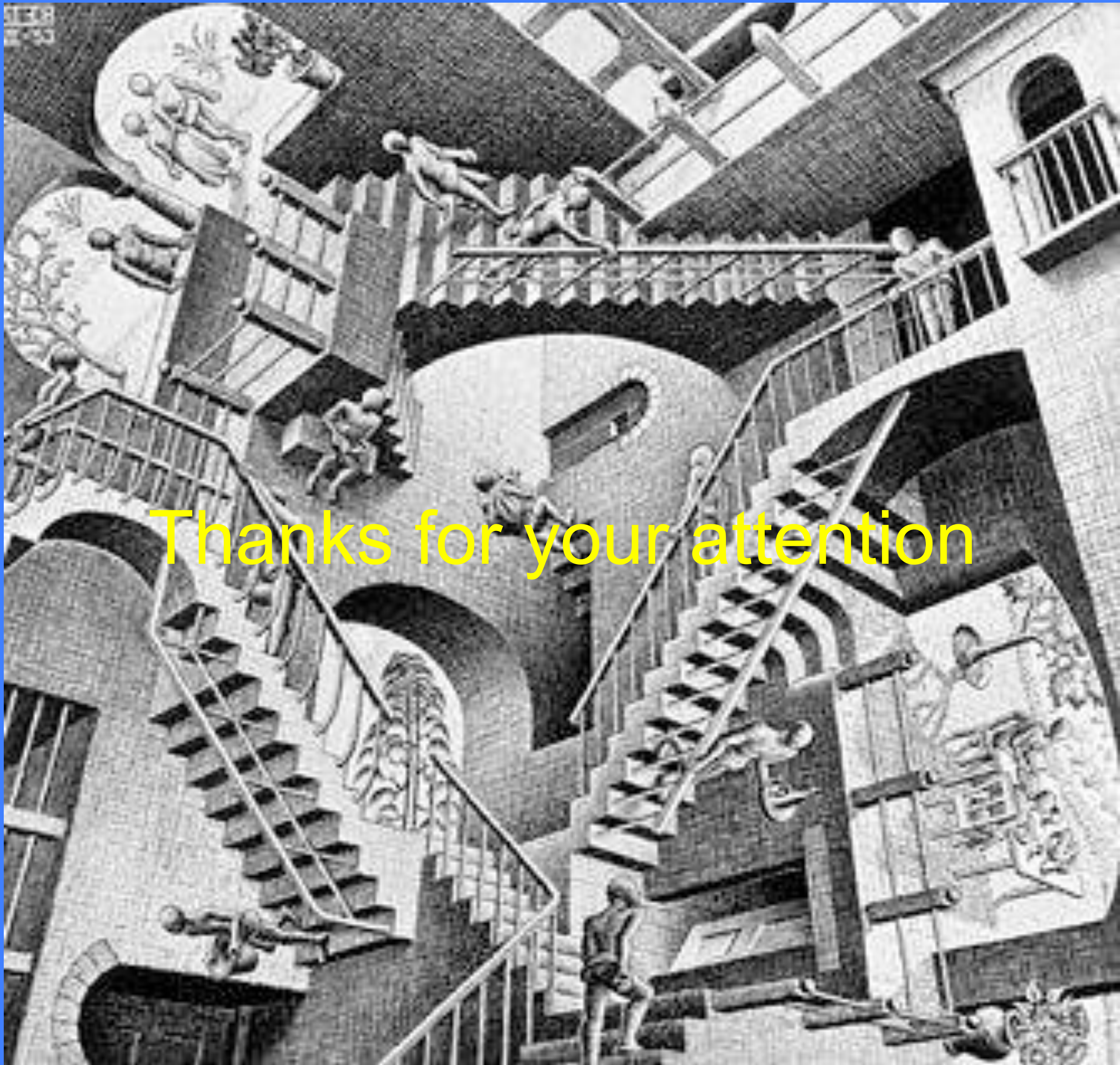

Thanks for your attention