2\textsuperscript{nd} part of this lecture covers:

- Transverse profile techniques
- Emittance determination and transfer lines
- Diagnostics for bunch length and momentum spread
The beam width can be changed by focusing via quadruples. Transverse matching between ascending accelerators is done by focusing. → Profiles have to be controlled at many locations.

**Synchrotrons:** Lattice functions $\beta(s)$ and $D(s)$ are fixed $\Rightarrow$ width $\sigma$ and emittance $\varepsilon$ are:

$$\sigma^2_x(s) = \varepsilon_x \beta_x(s) + \left( D(s) \frac{\Delta p}{p} \right)^2$$  and  $$\sigma^2_y(s) = \varepsilon_y \beta_y(s) \quad \text{(no vertical bend)}$$

**Transfer lines:** Lattice functions are ‘smoothly’ defined due to variable input emittance.

**Typical beam sizes:**

- $e^-$-beam: typically $\varnothing$ 0.1 to 3 mm,  
- protons: typically $\varnothing$ 1 to 30 mm

**A great variety of devices are used:**

- **Optical techniques:** Scintillating screens (all beams),  
  synchrotron light monitors ($e^-$), optical transition radiation ($e^-$, high energetic p),  
  ionization profile monitors (protons)

- **Electronics techniques:** Secondary electron emission SEM grids, wire scanners (all)
Scintillation Screen

Particle’s energy loss in matter produces light → the most direct way of profile observation as used from the early days on!

beam \rightarrow scintillation screen

\begin{itemize}
  \item window
  \item light
  \item CCD camera
\end{itemize}

Pneumatic feed-through with Ø70 mm screen:

Flange Ø200 mm & window

Screen Ø70 mm

Pneumatic drive

CCD
Example of Screen based Beam Profile Measurement

**Example**: GSI LINAC, 4 MeV/u, low current, YAG:Ce screen

### Advantage of screens:
- Direct 2-dim measurement
- High spatial resolution
- Cheap realization

⇒ widely used at transfer lines

### Disadvantage of screens:
- Intercepting device
- Some material might be brittle
- Low dynamic range
- Might be destroyed by the beam

Observation with a CCD, CMOS or video camera

Scintillation Screen (beam stopped)
Light output from various Scintillating Screens

Example: Color CCD camera: Images at different particle intensities determined for U at 300 MeV/u

- Very different light yield i.e. photons per ion’s energy loss
- Different wavelength of emitted light
Material Properties for Scintillating Screens

Some materials and their basic properties:

<table>
<thead>
<tr>
<th>Name</th>
<th>Type</th>
<th>Material</th>
<th>Activ.</th>
<th>Max. $\lambda$</th>
<th>Decay</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chromox</td>
<td>Ceramics</td>
<td>$\text{Al}_2\text{O}_3$</td>
<td>Cr</td>
<td>700 nm</td>
<td>$\approx$ 10 ms</td>
</tr>
<tr>
<td>Alumina</td>
<td></td>
<td>$\text{Al}_2\text{O}_3$</td>
<td>Non</td>
<td>380 nm</td>
<td>$\approx$ 10 ns</td>
</tr>
<tr>
<td>YAG:Ce</td>
<td>Crystal</td>
<td>$\text{Y}_3\text{Al}<em>5\text{O}</em>{12}$</td>
<td>Ce</td>
<td>550 nm</td>
<td>200 ns</td>
</tr>
<tr>
<td>P43</td>
<td>Powder</td>
<td>$\text{Gd}_2\text{O}_3\text{S}$</td>
<td>Tb</td>
<td>545 nm</td>
<td>1 ms</td>
</tr>
<tr>
<td>P46</td>
<td></td>
<td>$\text{Y}_3\text{Al}<em>5\text{O}</em>{12}$</td>
<td>Ce</td>
<td>530 nm</td>
<td>300 ns</td>
</tr>
<tr>
<td>P47</td>
<td></td>
<td>$\text{Y}_3\text{Si}<em>5\text{O}</em>{12}$</td>
<td>Ce&amp;Tb</td>
<td>400 nm</td>
<td>100 ns</td>
</tr>
</tbody>
</table>

Properties of a good scintillator:

- Large light output at optical wavelength
  → standard CCD camera can be used
- Large dynamic range → usable for different ions
- Short decay time → observation of variations
- Radiation hardness → long lifetime
- Good mechanical properties → typ. size up to Ø 10 cm
  (Phosphor Pxx grains of Ø $\approx$ 10 μm on glass or metal).
Outline:

- Scintillation screens:
  - emission of light, universal usage, limited dynamic range
- **SEM-Grid**: emission of electrons, workhorse, limited resolution
- Wire scanner
- Ionization Profile Monitor
- Optical Transition Radiation
- Synchrotron Light Monitors
- Summary
Secondary Electron Emission by Ion Impact

Energy loss of ions in metals close to a surface:
Closed collision with large energy transfer: \( \rightarrow \) fast e\(^{-}\) with \( E_{\text{kin}} \gg 100 \text{ eV} \)
Distant collision with low energy transfer \( \rightarrow \) slow e\(^{-}\) with \( E_{\text{kin}} \leq 10 \text{ eV} \)
\( \rightarrow \) ‘diffusion’ & scattering with other e\(^{-}\): scattering length \( L_s \approx 1 - 10 \text{ nm} \)
\( \rightarrow \) at surface \( \approx 90 \% \) probability for escape

Secondary electron yield and energy distribution comparable for all metals!

\[ \Rightarrow Y = \text{const.} \times \frac{dE}{dx} \quad (\text{Sternglass formula}) \]

Secondary Electron Emission Grids = SEM-Grid

Beam surface interaction: $e^-$ emission $\rightarrow$ measurement of current.

*Example: 15 wire spaced by 1.5 mm:*

*SEM-Grid feed-through on CF200:*

![Image of SEM-Grid feed-through on CF200](image-url)
Beam surface interaction: $e^{-}$ emission $\rightarrow$ measurement of current.

*Example: 15 wire spaced by 1.5 mm:*

Each wire is equipped with one I/U converter different ranges settings by $R_i$

$\rightarrow$ very large dynamic range up to $10^6$. 

Example of Profile Measurement with SEM-Grids

Even for low energies, several SEM-Grid can be used due to the $\approx 80\%$ transmission $\Rightarrow$ frequently used instrument beam optimization: setting of quadrupoles, energy….

**Example:** $C^6^+ \text{ beam of 11.4 MeV/u at different locations at GSI-LINAC}$
The Artist view of a SEM-Grid = Harp
Outline:

- Scintillation screens: emission of light, universal usage, limited dynamic range
- SEM-Grid: emission of electrons, workhorse, limited resolution
- **Wire scanner**: emission of electrons, workhorse, scanning method
- Ionization Profile Monitor
- Optical Transition Radiation
- Synchrotron Light Monitors
- **Summary**
Slow, linear Wire Scanner

Idea: One wire is scanned through the beam!
Slow, linear scanner are used for:

- low energy protons
- high resolution measurements e.g. usable at $e^+e^-$ colliders by de-convolution $\sigma^2_{beam}=\sigma^2_{meas}-d^2_{wire}$
  $\Rightarrow$ resolution down to 10 $\mu$m range can be reached
- detection of beam halo.
The Artist view of a Beam Scraper or Scanner
Fast, Flying Wire Scanner

In a synchrotron **one** wire is scanned though the beam as fast as possible.

Fast pendulum scanner for synchrotrons; sometimes it is called *flying wire*:

From https://twiki.cern.ch/twiki/bin/viewauth/BWSUpgrade/
Usage of Flying Wire Scanners

**Material:** carbon or SiC → low Z-material for low energy loss and high temperature.

**Thickness:** down to 10 μm → high resolution.

**Detection:** Either the secondary current (like SEM-grid) or high energy secondary particles (like beam loss monitor)

**flying wire:** only sec. particle detection due to induced current by movement.

**Secondary particles:**

**Proton beam** → hadrons shower (π, n, p...)

**Electron beam** → Bremsstrahlung photons.

**Kinematics of flying wire:**

Velocity during passage typically 10 m/s = 36 km/h and typical beam size \( \varnothing \) 10 mm ⇒ time for traversing the beam \( t \approx 1 \text{ ms} \)

**Challenges:** Wire stability for fast movement with high acceleration

![Graph of Proton impact on scanner at CERN-PS Booster]

- Rest mass:
  - \( m_{\pi^\pm} = 140 \text{ MeV/c}^2 \)
  - \( m_{\pi^0} = 135 \text{ MeV/c}^2 \)

U. Raich et al., DIPAC 2005
The Artist View of a Wire Scanner
## Comparison between SEM-Grid and Wire Scanners

<table>
<thead>
<tr>
<th>Grid:</th>
<th>Measurement at a single moment in time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scanner:</td>
<td>Fast variations can not be monitored</td>
</tr>
<tr>
<td></td>
<td>→ for pulsed LINACs precise synchronization is needed</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grid:</th>
<th>Not adequate at synchrotrons for stored beam parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scanner:</td>
<td>At high energy synchrotrons flying wire scanners are nearly non-destructive</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grid:</th>
<th>Resolution of a grid is fixed by the wire distance (typically 1 mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scanner:</td>
<td>For slow scanners the resolution is about the wire thickness (down to 10 μm)</td>
</tr>
<tr>
<td></td>
<td>→ used for e⁻-beams having small sizes (down to 10 μm)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grid:</th>
<th>Needs one electronics channel per wire</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>→ expensive electronics and data acquisition</td>
</tr>
</tbody>
</table>

| Scanner: | Needs a precise movable feed-through → expensive mechanics. |
Outline:

- Scintillation screens:
  - emission of light, universal usage, limited dynamic range
- SEM-Grid: emission of electrons, workhorse, limited resolution
- Wire scanner: emission of electrons, workhorse, scanning method
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  - secondary particle detection from interaction beam-residual gas
- Optical Transition Radiation
- Synchrotron Light Monitors
- Summary
**Ionization Profile Monitor at GSI Synchrotron**

**Non-destructive** device for proton synchrotron:
- beam ionizes the residual gas by electronic stopping
- gas ions or $e^-$ accelerated by $E$-field $\approx 1$ kV/cm
- spatial resolved single particle detection

**Realization at GSI synchrotron:**
- IPM with 175 x 175 mm clearance
- HV-electrode
- MCP: 100 x 30 mm$^2$
- 63 wires, 2 mm spacing
- 300 mm flange

**Typical vacuum pressure:**
- Transfer line: $N_2$ $10^{-8}...10^{-6}$ mbar $\cong 3\cdot10^8...3\cdot10^{10}$ cm$^{-3}$
- Synchrotron: $H_2$ $10^{-11}...10^{-9}$ mbar $\cong 3\cdot10^5...3\cdot10^7$ cm$^{-3}$
Ionization Profile Monitor Realization

The realization for the heavy ion storage ring ESR at GSI:

Realization at GSI synchrotron:

*IPM with 175 x 175 mm clearance*

*HV-electrode*

*MCP: 100 x 30 mm²*

*63 wires, 2 mm spacing*

*300 mm flange*

Horizontally IPM:

- E-field box
- Electodes
- MCP

**Horizontal camera**

- Insertion 650 mm
- Ø250 mm
- 175 mm

**Vertical IPM**

- E-field separation disks
- View port Ø150 mm
- Vertical camera
The realization for the heavy ion storage ring ESR at GSI: 

**Realization at GSI synchrotron:**

- **Horizontal IPM:**
  - E-field box
  - Electrodes
  - Beam
  - MCP
  - View port Ø150 mm

- **Vertical IPM:**
  - E-field box
  - Electrodes
  - View port Ø150 mm

**Ionization Profile Monitor Realization**
‘Adiabatic’ Damping during Acceleration

The emittance $\varepsilon = \int dx dx'$ is defined via the position deviation and angle in lab-frame

$$v_\perp \quad v \quad \varphi_{\text{slow}} \quad v_\parallel$$

before acceleration

$$v_\perp \quad v \quad \varphi_{\text{fast}} \quad v_\parallel$$

after acceleration

After acceleration the longitudinal velocity is increased $\Rightarrow$ angle $\varphi$ is smaller

The angle is expressed in momenta: $x' = p_\perp / p_\parallel$; the emittance is $<xx'> = 0$:

$$\varepsilon = x \cdot x' = x \cdot p_\perp / p_\parallel$$

$\Rightarrow$ under ideal conditions the emittance can be normalized to the momentum $p_\parallel = \gamma \cdot m \cdot \beta c$

$\Rightarrow$ normalized emittance $\varepsilon_{\text{norm}} = \beta \gamma \cdot \varepsilon$ is preserved with the Lorentz factor $\gamma$ and velocity $\beta = v/c$

**Example:** Acceleration in GSI-synchrotron for $\text{C}^{6+}$ from $6.7 \rightarrow 600 \text{ MeV/u}$ ($\beta = 12 \rightarrow 79 \%$) observed by IPM

theoretical width: $\langle x \rangle_f = \sqrt{\frac{\beta_i \cdot \gamma_i}{\beta_f \cdot \gamma_f}} \cdot \langle x \rangle_i$

$= 0.33 \cdot \langle x \rangle_i$

measured width: $\langle x \rangle_f \approx 0.37 \cdot \langle x \rangle_i$

IPM is well suited

for long time observations without beam disturbance

$\rightarrow$ mainly used at proton synchrotrons.

---

GSI - Palaver, Dec. 10th, 2003, A dedicated proton accelerator for $p$-physics at the future GSI facilities

Peter Forck, CAS 2018, Constanta
**Ideal case of injection matching:**
Orientation of injected beam matches phase space as given by synchrotron Twiss parameters $\alpha$, $\beta$, and $\gamma$ i.e. ‘machine emittance’
$\iff$ no change after each turn $\iff$ stable storage
$\iff$ The beam ellipse $\sigma_{beam}$ correspond to the machine ellipse at injection point for $N=0$ i.e.
\[
\sigma_{beam}(N = 0) = \epsilon_{beam} \begin{pmatrix} \beta_{synchronous} & -\alpha_{synchronous} \\ -\alpha_{synchronous} & \gamma_{synchronous} \end{pmatrix}
\]
$\Rightarrow$ only in this case stable storage (math: $t \to \infty$)
\[
\sigma_{beam}(N = 0) = \sigma_{beam}(N \to \infty)
\]
**Injection Matching into a Synchrotron: Phase Space Mismatch**

**Ideal case of injection matching:**
Orientation of injected beam matches phase space as given by synchrotron
Twiss parameters $\alpha, \beta$, and $\gamma$ i.e. ‘machine emittance’
$\Leftrightarrow$ no change after each turn $\Leftrightarrow$ stable storage

**Mismatched case:**
- The beam ellipse $\sigma_{beam}$ has different orientation as machine ellipse at injection point for $N=0$ i.e.
- Transformation after one turn
  $\sigma_{beam}(N = 1) = M\sigma_{beam}(N = 0) M^T$
  $\neq \sigma_{beam}(N = 0)$
i.e. rotation in phase space by the tune
i.e. phase advance per turn

Depictive argument: A particle on both ellipses

**Observable quantity:** Beam profile oscillates

**After many turns:**
Particle have different tunes e.g. by longitudinal momentum deviation and chromaticity $\frac{\Delta Q}{Q_0} = \xi \cdot \frac{\Delta p}{p_0}$
or space charge $\Delta Q_{incoh}$ $\Rightarrow$ Entire transverse phase space is filled i.e. beam with enlarged emittance
Mis-matched injection into a synchrotron:

Can be monitored by beam profile measurement:

**Example:** Injection of a 80 ns bunch of protons into CERN PS at 1.4 GeV/u (2.2 µs revolution time)

Profile measurement by SEM-Grid

- Turn-by-turn profile variation related to tune
- Used for improvement of injection parameters

From M. Benedikt et al., DIPAC 2001
Outline:

- Scintillation screens:
  emission of light, universal usage, limited dynamic range
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- Wire scanner: emission of electrons, workhorse, scanning method
- Ionization Profile Monitor:
  secondary particle detection from interaction beam-residual gas
- **Optical Transition Radiation:**
  crossing material boundary, for relativistic beams only
- Synchrotron Light Monitors
- Summary
Optical Transition Radiation OTR for a single charge $e$:

Assuming a charge $e$ approaches an ideal conducting boundary e.g. metal foil

- image charge is created by electric field
- dipole type field pattern
- field distribution depends on velocity $\beta$ and Lorentz factor $\gamma$ due to relativistic trans. field increase
- penetration of charge through surface within $t < 10$ fs: sudden change of source distribution
- emission of radiation with dipole characteristic

Other physical interpretation: Impedance mismatch at boundary leads to radiation
Optical Transition Radiation: Depictive Description

Optical Transition Radiation OTR can be described in classical physics:

approximated formula for normal incidence & in-plane polarization:

\[ \frac{d^2W}{d\theta \, d\omega} \approx \frac{2e^2 \beta^2}{\pi \, c} \cdot \frac{\sin^2 \theta \cdot \cos^2 \theta}{\left(1 - \beta^2 \cos^2 \theta\right)^2} \]

\( W \): radiated energy
\( \omega \): frequency of wave

Angular distribution of radiation in optical spectrum:

- lope emission pattern depends on velocity or Lorentz factor \( \gamma \)
- peak at angle \( \theta \approx 1/\gamma \)
- emitted energy i.e. amount of photons scales with \( W \propto \beta^2 \)
- broad wave length spectrum (i.e. no dependence on \( \omega \))
- suited for high energy electrons
Technical Realization of Optical Transition Radiation OTR

**OTR is emitted by charged particle passage through a material boundary.**

Photon distribution:
\[
\frac{dN_{\text{photon}}}{d\Omega} = N_{\text{beam}} \cdot \frac{2e^2 \beta^2}{\pi c} \cdot \log\left(\frac{\lambda_{\text{begin}}}{\lambda_{\text{end}}}\right) \cdot \frac{\theta^2}{(\gamma^{-2} + \theta^2)^2}
\]

- **Detection:** Optical 400 nm < \( \lambda \) < 800 nm using image intensified CCD
- Larger signal for relativistic beam \( \gamma \gg 1 \)
- Low divergence for \( \gamma \gg 1 \) \( \Rightarrow \) large signal

**⇒ well suited for e\(^{-}\) beams**

**⇒ p-beam only for** \( E_{\text{kin}} > 10 \text{ GeV} \) \( \Leftrightarrow \gamma > 10 \)

- Insertion of thin Al-foil under 45°
- Observation of low light by CCD.
Example of realization at TERATRON:
➤ Insertion of foil
e.g. 5 µm Kapton coated with 0.1 µm Al

Advantage: thin foil ⇒ low heating & straggling
2-dim image visible

Results at FNAL-TEVATRON synchrotron
with 150 GeV proton
Using fast camera: Turn-by-turn measurement

Courtesy V.E. Scarpine (FNAL) et al., BIW’06
Optical Transition Radiation compared to Scintillation Screen

Installation of OTR and scintillation screens on same drive:

Results:
- Much more light from YAG:Ce for 100 MeV (γ=200) electrons light output $I_{YAG} \approx 10^5 I_{OTR}$
- Broader image from YAG:Ce due to finite shoulders or CCD saturation(?)

Courtesy of U. Iriso et al., DIPAC’09

Example: ALBA LINAC 100 MeV
Comparison between Scintillation Screens and OTR

**OTR**: electrodynamic process → beam intensity linear to # photons, high radiation hardness

**Scint. Screen**: complex atomic process → saturation possible, for some low radiation hardness

**OTR**: thin foil Al or Al on Mylar, down to 0.25 μm thickness
→ minimization of beam scattering (Al is low Z-material)

**Scint. Screen**: thickness ≈ 1 mm inorganic, fragile material, not radiation hard

**OTR**: low number of photons → expensive image intensified CCD

**Scint. Screen**: large number of photons → simple CCD sufficient

**OTR**: complex angular photon distribution → resolution limited

**Scint. Screen**: isotropic photon distribution → simple interpretation

**OTR**: large γ needed → e^{-}-beam with $E_{kin} > 100$ MeV, proton-beam with $E_{kin} > 100$ GeV

**Scint. Screen**: for all beams

**Remark**: OTR not suited for LINAC-FEL due to coherent light emission (not covered here) but scintilation screens can be used.
Measurement of Beam Profile

Outline:

- Scintillation screens:
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- Wire scanner: emission of electrons, workhorse, scanning method
- Ionization Profile Monitor:
  - secondary particle detection from interaction beam-residual gas
- Optical Transition Radiation:
  - crossing optical boundary, for relativistic beams only
- **Synchrotron Light Monitors**
  - photon detection of emitted synchrotron light in optical and X-ray range
- Summary
An electron bent (i.e. accelerated) by a dipole magnet emit synchrotron light.

This light is emitted into a cone of opening $2/\gamma$ in lab-frame.

$\Rightarrow$ Well suited for rel. $e^-$

For protons:

Only for energies $E_{kin} > 100$ GeV

The light is focused to a intensified CCD.

**Advantage:**

Signal anyhow available!
Realization of a Synchrotron Light Monitor

Extracting out of the beam’s plane by a (cooled) mirror
→ Focus to a slit + wavelength filter for optical wavelength
→ Image intensified CCD camera

*Example:* CERN LEP-monitor with bending radius 3.1 km (blue or near UV)

![Diagram of synchrotron light monitor](image)

Courtesy C. Bovet (CERN) et al., PAC’91
Example: Synchrotron radiation facility APS accumulator ring and blue wavelength:

**Advantage:** Direct measurement of 2-dim distribution, only mirror installed in the vacuum pipe

**Realization:** Optics outside of vacuum pipe

**Disadvantage:** Resolution limited by the diffraction due to finite apertures in the optics.

B.X. Yang (ANL) et al. PAC’97

---

**Result from a Synchrotron Light Monitor**

**Synchrotron**

**SRM**

**injection**

**extraction**

\[ \sigma_y = 0.165 \text{ mm} \]

\[ \sigma_x = 0.797 \text{ mm} \]
Adiabatic Damping for an Electron Beam

*Example:* Booster at the light source ALBA acceleration from 0.1 → 3 GeV within 130 ms

Profile measure by synchrotron light monitor:

The beam emittance in influenced by:

- Adiabatic damping
- Longitudinal momentum contribution via dispersion
  \[ \Delta x_D(s) = D(s) \cdot \frac{\Delta p}{p} \]
- Total width
  \[ \Delta x_{tot}(s) = \sqrt{\varepsilon \beta(s) + D(s) \cdot \frac{\Delta p}{p}} \]
- Quantum fluctuation due to light emission

![Beam profile images](image)

Courtesy U. Iriso & M. Pont (ALBA) et al. IPAC 2011
The Artist View of a Synchrotron Light Monitor

$5000$ Prize

Purpose. To recognize and encourage innovative achievements in the field of accelerator beam instrumentation.

Awarded. The Faraday Cup Award consists of a US$ 5000 prize and a certificate to be presented at the annual Rules. The Faraday Cup shall be awarded for an outstanding contribution to the development of an innovative beam diagnostics instrument of proven workability. The Faraday Cup is only awarded for published contribution and delivered performance - as opposed to theoretical performance.
Diffraction Limit of Synchrotron Light Monitor

**Limitations:**
Diffraction limits the resolution due to Fraunhofer diffraction

\[ \sigma \approx 0.6 \cdot \left( \frac{\lambda^2}{\rho} \right)^{1/3} \]

\[ \approx 100 \ \mu\text{m} \] for typical case

**Improvements:**

- **Shorter wavelength:**
  Using X-rays and an aperture of Ø 1mm
  → ‘X-ray pin hole camera’, achievable resolution \( \sigma \approx 10 \ \mu\text{m} \)

- **Interference technique:**
  At optical wavelength using a double slit
  → interference fringes achievable resolution \( \sigma \approx 1 \ \mu\text{m} \).
X-ray Pin-Hole Camera

The diffraction limit is \[ \sigma \approx 0.6 \cdot \left( \lambda^2 / \rho \right)^{1/3} \Rightarrow \text{shorter wavelength by X-rays.} \]

Example: PETRA III

 Courtesy K. Wittenburg, DESY

Example: PETRA III result:

\[ \sigma = 44 \mu m \]

\[ \sigma = 32 \mu m \]
Summary for Beam Profile Measurement

Different techniques are suited for different beam parameters:

- **e\(^-\)-beam:** typically Ø 0.1 to 3 mm, **protons:** typically Ø 3 to 30 mm

Intercepting ↔ non-intercepting methods

Direct observation of electrodynamics processes:
- Optical synchrotron radiation monitor: non-destructive, for e\(^-\)-beams, complex, limited res.
- X-ray synchrotron radiation monitor: non-destructive, for e\(^-\)-beams, very complex
- OTR screen: nearly non-destructive, large relativistic γ needed, e\(^-\)-beams mainly

Detection of secondary photons, electrons or ions:
- Scintillation screen: destructive, large signal, simple setup, all beams
- Ionization profile monitor: non-destructive, expensive, limited resolution, for protons
- **xxx** Residual fluorescence monitor: non-destructive, limited signal strength, for protons

Wire based electronic methods:
- SEM-grid: partly destructive, large signal and dynamic range, limited resolution
- Wire scanner: partly destructive, large signal and dynamics, high resolution, slow scan.
**Measurement of transverse Emittance**

The emittance characterizes the whole beam quality, assuming linear behavior as described by second order differential equation.

It is defined within the phase space as:

\[ \varepsilon_x = \frac{1}{\pi} \int_A dx dx' \]

The measurement is based on determination of:

- **either** profile width \( \sigma_x \) and angular width \( \sigma_x' \) at one location
- **or** \( \sigma_x \) at different locations and linear transformations.

Different devices are used at transfer lines:

- Lower energies \( E_{kin} < 100 \text{ MeV/u} \): slit-grid device, pepper-pot (suited in case of non-linear forces).
- All beams: Quadrupole variation, 'three grid' method using linear transformations (not well suited in the presence of non-linear forces).

**Synchrotron:** lattice functions results in stability criterion

\[ \Delta \varepsilon_x = \frac{1}{\beta_x(s)} \left[ \sigma_x^2 - \left( D(s) \frac{\Delta p}{p} \right) \right] \]  \[ \varepsilon_y = \frac{\sigma_y^2}{\beta_y(s)} \]

\( \Rightarrow \) beam width delivers emittance:
Trajectory and Characterization of many Particles

- Single particle trajectories are forming a beam
- They have a distribution of start positions and angles

⇒ Characteristic quantity is the **beam envelope**

**Goal:**
Transformation of envelope
⇔ behavior of whole ensemble

![Diagram showing single and combined particle trajectories with envelopes for Focus. quad. and drift sections.](image-url)
Definition of Coordinates and basic Equations

The basic vector is 6 dimensional:

\[
\vec{x}(s) = \begin{pmatrix} x \\ x' \\ y \\ y' \\ l \\ \delta \end{pmatrix} = \begin{pmatrix} \text{hori. spatial deviation} \\ \text{horizontal divergence} \\ \text{vert. spatial deviation} \\ \text{vertical divergence} \\ \text{longitudinal deviation} \\ \text{momentum deviation} \end{pmatrix} = \begin{pmatrix} [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [\text{mrad}] \\ [\text{mm}] \\ [\%] \end{pmatrix}
\]

The transformation of a single particle from a location \( s_0 \) to \( s_1 \) is given by the Transfer Matrix \( R \):

\[
x(s_1) = R(s) \cdot x(s_0)
\]

The transformation of a the envelope from a location \( s_0 \) to \( s_1 \) is given by the Beam Matrix \( \sigma \):

\[
\sigma(s_1) = R(s) \cdot \sigma(s_0) \cdot R^T(s)
\]

6-dim Beam Matrix with decoupled hor. & vert. plane:

\[
\sigma = \begin{pmatrix}
\sigma_{11} & \sigma_{12} & 0 & 0 & \sigma_{15} & \sigma_{16} \\
\sigma_{12} & \sigma_{22} & 0 & 0 & \sigma_{25} & \sigma_{26} \\
0 & 0 & \sigma_{33} & \sigma_{34} & 0 & 0 \\
0 & 0 & \sigma_{34} & \sigma_{44} & 0 & 0 \\
\sigma_{15} & \sigma_{25} & 0 & 0 & \sigma_{55} & \sigma_{56} \\
\sigma_{16} & \sigma_{26} & 0 & 0 & \sigma_{56} & \sigma_{66}
\end{pmatrix}
\]

Beam width for the three horizontal coordinates: Horizontal beam matrix:

\[
x_{\text{rms}} = \sqrt{\sigma_{11}} \quad \sigma_{11} = \langle x^2 \rangle
\]

\[
y_{\text{rms}} = \sqrt{\sigma_{33}} \quad \sigma_{12} = \langle xx' \rangle
\]

\[
l_{\text{rms}} = \sqrt{\sigma_{55}} \quad \sigma_{22} = \langle x'^2 \rangle
\]

Peter Forck, CAS 2018, Constanta
The Emittance for Gaussian and non-Gaussian Beams

The beam distribution can be non-Gaussian, e.g. at:

- beams behind ion source
- space charged dominated beams at LINAC & synchrotron
- cooled beams in storage rings

General description of emittance using terms of 2-dim distribution:

\[ \varepsilon_{\text{rms}} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2} \]

It describes the value for 1 standard derivation

For **Gaussian beams only**: \( \varepsilon_{\text{rms}} \) interpreted as area containing a fraction \( f \) of ions:

\[ \varepsilon(f) = -2\pi \varepsilon_{\text{rms}} \cdot \ln(1 - f) \]

**Care:**

No common definition of emittance concerning the fraction \( f \)

<table>
<thead>
<tr>
<th>Emittance ( \varepsilon(f) )</th>
<th>Fraction ( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 1 \cdot \varepsilon_{\text{rms}} )</td>
<td>15 %</td>
</tr>
<tr>
<td>( \pi \cdot \varepsilon_{\text{rms}} )</td>
<td>39 %</td>
</tr>
<tr>
<td>( 2\pi \cdot \varepsilon_{\text{rms}} )</td>
<td>63 %</td>
</tr>
<tr>
<td>( 4\pi \cdot \varepsilon_{\text{rms}} )</td>
<td>86 %</td>
</tr>
<tr>
<td>( 8\pi \cdot \varepsilon_{\text{rms}} )</td>
<td>98 %</td>
</tr>
</tbody>
</table>
Outline:

- Definition and some properties of transverse emittance
- Slit-Grid device: scanning method
  - scanning slit → beam position & grid → angular distribution
- Quadrupole strength variation and position measurement
- Summary
The Slit-Grid Measurement Device

Slit-Grid: Direct determination of position and angle distribution. Used for protons with $E_{kin} < 100 \text{ MeV/u} \Rightarrow \text{range } R < 1 \text{ cm}.$

**Hardware**

- **Slit:** position $P(x)$ with typical width: 0.1 to 0.5 mm
- **Distance:** typ. 0.5 to 5 m (depending on beam energy 0.1 ... 100 MeV)
- **SEM-Grid:** angle distribution $P(x')$

**Analysis**

phase space

emittance ellipse

0.5...5 m
The distribution is depicted as a function of position [mm] & angle [mrad]

**The distribution can be visualized by**
- Mountain plot
- Contour plot

**Calc. of 2nd moments** $<x^2>$, $<x'^{2}>$ & $<xx'>$

**Emittance value $\varepsilon_{rms}$ from**

$$\varepsilon_{rms} = \sqrt{\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2}$$

⇒ **Problems:**
- Finite **binning** results in limited resolution
- **Background** → large influence on $<x^2>$, $<x'^2>$ and $<xx'>$

**Or fit of distribution with an ellipse**

⇒ **Effective emittance only**

**Remark:** Behind a ion source the beam might very non-Gaussian due to plasma density and aberration at quadrupoles

Beam: Ar$^{4+}$, 60 keV, 15 μA at Spiral2 Phoenix ECR source.

P. Ausset, DIPAC 2009
Outline:

- **Definition and some properties of transverse emittance**
- **Slit-Grid device: scanning method**
  - scanning slit → beam position & grid → angular distribution
- **Quadrupole strength variation and position measurement**
  - emittance from several profile measurement and beam optical calculation
- **Summary**
Emittance Measurement by Quadrupole Variation

From a profile determination, the emittance can be calculated via linear transformation, if a well known and constant distribution (e.g. Gaussian) is assumed.

- Measurement of beam width
  \[ x_{\text{max}}^2 = \sigma_{11}(1, k) \]

  Matrix \( R(k) \) describes the focusing.

- With the drift matrix the transfer is
  \[ R(k_i) = R_{\text{drift}} \cdot R_{\text{focus}}(k_i) \]

- Transformation of the beam matrix
  \[ \sigma(1,k_i) = R(k_i) \cdot \sigma(0) \cdot R^T(k_i) \]

  **Task:** Calculation of \( \sigma(0) \) at entrance \( s_0 \) i.e. all three elements

\[ x^2(k) = \sigma_{11}(1,k) \]
Measurement of transverse Emittance

Using the ‘thin lens approximation’ i.e. the quadrupole has a focal length of \( f \):

\[
R_{\text{focus}}(K) = \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ K & 1 \end{pmatrix} \implies R(L, K) = R_{\text{drift}}(L) \cdot R_{\text{focus}}(K) = \begin{pmatrix} 1 + LK & L \\ K & 1 \end{pmatrix}
\]

Measurement of the matrix-element \( \sigma_{11}(l, K) \) from \( \sigma(l, K) = R(K) \cdot \sigma(0) \cdot R^T(K) \)

**Example:** Square of the beam width at ELETTRA 100 MeV e⁺ Linac, YAG:Ce:

![Graph showing measurement of the matrix-element \( \sigma_{11}(l, K) \) from \( \sigma(l, K) = R(K) \cdot \sigma(0) \cdot R^T(K) \).]

**For completeness:** The relevant formulas

\[
\sigma_{11}(l, K) = L^2 \sigma_{11}(0) \cdot K^2 + 2 \cdot (L \sigma_{11}(0) + L^2 \sigma_{12}(0)) \cdot K + L^2 \sigma_{22}(0) + \sigma_{11}(0)
\]

\[
\equiv a \cdot K^2 - 2ab \cdot K + ab^2 + c
\]

The three matrix elements at the quadrupole:

\[
\sigma_{11}(0) = \frac{a}{L^2}
\]

\[
\sigma_{12}(0) = -\frac{a}{L^2} \left( \frac{1}{L} + b \right)
\]

\[
\sigma_{22}(0) = \frac{1}{L^2} \left( ab^2 + c + \frac{2ab}{L} + \frac{a}{L^2} \right)
\]

\[
\varepsilon_{\text{rms}} \equiv \sqrt{\det \sigma(0)} = \sqrt{\sigma_{11}(0) \cdot \sigma_{22}(0) - \sigma_{12}^2(0)} = \sqrt{ac / L^2}
\]

G. Penco (ELETTRA) et al., EPAC’08
Summary for transverse Emittance Measurement

Emittance is the important quantity for comparison to theory.
It includes size (value of \( \varepsilon \)) and orientation in phase space (\( \sigma_{ij} \) or \( \alpha, \beta \) and \( \gamma \))

three independent values \( \varepsilon_{rms} = \sqrt{\sigma_{11} \cdot \sigma_{22} - \sigma_{12}^2} = \sqrt{<x^2> \cdot <x'^2> - <xx'>^2} \)
assuming no coupling between horizontal, vertical and longitudinal planes

Transfer line, low energy beams → direct measurement of x- and x'-distribution

- **Slit-grid**: movable slit → \( x \)-profile, grid → \( x' \)-profile

Transfer line, all beams → profile measurement + linear transformation:

- **Quadrupole variation**: one location, different setting of a quadrupole
  
  **Assumptions**: 
  - well aligned beam, no steering
  - no emittance blow-up due to space charge.

**Important remark**: For a synchrotron with a stable beam storage,

width measurement is sufficient using \( x_{rms} = \sqrt{\varepsilon_{rms} \cdot \beta} \)
Measurement of longitudinal Parameters

**Measurement of longitudinal parameter:**

Bunch length measurement at

- Synchrotron light sources
- Linear light sources
- Summary

**Longitudinal ↔ transverse correspondences:**

- position relative to rf ↔ transverse center-of-mass
- bunch structure in time ↔ transverse profile
- momentum or energy spread ↔ transverse divergence
- longitudinal emittance ↔ transverse emittance.
Bunch Length Measurement for relativistic Electrons

Electron bunches are too short ($\sigma_t < 300$ ps) to be covered by the bandwidth of pick-ups ($f < 1$ GHz $\Leftrightarrow t_{\text{rise}} > 300$ ps) for structure determination.

→ Time resolved observation of synchr. light with a streak camera: Resolution $\approx 1$ ps.
Technical Realization of Streak Camera

Hardware of a streak camera

Time resolution down to 0.5 ps:

≈ 30 cm

≈ 60 cm
Technical Realization of Streak Camera

Hardware of a streak camera

Time resolution down to 0.5 ps:

The Streak Camera setup at ELETTRA, Trieste, Italy
Results of Bunch Length Measurement by a Streak Camera

The streak camera delivers a fast scan in vertical direction (here 360 ps full scale) and a slower scan in horizontal direction (24 μs).

**Example:** Bunch length at the synchrotron light source SOLEIL for $U_{rf} = 2$ MV for slow direction 24 μs and scaling for fast scan 360 ps: measure $\sigma_t = 35$ ps.

Short bunches are desired by the users

**Example:** Bunch length $\sigma_t$ as a function of stored current (i.e. space charge de-focusing) at SOLEIL

Courtesy of M. Labat et al., DIPAC’07
The Artist View of a Streak Camera
Excursus: 4th Generation Light Sources & Beam Delivery

4th Generation Light Sources: LINAC based, single pass with large energy loss. $E_{\text{electron}} \approx 1 \ldots 18 \text{ GeV}$, coherent light from undulator, $E_{\gamma} < 1000 \text{ keV}$, temporally short pulse.

**Goal:** Short bunches with high number of particles → short, intense laser pulses for electron generation.

**Requirement:** Position stability ⇒ resolution < 1 μm.

Flash, Hamburg
**Bunch length measurement by electro-optical Method**

For Free Electron Lasers → bunch length below 1 ps is achieved

→ below resolution of streak camera
→ short laser pulses with $t \approx 10 \text{ fs}$ and electro-optical modulator

Electro optical modulator: birefringent, rotation angle depends on external electric field

Relativistic electron bunches: transverse field $E_{\perp, \text{lab}} = \gamma E_{\perp, \text{rest}}$ carries the time information

Scanning of delay between bunch and laser → time profile after several pulses.

From S.P.Jamison et al., EPAC 2006
Realization of EOS Scanning

Setup of a scanning EOS method

Using 12fs pulses from Ti:Al₂O₃ laser at 800nm and ZnTe crystal 0.5mm thick with a e⁻ beam 46MeV of 200pC

X. Yan et al, PRL 85, 3404 (2000)
**Example**: Bunch length at FLASH
(1 ps bunch duration = 300 μm length)

B. Steffen (PSI) et al., DIPAC 2009
B. Steffen et al., FEL Conf. 2005
Summary of longitudinal Measurements

Devices for bunch length at light sources:

*Streak cameras:*

- Time resolved monitoring of synchrotron radiation
  - for relativistic e\(^{-}\)-beams, \(t_{\text{bunch}} < 1\) ns, injection matching
  reason: too short bunches for rf electronics.

*Laser scanning:*

- Electro-optical modulation of short laser pulse
  - very high time resolution down to some fs
Diagnostics is the ’sensory organ’ for the beam.

It required for operation and development of accelerators

Several categories of demands leads to different installations:

- Quick, non-destructive measurements leading to a single number or simple plots
- Complex instrumentation used for hard malfunction and accelerator development
- Automated measurement and control of beam parameters i.e. feedback

A clear interpretation of the results is a important design criterion.

General comments:

- Quite different technologies are used, based on various physics processes
- Accelerator development goes parallel to diagnostics development
Conclusion for Beam Diagnostics Course

Thank you for your attention!

For a successful construction and operation of an accelerator, the understand and right balance of all disciplines is required!
Backup slides
Results:

- Several orders of magnitude different light output
- Material matched to beam intensity must be chosen
- Well suited: powder phosphor screens P43 and P46
- Cheap, can be sedimented on large substrates of nearly any shape
- Light output linear with respect to particles per pulse

Example: Beam images for various scintillators irradiated by Uranium at \( \approx 300 \text{ MeV/u} \) at GSI

![Graph showing light output from various screens](image_url)

Courtesy P. Forck et al., IPAC’14, A. Lieberwirth et al., NIM B 2015
Broadening due to the Beam’s Space Charge: Ion Detection

Influence of the residual gas ion trajectory by:
- External electric field $E_{ex}$
- Electric field of the beam’s space charge $E_{space}$

E.g. Gaussian density distribution for round beam: 

$$E_{space}(r) = \frac{1}{\pi \sigma_{true}} \cdot \frac{\sigma_{true}}{r} \cdot \frac{1}{l} \cdot \left[ 1 - \exp \left( -\frac{r^2}{2\sigma_{true}^2} \right) \right]$$

Estimation of correction: 

$$\sigma_{corr}^2 \approx \frac{e^2 \ln 2}{4\pi \varepsilon_0 m c^2} \cdot \frac{qN}{l} \cdot d_{gap} \cdot \sqrt{\frac{1}{eU_{ex}}} \cdot 1 \cdot \frac{1}{\sqrt{U_{ex}}} \cdot N \cdot d_{gap}$$

With the measured beam width is given by convolution: 

$$\sigma_{meas}^2 = \sigma_{true}^2 + \sigma_{corr}^2$$

Example: $U^{73+}$, $10^9$ particles per 3 m bunch length, cooled beam with $\sigma_{true} = 1$ mm FWHM.
Electron Detection and Guidance by Magnetic Field

Alternative: $e^-$ detection in an external magnetic field

$\rightarrow$ cyclotron radius $r_c = \sqrt{\frac{2m_e E_{\text{kin},\perp}}{eB}} \Rightarrow r_c < 0.1 \text{ mm for } B = 0.1 \text{ T}$

$E_{\text{kin}}$, given by atomic physics, 0.1 mm is internal resolution of MCP.

Time-of-flight: $\approx 1 \text{ ns} \rightarrow 2$ or $3$ cycles.

B-field: By dipole magnets with large aperture $\rightarrow$ IPM is expensive device.
**Magnetic field for electron guidance:**

Maximum image distortion:

5% of beam width \( \Rightarrow \frac{\Delta B}{B} < 1\% \)

**Challenges:**

- High \( B \)-field homogeneity of 1%
- Clearance up to 500 mm
- Correctors required to compensate beam steering
- Insertion length 2.5 m incl. correctors

For MCP wire-array readout lower clearance required
Emittance Enlargement by Injection Mis-steering

Emittance conservation requires precise injection matching

Wrong angle of injected beam:

- injection into outer phase space → large $\beta$-amplitude i.e. large beam
- might result in ‘hollow’ beam
- filling of acceptance i.e. loss of particles

⇒ Hadron beams: larger emittance after acceleration

Example: Variation of vertical injection angle by magnetic steerer
Beam: C$^{6+}$ at 6.7 MeV/u acc. to 600 MeV/u, up to $6 \times 10^9$ ions per fill with multi-turn injection, IPM integration 0.5 ms i.e. ≈ 100 turns

Vertical profile at injection:

Vertical profile after acc.:

Horizontal profile at injection:

Horizontal profile after acc.:

Emittance enlargement by injection missteering

Horizontal profile:

Vertical profile:

Schematic simulation:

Courtesy M. Syphers

Peter Forck, CAS 2018, Constanta
Beam Induced Fluorescence for intense Profiles

Large beam power → Non-intercepting method: 
⇒ Beam Induced Fluorescence BIF

\[ \text{N}_2 + \text{Ion} \rightarrow (\text{N}_2^+) + \text{Ion} \rightarrow \text{N}_2^+ + \gamma + \text{Ion} \]

With single photon detection scheme

\[ 390 \text{ nm} < \lambda < 470 \text{ nm} \]

⇒ non-destructive, compact installation.

Installation of hor&vert. BIF Monitor:

- Horizontal BIF
- Vertical BIF
- Camera
- Image intensifier
- Gas inlet
- Lens
- Vacuum gauge
- Beam
- Vacuum window
- Gas inlet
- Pulse generator
- Image intensifier & CCD
- Vacuum chamber
- N₂ atmosphere
- 150 mm flange
**Beam Induced Fluorescence Monitor BIF: Image Intensifier**

**Scheme of Image intensifier:**
- **Photocathode**
- **double MCP**
- **Phosphor**

**Image intensifier:**
- Photo cathode $\rightarrow$ creation of photo-$e^-$
- Accelerated toward MCP for amplification
- Detection of ampl. $e^-$ by phosphor screen
- Image recorded by CCD
- Low light amplification
  (commercially used for night vision devices)

**A BIF monitor consists of only:**
- optics outside beam pipe
- image intensifier + camera
- gas-inlet for pressure increase
  $\Rightarrow$ nearly no installation inside vacuum.
  - only LEDs for calibration
  $\Rightarrow$ cheaper than IPM, but lower signal.
Beam Induced Fluorescence Monitor BIF: Image Intensifier

‘Single photon counting’:

Example at GSI-LINAC:
4.7 MeV/u Ar\(^{10+}\) beam
I=2.5 mA equals to \(10^{11}\) particle
One single macro pulse of 200 \(\mu\)s
Vacuum pressure: \(p=10^{-5}\) mbar (N\(_2\))

A BIF monitor consists of only:
- optics outside beam pipe
- image intensifier + camera
- gas-inlet for pressure increase
\(\Rightarrow\) nearly no installation inside vacuum.
- only LEDs for calibration
\(\Rightarrow\) cheaper than IPM, but lower signal.
Comparison between IPM and BIF

Non-destructive methods preferred:

Beam is not influenced and diagnostics device is not destroyed!

IPM: Beam ionizes the residual gas
→ measurement of all ionization products, $\Omega = 4\pi$-geometry due to E-field

BIF: Beam ionizes and excites the residual gas
→ measurement of photons emitted toward camera, solid angle $\Omega \approx 10^{-4}$

IPM: Higher efficiency than BIF

BIF: Low detection efficiency, only $\approx 10^{-4}$ of IPM
⇒ longer observation time or higher pressure required

IPM: Complex installation inside vacuum

BIF: Nearly no installation inside vacuum

IPM: More expensive, for some beam parameters even guiding magnetic field required

BIF: More sensitive to external parameters like radiation stray light
Optical Transition Radiation with 45° incidence

**OTR with 45° beam incidence and observation at 90°:**
approximated formula for 45° incidence & in plane polarization:

\[
\frac{d^2W}{d\theta d\omega} \approx \frac{2e^2 \beta^2}{\pi c} \cdot \left( \frac{\sin \theta}{1 - \beta \cos \theta} + \frac{\cos \theta}{1 - \beta \sin \theta} \right)^2
\]

Angular distribution of radiation in optical spectrum:
- emission pattern depends on velocity
- peak at angle \( \theta \approx 1/\gamma \)
- emitted energy scales with \( W \propto \beta^2 \)
- symmetric with respect to \( \theta \) for \( \gamma > 100 \)

Remark: polarization of emitted light:
- in scattering plane \( \rightarrow \) parallel E-vector
- perpendicular plane \( \rightarrow \) rectangular E-vector
Definition of transverse Emittance

The emittance characterizes the whole beam quality:

\[ \varepsilon_x = \frac{1}{\pi} \int_A dxdx' \]

**Ansatz:**

**Beam matrix** at one location:

\[ \mathbf{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} = \varepsilon \cdot \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix} \]

It describes a 2-dim probability distr.

The value of emittance is:

\[ \varepsilon_x = \sqrt{\det \mathbf{\sigma}} = \sqrt{\sigma_{11}\sigma_{22} - \sigma_{12}^2} \]

For the profile and angular measurement:

\[ x_\sigma = \sqrt{\sigma_{11}} = \sqrt{\varepsilon \beta} \quad \text{and} \quad x'_\sigma = \sqrt{\sigma_{22}} = \sqrt{\varepsilon \gamma} \]

Geometrical interpretation:

All points \( \mathbf{x} \) fulfilling \( \mathbf{x}^t \cdot \mathbf{\sigma}^{-1} \cdot \mathbf{x} = 1 \) are located on an **ellipse**

\[ \sigma_{22}x^2 - 2\sigma_{12}xx' + \sigma_{11}x'^2 = \det \mathbf{\sigma} = \varepsilon_x^2 \]