**Introduction to Transverse Beam Dynamics**

Bernhard Holzer,  
CERN-LHC

**The Ideal World**

I.) Magnetic Fields and Particle Trajectories

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**Luminosity Run of a typical storage ring:**

LHC Storage Ring: Protons accelerated and stored for 12 hours  
distance of particles travelling at about \( v \approx c \)  
\( L = 10^{10} - 10^{11} \) km

... several times Sun - Pluto and back

<table>
<thead>
<tr>
<th>Energy (GeV)</th>
<th>0</th>
<th>500</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
<th>2500</th>
<th>3000</th>
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<td>intensity ((10^{15}))</td>
<td>1</td>
<td></td>
<td></td>
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→ guide the particles on a well defined orbit ("design orbit")  
→ focus the particles to keep each single particle trajectory  
  within the vacuum chamber of the storage ring, i.e. close to the design orbit.
**Transverse Beam Dynamics:**

0.) Introduction and Basic Ideas

"... in the end and after all it should be a kind of circular machine"  
⇒ need transverse deflecting force

Lorentz force  
\[ F = q \left( \dot{x} + \mathbf{v} \times \mathbf{B} \right) \]

typical velocity in high energy machines:  
\[ v \approx c \approx 3 \times 10^8 \frac{m}{s} \]

old greek dictum of wisdom:  
if you are clever, you use magnetic fields in an accelerator wherever it is possible.

But remember: magn. fields act always perpendicular to the velocity of the particle  
⇒ only bending forces,  
⇒ no „beam acceleration”

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The ideal circular orbit

condition for circular orbit:

Lorentz force  
\[ F_L = e \mathbf{v} \mathbf{B} \]

centrifugal force  
\[ F_{cent} = \frac{\gamma m_0 v^2}{\rho} \]

\[ \frac{\gamma m_0 v^2}{\rho} = e \mathbf{v} \mathbf{B} \]

\[ \frac{p}{e} = B \rho \]

\[ B \rho = "beam rigidity" \]
1. The Magnetic Guide Field

Dipole Magnets:
define the ideal orbit
homogeneous field created
by two flat pole shoes

\[ B = \frac{\mu_0 n I}{h} \]

convenient units:

\[ B = [T] = \frac{V_s}{m^2} \quad p = \frac{[GeV]}{c} \]

Example LHC:

\[ B = 8.3T \]
\[ p = 7000 \frac{GeV}{c} \]

Normalise magnetic field to momentum:

\[ \frac{p}{e} = B \rho \quad \rightarrow \quad \frac{1}{\rho} = \frac{eB}{p} \]

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**The Magnetic Guide Field**

\[ \frac{1}{\rho} = 0.3 \frac{8.3 V_s}{7000 * 10^6 eV / c} = 8.3 \times 10^4 m / \]

\[ \frac{1}{\rho} = 0.3 \frac{8.3}{7000 / m} \]

\[ \rho = 2.53 \text{ km} \quad \rightarrow \quad 2\pi \rho = 17.6 \text{ km} \]

\[ \approx 66\% \]

\[ B = \cdots 8 \text{ T} \]

\[ \text{"normalised bending strength"} \]

rule of thumb:

\[ \frac{1}{\rho} = 0.3 \frac{B}{p} \frac{T}{[GeV / c]} \]
2.) **Quadrupole Magnets:**

required: focusing forces to keep trajectories in vicinity of the ideal orbit
linear increasing Lorentz force
linear increasing magnetic field \( B_y = g \cdot x \quad B_x = g \cdot y \)

normalised quadrupole field:

gradient of a quadrupole magnet: \( g = \frac{2 \mu_0 n I}{r^3} \)

\[ k = \frac{g}{p/e} \]

simple rule:

\[ k = 0.3 \frac{g(T/m)}{p(GeV/c)} \]

LHC main quadrupole magnet \( g = 25 \ldots 220 \ T/m \)

what about the vertical plane: Maxwell

\[ \nabla \times B = \epsilon_0 \frac{\partial E}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \]

3.) **The equation of motion:**

Linear approximation:

* ideal particle \( \rightarrow \) design orbit

* any other particle \( \rightarrow \) coordinates \( x, y \) small quantities \( x, y \ll \rho \)

\( \rightarrow \) magnetic guide field: only linear terms in \( x \) & \( y \) of \( B \)

have to be taken into account

Taylor Expansion of the \( B \) field:

\[ B_y (x) = B_y^\circ + \frac{dB_y}{dx} \cdot x + \frac{1}{2!} \frac{d^2 B_y}{dx^2} \cdot x^2 + \frac{1}{3!} \frac{d^3 B_y}{dx^3} \cdot x^3 + \ldots \]

normalise to momentum \( p/e = B_0 \rho \)

\[ \frac{B(x)}{p/e} = \frac{B_0}{B_0 \rho} \cdot \frac{g \cdot x}{p/e} + \frac{1}{2!} \frac{eg'}{p/e} + \frac{1}{3!} \frac{eg''}{p/e} + \ldots \]

\[ \frac{B(x)}{p/e} = \frac{B_0}{B_0 \rho} \cdot \frac{g \cdot x}{p/e} + \frac{1}{2!} \frac{eg'}{p/e} + \frac{1}{3!} \frac{eg''}{p/e} + \ldots \]
The Equation of Motion:

\[
\frac{B(x)}{\rho / e} = \frac{1}{\rho} + k \, x + \frac{1}{2!} \, x^2 + \frac{1}{3!} \, x^3 + \ldots
\]

only terms linear in \(x, y\) taken into account  dipole fields quadrupole fields

Separate Function Machines:

Split the magnets and optimise them according to their job:
bending, focusing etc

Example:
heavy ion storage ring TSR

\[\text{...}
\]

Equation of Motion:

Consider local segment of a particle trajectory ...
and remember the old days:
(Goldstein page 27)

radial acceleration:

\[a_r = \frac{d^2 \rho}{dt^2} - \rho \left( \frac{d \theta}{dt} \right)^2\]

Ideal orbit: \(\rho = \text{const}, \frac{d \rho}{dt} = 0\)

Force:
\[F = m \rho \left( \frac{d \theta}{dt} \right)^2 = m \rho \omega^2\]
\[F = m \frac{v^2}{\rho}\]

general trajectory: \(\rho \rightarrow \rho + x\)

\[F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = e B_y v\]
\[ F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^2}{x + \rho} = eB_y v \]

1. \[ \frac{d^2}{dt^2} (x + \rho) = \frac{d^2}{dt^2} x \quad \text{... as } \rho = \text{const} \]

2. remember: \( x \approx mm, \rho \approx m \quad \rightarrow \text{ develop for small } x \)

\[ \frac{1}{x + \rho} = \frac{1}{\rho} \left( 1 - \frac{x}{\rho} \right) \]

Taylor Expansion

\[ f(x) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots \]

\[ m \frac{d^2x}{dt^2} - \frac{mv^2}{\rho} (1 - \frac{x}{\rho}) = eB_y v \]

---

guide field in linear approx.

\[ B_y = B_0 + x \frac{\partial B_y}{\partial x} \]

\[ m \frac{d^2x}{dt^2} - \frac{mv^2}{\rho} (1 - \frac{x}{\rho}) = ev \left( B_0 + x \frac{\partial B_y}{\partial x} \right) \quad m^2 \]

\[ \frac{d^2x}{dt^2} \left( \frac{v^2}{\rho} (1 - \frac{x}{\rho}) \right) = e \frac{v B_0}{m} + e v x \frac{g}{m} \]

independent variable: \( t \rightarrow s \)

\[ \frac{dx}{dt} = \frac{dx}{ds} \frac{ds}{dt} \]

\[ \frac{d^2x}{dt^2} = \frac{d}{ds} \left( \frac{dx}{ds} \right) \frac{ds}{dt} \]

\[ \frac{d^2x}{dt^2} = x^\prime \left( \frac{v^2}{\rho} (1 - \frac{x}{\rho}) \right) = \frac{e v B_0}{m} + \frac{e v x g}{m} \quad v^2 \]
\[ x' - \frac{1}{\rho} (1 - \frac{x}{\rho}) = \frac{e B_0}{m v} + \frac{e x g}{m v} \]

\[ x'' - \frac{1}{\rho^2} + \frac{x}{\rho^2} = \frac{B_0}{p/e} + \frac{x g}{p/e} \]

\[ x'' - \frac{1}{\rho^2} + \frac{x}{\rho^2} = -\frac{1}{\rho^2} + k x \]

\[ x'' + x \left( \frac{1}{\rho^2} - k \right) = 0 \]

Equation for the vertical motion:

\[ \frac{1}{\rho^2} = 0 \] no dipoles ... in general ...

\[ k \leftrightarrow -k \] quadrupole field changes sign

\[ y'' + k y = 0 \]

\[ m v = p \]

normalize to momentum of particle

\[ \frac{B_0}{p/e} = \frac{1}{\rho} \]

\[ \frac{g}{p/e} = k \]

Remarks:

\[ x' + \left( \frac{1}{\rho^2} - k \right) \cdot x = 0 \]

... there seems to be a focusing even without a quadrupole gradient

"weak focusing of dipole magnets"

\[ k = 0 \] even without quadrupoles there is a retriving force (i.e. focusing) in the bending plane of the dipole magnets

... in large machines it is weak. (!)

Mass spectrometer: particles are separated according to their energy and focused due to the \(1/\rho\) effect of the dipole
Hard Edge Model:

\[ x'' + \left( \frac{1}{\rho^2} - k \right) x = 0 \]

... this equation is not correct !!!

\[ x''(s) + \left( \frac{1}{\rho^2(s)} - k(s) \right) x(s) = 0 \]

bending and focusing fields ... are functions of the independent variable \( s \)

Inside a magnet we assume constant focusing properties!

\[ \frac{1}{\rho} = \text{const} \quad k = \text{const} \]

4.) Solution of Trajectory Equations

Define ... hor. plane: \( K = 1/\rho^2 - k \)

... vert. Plane: \( K = k \)

Differential Equation of harmonic oscillator ... with spring constant \( K \)

\[ x'' + K x = 0 \]

Ansatz:

\[ x(s) = a_1 \cdot \cos(\omega s) + a_2 \cdot \sin(\omega s) \]

general solution: linear combination of two independent solutions

\[ x'(s) = -a_1 \omega \sin(\omega s) + a_2 \omega \cos(\omega s) \]

\[ x''(s) = -a_1 \omega^2 \cos(\omega s) - a_2 \omega^2 \sin(\omega s) = -\omega^2 x(s) \]

\[ \omega = \sqrt{K} \]

general solution:

\[ x(s) = a_1 \cos(\sqrt{K} s) + a_2 \sin(\sqrt{K} s) \]
Hor. Focusing Quadrupole $K > 0$:

For convenience expressed in matrix formalism:

\[
\begin{align*}
\begin{bmatrix} x' \end{bmatrix}_{s=0} &= s_0, \\
\begin{bmatrix} x' \end{bmatrix}_{s=s_1} &= s_1
\end{align*}
\]

\[
x(s) = x_0 \cdot \cos(\sqrt{|K|} s) + x_0' \cdot \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|} s)
\]

\[
x'(s) = -x_0 \cdot \sqrt{|K|} \cdot \sin(\sqrt{|K|} s) + x_0' \cdot \cos(\sqrt{|K|} s)
\]

For convenience expressed in matrix formalism:

\[
M_{fr} = \begin{pmatrix} \cos(\sqrt{|K|} s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|} s) \\
-\sqrt{|K|} \sin(\sqrt{|K|} s) & \cos(\sqrt{|K|} s) \end{pmatrix}
\]

hor. defocusing quadrupole:

\[
x' - K x = 0
\]

Remember from school:

\[
f(x) = \cosh(x), \quad f'(x) = \sinh(x)
\]

Ansatz:

\[
x(s) = a_1 \cdot \cosh(\omega s) + a_2 \cdot \sinh(\omega s)
\]

\[
M_{def} = \begin{pmatrix} \cosh(\sqrt{|K|} s) & \frac{1}{\sqrt{|K|}} \sinh(\sqrt{|K|} s) \\
\sqrt{|K|} \sinh(\sqrt{|K|} s) & \cosh(\sqrt{|K|} s) \end{pmatrix}
\]

drift space:

\[
K = 0 \quad \Rightarrow \quad M_{def} = \begin{pmatrix} 1 & 0 \\
0 & 1 \end{pmatrix}
\]

\[
! \text{ with the assumptions made, the motion in the horizontal and vertical planes are independent \,. \, \text{... the particle motion in } x \& y \text{ is uncoupled}!}
\]
**Thin Lens Approximation:**

Matrix of a quadrupole lens

\[
M = \begin{pmatrix}
\cos \sqrt{|k|} l & \frac{1}{\sqrt{|k|}} \sin \sqrt{|k|} l \\
-\frac{1}{\sqrt{|k|}} \sin \sqrt{|k|} l & \cos \sqrt{|k|} l
\end{pmatrix}
\]

In many practical cases we have the situation:

\[f = \frac{1}{k l_q} \gg l_q\] ... focal length of the lens is much bigger than the length of the magnet

Limes: \(l_q \to 0\) while keeping \(k l_q = \text{const}\)

\[
M_x = \begin{pmatrix}
1 & 0 \\
\frac{1}{f} & 1
\end{pmatrix} \quad M_z = \begin{pmatrix}
1 & 0 \\
-\frac{1}{f} & 1
\end{pmatrix}
\]

... useful for fast (and in large machines still quite accurate) "back on the envelope calculations" ... and for the guided studies!

**Transformation through a system of lattice elements**

Combine the single element solutions by multiplication of the matrices

\[
M_{\text{total}} = M_{\text{QP}} \cdot M_{\text{D}} \cdot M_{\text{OD}} \cdot M_{\text{geo}} \cdot M_{\text{DP}}
\]

\[
\begin{pmatrix}
x' \\
x''
\end{pmatrix}_{s_2} = M(s_2, s_1) \cdot \begin{pmatrix}
x \\
x'
\end{pmatrix}_{s_1}
\]

Typical values in a strong foc. machine: \(x \approx \text{mm}, x' \leq \text{mrad}\)
5.) Orbit & Tune:

Tune: number of oscillations per turn

64.31
59.32

Relevant for beam stability:
non integer part

LHC revolution frequency: 11.3 kHz

\[ 0.31 \times 11.3 \text{ kHz} = 3.5 \text{ kHz} \]

Question: what will happen, if the particle performs a second turn?

... or a third one or ... 10^9 turns
Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill’s equation“

Example: particle motion with periodic coefficient

equation of motion: \( x''(s) - k(s)x(s) = 0 \)

restoring force \( \neq \text{const}, \)

\( k(s) = \text{depending on the position } s \)

\( k(s+L) = k(s), \text{ periodic function} \)

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position \( s \) in the ring.

6.) The Beta Function

General solution of Hill’s equation:

\[ (i) \quad x(s) = e^{\sqrt{\epsilon} \int \beta(s) \cdot \cos(\psi(s) + \phi)} \]

\( \epsilon, \phi = \text{integration constants determined by initial conditions} \)

\( \beta(s) = \text{periodic function given by focusing properties of the lattice ↔ quadrupoles} \)

\( \beta(s + L) = \beta(s) \)

Inserting (i) into the equation of motion …

\[ \psi(s) = \int_{0}^{s} \frac{ds}{\beta(s)} \]

\( \Psi(s) = \text{„phase advance“ of the oscillation between point „0“ and „s“ in the lattice.} \)

For one complete revolution: number of oscillations per turn „Tune“

\[ Q_J = \frac{1}{2\pi} \int \frac{ds}{\beta(s)} \]
7.) Beam Emittance and Phase Space Ellipse

general solution of Hill equation

\begin{align*}
(1) \quad x(s) &= \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \\
(2) \quad x'(s) &= -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \left\{ \alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \right\}
\end{align*}

from (1) we get

\[
\cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} \sqrt{\beta(s)}}
\]

\[
\alpha(s) = \frac{-1}{2} \beta'(s)
\]

\[
\gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)}
\]

Insert into (2) and solve for \( \varepsilon \)

\[
\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)
\]

* \( \varepsilon \) is a constant of the motion ... it is independent of "s"*

* parametric representation of an ellipse in the \( x, x' \) space

* shape and orientation of ellipse are given by \( \alpha, \beta, \gamma \)

---

Beam Emittance and Phase Space Ellipse

\[
\varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s)
\]

**Liouville:** in reasonable storage rings, area in phase space is constant.

\[
A = \pi \varepsilon = \text{const}
\]

\( \varepsilon \): beam emittance = wiggilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.

Scientifically speaking: area covered in transverse \( x, x' \) phase space ... and it is constant !!!!
**Phase Space Ellipse**

**Particle trajectory:** \( x(s) = \sqrt{s} \sqrt{\beta(s)} \cos \left\{ \varphi(s) + \phi \right\} \)

**Max. Amplitude:** \( \ddot{x}(s) = \sqrt{\epsilon} \beta \rightarrow x' \) at that position ...?

... put \( \ddot{x}(s) \) into \( \epsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s) \) and solve for \( x' \)

\( \epsilon = \gamma \cdot \epsilon \beta + 2\alpha \sqrt{\epsilon \beta} \cdot x' + \beta \epsilon x'^2 \)

\( x' = -\alpha \sqrt{\epsilon / \beta} \)

\*: A high \( \beta \)-function means a large beam size and a small beam divergence. ... et vice versa !!!

\*: In the middle of a quadrupole \( \beta = \) maximum, \( \alpha = \) zero \( \} x' = 0 \) ... and the ellipse is flat

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**Phase Space Ellipse**

\( \epsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s) x'^2(s) \)

\( \epsilon = \frac{x^2}{\beta} + \frac{\alpha^2 x^2}{\beta} + 2\alpha \cdot x x' + \beta \cdot x'^2 \)

... solve for \( x' \)

\( x'_{1,2} = \frac{-\alpha \cdot x \pm \sqrt{\epsilon \beta - x^2}}{\beta} \)

... and determine \( \ddot{x} \) via: \( \frac{dx'}{dx} = 0 \)

\( \ddot{x}' = \sqrt{\epsilon \gamma} \)

\( \ddot{x} = \alpha \sqrt{\frac{\gamma}{\epsilon}} \)

Shape and orientation of the phase space ellipse depend on the Twiss parameters \( \beta \alpha \gamma \)
**Emittance of the Particle Ensemble:**

\[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\Psi(s) + \phi) \]

**Gauss Particle Distribution:**

\[ \rho(x) = \frac{N}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2} \frac{x^2}{\sigma^2}} \]

Particle at distance 1 \( \sigma \) from centre ↔ 68.3% of all beam particles

Aperture requirements: \( r_0 = 10^* \sigma \)

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**8.) Transfer Matrix \( M \) ... yes we had the topic already**

*general solution of Hill’s equation*

\[
\begin{align*}
\Psi(s) &= x(s) - \sqrt{\varepsilon} \sqrt{\beta(s)} \left( \cos\psi \cos\phi - \sin\psi \sin\phi \right) \\
\Phi(s) &= x(s) - \sqrt{\varepsilon} \sqrt{\beta(s)} \left( \alpha \sin\psi \cos\phi - \alpha \sin\psi \sin\phi + \sin\psi \cos\phi + \cos\psi \sin\phi \right)
\end{align*}
\]

Remember the trigonometrical gymnastics: \( \sin(a + b) = \ldots \) etc

\[
\begin{align*}
x(s) &= \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \\
x'(s) &= \sqrt{\varepsilon} \sqrt{\beta(s)} \left[ \alpha \cos\psi + \cos\phi - \alpha \sin\psi \sin\phi + \sin\psi \cos\phi + \cos\psi \sin\phi \right]
\end{align*}
\]

Starting at point \( s(0) = s_0 \), where we put \( \Psi(0) = 0 \)

\[
\begin{align*}
\cos\phi &= \frac{x_0}{\sqrt{\varepsilon} \beta_0} \\
\sin\phi &= -\frac{1}{\sqrt{\varepsilon}} \left( x_0 \sqrt{\beta_0} + \frac{\alpha \beta_0}{\sqrt{\beta_0}} \right)
\end{align*}
\]

Inserting above …
which can be expressed ... for convenience ... in matrix form

\[
\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} M \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0
\]

\[
M = \begin{pmatrix}
\sqrt{\frac{\beta_x}{\beta_0}} \left( \cos \psi_x + \alpha_0 \sin \psi_x \right) & \sqrt{\beta_x} \beta_0 \sin \psi_x \\
(\alpha_0 - \alpha_x) \cos \psi_x - (1 + \alpha_0 \alpha_x) \sin \psi_x & \sqrt{\beta_x} \left( \cos \psi_x - \alpha_x \sin \psi_x \right)
\end{pmatrix}
\]

* we can calculate the single particle trajectories between two locations in the ring, if we know the \( \alpha \beta \gamma \) at these positions.
* and nothing but the \( \alpha \beta \gamma \) at these positions.

\[ * \quad \cdots \quad ! \]

... Äquivalenz der Matrizen

9.) Periodic Lattices

\[
M = \begin{pmatrix}
\sqrt{\frac{\beta_x}{\beta_0}} \left( \cos \psi_x + \alpha_0 \sin \psi_x \right) & \sqrt{\beta_x} \beta_0 \sin \psi_x \\
(\alpha_0 - \alpha_x) \cos \psi_x - (1 + \alpha_0 \alpha_x) \sin \psi_x & \sqrt{\beta_x} \left( \cos \psi_x - \alpha_x \sin \psi_x \right)
\end{pmatrix}
\]

„This rather formidable looking matrix simplifies considerably if we consider one complete revolution …“

\[
M(s) = \begin{pmatrix}
\cos \psi_{\text{norm}} + \alpha_0 \sin \psi_{\text{norm}} & \beta_x \sin \psi_{\text{norm}} \\
-\gamma_0 \sin \psi_{\text{norm}} & \cos \psi_{\text{norm}} - \alpha_0 \sin \psi_{\text{norm}}
\end{pmatrix}
\]

\[
\psi_{\text{norm}} = \int_{s_0}^{s} \frac{dx}{\beta(s)} \quad \psi_{\text{norm}} \text{ = phase advance per period}
\]

\[
Q = \frac{1}{2\pi} \int \frac{ds}{\beta(s)}
\]

Tune: Phase advance per turn in units of \( 2\pi \)
Stability Criterion:

Question: what will happen, if we do not make too many mistakes and your particle performs one complete turn?

Matrix for 1 turn:

\[
M = \begin{pmatrix}
\cos \psi_{\text{mm}} + \alpha_1 \sin \psi_{\text{mm}} & \beta_1 \sin \psi_{\text{mm}} \\
-\gamma_1 \sin \psi_{\text{mm}} & \cos \psi_{\text{mm}} - \alpha_1 \sin \psi_{\text{mm}}
\end{pmatrix} = \cos \psi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin \psi \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}
\]

Matrix for N turns:

\[M^N = \left( (1 \cdot \cos \psi + J \cdot \sin \psi) \right)^N = 1 \cdot \cos N \psi + J \cdot \sin N \psi\]

The motion for N turns remains bounded, if the elements of \( M^N \) remain bounded:

\[\psi = \text{real} \iff \cos \psi \leq 1 \iff \text{Tr}(M) \leq 2\]

stability criterion ... proof for the disbelieving colleagues !!

Matrix for 1 turn:

\[
M = \begin{pmatrix}
\cos \psi_{\text{mm}} + \alpha_1 \sin \psi_{\text{mm}} & \beta_1 \sin \psi_{\text{mm}} \\
-\gamma_1 \sin \psi_{\text{mm}} & \cos \psi_{\text{mm}} - \alpha_1 \sin \psi_{\text{mm}}
\end{pmatrix} = \cos \psi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin \psi \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix}
\]

Matrix for 2 turns:

\[M^2 = \left( (I \cos \psi_1 + J \sin \psi_1) (I \cos \psi_2 + J \sin \psi_2) \right) \]

\[= I^2 \cos \psi_1 \cos \psi_2 + IJ \cos \psi_1 \sin \psi_2 + JI \sin \psi_1 \cos \psi_2 + J^2 \sin \psi_1 \sin \psi_2 \]

\[\text{now} \ldots\]

\[I^2 = I\]

\[IJ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \]

\[JI = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \]

\[J^2 = \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ -\gamma & -\alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 + \gamma \beta & \alpha \beta - \gamma \alpha \\ -\gamma \alpha + \gamma \beta & \gamma^2 + \alpha \beta \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I \]

\[M^2 = I \cos(\psi_1 + \psi_2) + J \sin(\psi_1 + \psi_2)\]

\[M^2 = I \cos(2\psi) + J \sin(2\psi)\]
10.) Transformation of $\alpha$, $\beta$, $\gamma$

consider two positions in the storage ring: $s_0$, $s$

\[
\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0}
\]

\[
M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}
\]

since $\varepsilon = \text{const (Liouville)}$:

\[
\varepsilon = \beta_0 x'^2 + 2\alpha_0 xx' + \gamma_0 x^2
\]

\[
\varepsilon = \beta_0^2 x'^2 + 2\alpha_0 x_0 x'_0 + \gamma_0 x_0^2
\]

\[
\begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} = M^{-1} \begin{pmatrix} x \\ x' \end{pmatrix}_s
\]

\[
M^{-1} = \begin{pmatrix} m_{11} & -m_{12} \\ -m_{21} & m_{22} \end{pmatrix}
\]

\[
\varepsilon = \beta_0 (m_{11} x' - m_{22} x)^2 + 2\alpha_0 (m_{22} x - m_{12} x')(m_{11} x' - m_{22} x) + \gamma_0 (m_{22} x - m_{12} x')^2
\]

sort via $x$, $x'$ and compare the coefficients to get ...

---

The Twiss parameters $\alpha$, $\beta$, $\gamma$ can be transformed through the lattice via the matrix elements defined above.

\[
\beta(s) = m_{11}^2 \beta_0 - 2m_{11}m_{12} \alpha_0 + m_{12}^2 \gamma_0
\]

\[
\alpha(s) = -m_{11}m_{22} \beta_0 + (m_{12}m_{21} + m_{11}m_{22}) \alpha_0 - m_{12}m_{22} \gamma_0
\]

\[
\gamma(s) = m_{21} \beta_0 - 2m_{22} \alpha_0 + m_{22} \gamma_0
\]

in matrix notation:

\[
\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_{s_0} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{22} & (m_{12}m_{21} + m_{11}m_{22}) & -m_{12}m_{22} \\ m_{21} & -2m_{22} & m_{22} \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_s
\]

1.) this expression is important

2.) given the twiss parameters $\alpha$, $\beta$, $\gamma$ at any point in the lattice we can transform them and calculate their values at any other point in the ring.

3.) the transfer matrix is given by the focusing properties of the lattice elements, the elements of $M$ are just those that we used to calculate single particle trajectories.

4.) go back to point 1.)
II.) Acceleration and Momentum Spread

The "not so ideal world"

Remember:

Beam Emittance and Phase Space Ellipse:

- equation of motion: \( x''(s) - k(s) x(s) = 0 \)
- general solution of Hills equation: \( x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \varphi) \)
- beam size: \( \sigma = \sqrt{\varepsilon \beta} \text{ mm} \)

\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \]

* \( \varepsilon \) is a constant of the motion … it is independent of \( x \)\n* parametric representation of an ellipse in the \( x x' \) space \n* shape and orientation of ellipse are given by \( \alpha, \beta, \gamma \)
11.) Liouville during Acceleration

\[ \varepsilon = \gamma(s) x'(s) + 2 \alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \]

Beam Emittance corresponds to the area covered in the \( x, x' \) Phase Space Ellipse

Liouville: Area in phase space is constant.

**But so sorry ... \( \varepsilon \neq \text{const} \) !**

Classical Mechanics:

phase space = diagram of the two canonical variables
position & momentum  \( x, p_x \)

\[ p_j = \frac{\alpha L}{\partial q_j}; \quad L = T - V = \text{kin. Energy} - \text{pot. Energy} \]

According to Hamiltonian mechanics:

phase space diagram relates the variables \( q \) and \( p \)

\[ q = \text{position} = x \]
\[ p = \text{momentum} = \gamma mv = mc\beta_x \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}; \quad \beta_x = \frac{v}{c} \]

Liouville's Theorem: \( \int p \, dq = \text{const} \)

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

\[ x' = \frac{dx}{ds} = \frac{dx}{dt} \frac{dt}{ds} = \frac{\beta_x}{\beta} \]

where \( \beta_x = v_x / c \)

\[ \int p \, dq = mc \gamma \beta_x \int dx \]

\[ \int p \, dq = mc \gamma \beta \int x' \, dx \]

\[ \Rightarrow \varepsilon = \int x' \, dx \approx \frac{1}{\beta \gamma} \]

the beam emittance shrinks during acceleration \( \varepsilon \sim 1 / \gamma \)
Nota bene:

1.) A proton machine ... or an electron linac ... needs the highest aperture at injection energy !!!
   as soon as we start to accelerate the beam size shrinks as $\gamma^{-1/2}$ in both planes.

   $$\sigma = \sqrt{\epsilon \beta}$$

2.) At lowest energy the machine will have the major aperture problems,
   $\Rightarrow$ here we have to minimise $\beta$

3.) we need different beam optics adopted to the energy:
   A Mini Beta concept will only be adequate at flat top.

Example: HERA proton ring

- injection energy: 40 GeV $\gamma = 43$
- flat top energy: 920 GeV $\gamma = 980$

- emittance $\epsilon$ (40 GeV) $= 1.2 \times 10^{-7}$
- $\epsilon$ (920 GeV) $= 5.1 \times 10^{-9}$
12.) The "Δp/p ≠ 0" Problem

Linear Accelerator

Energy Gain per \(\text{Gap}\):
\[
W = q U \sin \omega \tau t
\]

*RF Acceleration: multiple application of the same acceleration voltage; brilliant idea to gain higher energies ... but changing acceleration voltage

Problem: panta rhei !!!

(\text{Heraclitus: 540-480 v. Chr.})

Example: HERA RF:

\[
\begin{align*}
\nu &= 500\, \text{MHz} \\
\lambda &= 60\, \text{cm}
\end{align*}
\]

\[
\sin(90^\circ) = 1 \\
\sin(84^\circ) = 0.994 \\
\frac{\Delta U}{U} = 6.0 \times 10^{-3}
\]

Bunch length of Electrons ≈ 1 cm

\[
\frac{\Delta \nu}{\nu} = 1.0 \times 10^{-2}
\]

typical momentum spread of an electron bunch:
13.) Dispersion: trajectories for $\Delta p / p \neq 0$

Force acting on the particle

$$F = m \frac{d^2}{dt^2} (x + \rho) - \frac{mv^3}{x + \rho} = e B_x y$$

remember: $x \approx mm$, $\rho \approx m$ … develop for small $x$

$$m \frac{d^3x}{dt^3} \frac{mv^3}{\rho} (1 - \frac{x}{\rho}) = eB_y y$$

consider only linear fields, and change independent variable: $t \rightarrow s$  \( B_y = B_y + x \frac{\partial B_y}{\partial x} \)

$x'' - \frac{1}{\rho} \left(1 - \frac{x}{\rho}\right) = \frac{eB_0}{mv} + \frac{eg}{mv}$

$p = p_0 + \Delta p$

… but now take a small momentum error into account !!!

Dispersion:

develop for small momentum error

$$\Delta p \ll p_0 \Rightarrow \frac{1}{p_0 + \Delta p} = \frac{1}{p_0} \frac{\Delta p}{p_0}$$

$$x'' + \frac{x}{\rho^2} = \frac{eB_0}{p_0} \frac{\Delta p}{p_0} + \frac{eg}{p_0} \left( \frac{\Delta p}{p_0} \right) - \frac{k}{p_0} x = 0$$

$$x'' + \frac{x}{\rho^2} - kx = \frac{\Delta p}{p_0} \frac{1}{\rho} \frac{\Delta p}{p_0} \frac{1}{\rho}$$

Momentum spread of the beam adds a term on the r.h.s. of the equation of motion.

⇒ inhomogeneous differential equation.
Dispersion:

\[ x^* + x \left( \frac{1}{\rho^2} - k \right) = \Delta \frac{p}{p} \cdot \frac{1}{\rho} \]

General solution:

\[ x(s) = x_h(s) + x_i(s) \]

\[ \begin{cases} 
  x_i'(s) + K(s) \cdot x_i(s) = 0 \\
  x_i'(s) + K(s) \cdot x_i(s) = \frac{1}{\rho} \cdot \Delta \frac{p}{p} 
\end{cases} \]

Normalise with respect to \( \Delta p/p \):

\[ D(s) = \frac{x_i(s)}{\Delta p/p} \]

Dispersion function \( D(s) \)

* is that special orbit, an ideal particle would have for \( \Delta p/p = 1 \)

* the orbit of any particle is the sum of the well known \( x_p \) and the dispersion

* as \( D(s) \) is just another orbit it will be subject to the focusing properties of the lattice

Dispersion

Example: homogeneous dipole field

Matrix formalism:

e.g. matrix for a quadrupole lens:

\[ M_{xy} = \begin{pmatrix} 
  \cos(\sqrt{|k|} s) & -\frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|} s) \\
  -\frac{1}{\sqrt{|k|}} \sin(\sqrt{|k|} s) & \cos(\sqrt{|k|} s) 
\end{pmatrix} \begin{pmatrix} C \\ S \end{pmatrix} \]

\[ x(s) = x_p(s) + D(s) \cdot \Delta \frac{p}{p} \]

\[ x(s) = C(s) \cdot x_0 + S(s) \cdot x'_0 + D(s) \cdot \Delta \frac{p}{p} \]

\[ \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} C & S \\ S' & C' \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} + \Delta \frac{p}{p} \begin{pmatrix} D' \end{pmatrix} \]
or expressed as 3x3 matrix

\[
\begin{pmatrix}
  x' \\
  y' \\
  \rho y/\rho_0
\end{pmatrix} =
\begin{pmatrix}
  C & S & D \\
  C' & S' & D' \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y' \\
  \rho y/\rho_0
\end{pmatrix}
\]

Example HERA

\[
x_0 = 1...2 \text{ mm} \\
D(s) = 1...2 \text{ m} \\
\Delta p/p = 1\cdot10^{-3}
\]

Amplitude of Orbit oscillation
contribution due to Dispersion = beam size
\(\Rightarrow\) Dispersion must vanish at the collision point

Calculate \(D, D'\)

\[
D(s) = S(s) \int_{s_0}^{s} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}
\]
(proof: see appendix)

Example: Drift

\[
M_{\text{Drift}} = \begin{pmatrix}
  1 & l \\
  0 & 1
\end{pmatrix}
\]

\[
D(s) = S(s) \int_{s_0}^{s} \frac{1}{\rho} C(\tilde{s}) d\tilde{s} - C(s) \int_{s_0}^{s} \frac{1}{\rho} S(\tilde{s}) d\tilde{s}
\]

\(= 0\)

Example: Dipole

\[
M_{\text{Dipole}} = \begin{pmatrix}
  \cos(\sqrt{|K|} s) & \frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|} s) \\
  -\frac{1}{\sqrt{|K|}} \sin(\sqrt{|K|} s) & \cos(\sqrt{|K|} s)
\end{pmatrix}
\]

\(K = \frac{1}{\rho} s = l_s\)

\[
D(s) = \rho \cdot \left(1 - \cos \frac{l}{\rho}\right)
\]

\[D'(s) = \sin \frac{l}{\rho}\]
Example: Dispersion, calculated by an optics code for a real machine

\[ x_p = D(s) \frac{\Delta p}{p} \]

* \( D(s) \) is created by the dipole magnets 

... and afterwards focused by the quadrupole fields

\[ D(s) \approx 1 \ldots 2 \text{ m} \]

Mini Beta Section,  
\[ \rightarrow \text{no dipoles} \text{!!!} \]

Dispersion is visible

\[ x_s = D(s) \frac{\Delta p}{p} \]

Attention: at the Interaction Points

we require \( D = D' = 0 \)
14.) Momentum Compaction Factor: $\alpha_p$

**Particle with a displacement $x$ to the design orbit $\rightarrow$ path length $dl$ ...**

\[
\frac{dl}{ds} = \frac{\rho + x}{\rho}
\]

\[\rightarrow dl = \left(1 + \frac{x}{\rho(s)}\right)ds\]

**Circumference of an off-energy closed orbit**

\[l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right)ds\]

remember:

\[x_{\Delta E}(s) = D(s) \cdot \frac{\Delta \rho}{\rho}\]

\[\delta l_{\Delta E} = \frac{\Delta \rho}{\rho} \oint \left(\frac{D(s)}{\rho(s)}\right)ds\]

*The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.*

**Definition:**

\[\frac{\delta l}{l} = \alpha_p \cdot \frac{\Delta \rho}{\rho}\]

\[\rightarrow \alpha_p = \frac{1}{l} \oint \left(\frac{D(s)}{\rho(s)}\right)ds\]

For first estimates assume:

\[\frac{1}{\rho} = \text{const.}\]

\[\oint D(s)ds = l_{dipole} \cdot \left\langle \frac{D}{dipole}\right\rangle\]

\[\alpha_p = \frac{1}{L} l_{dipole} \cdot \left\langle \frac{D}{dipole}\right\rangle = \frac{1}{L} \cdot \frac{1}{\rho} \cdot \left\langle \frac{D}{dipole}\right\rangle \rightarrow \alpha_p = \frac{2\pi}{L} \cdot \left\langle \frac{D}{R}\right\rangle\]

**Assume:**

\[v = c\]

\[\rightarrow \frac{\delta T}{T} = \frac{\delta l}{l} = \alpha_p \frac{\Delta \rho}{\rho}\]

$\alpha_p$ combines via the dispersion function the momentum spread with the longitudinal motion of the particle.
15.) Gradient Errors

Matrix in Twiss Form

Transfer Matrix from point "0" in the lattice to point "s":

\[
M(s) = \begin{pmatrix}
\sqrt{\beta_s} (\cos \psi_s + \alpha_s \sin \psi_s) & \sqrt{\beta_s} \beta_0 \sin \psi_s \\
(\alpha_s - \alpha_0) \cos \psi_s - (1 + \alpha_s \alpha_0) \sin \psi_s & \sqrt{\beta_s} (\cos \psi_s - \alpha_s \sin \psi_s)
\end{pmatrix}
\]

For one complete turn the Twiss parameters have to obey periodic boundary conditions:

\[
\begin{align*}
\beta(s + L) &= \beta(s) \\
\alpha(s + L) &= \alpha(s) \\
\gamma(s + L) &= \gamma(s)
\end{align*}
\]

\[
M(s) = \begin{pmatrix}
\cos \psi_{\text{term}} + \alpha \sin \psi_{\text{term}} & \beta \sin \psi_{\text{term}} \\
-\gamma \sin \psi_{\text{term}} & \cos \psi_{\text{term}} - \alpha \sin \psi_{\text{term}}
\end{pmatrix}
\]

Quadrupole Error in the Lattice

optic perturbation described by thin lens quadrupole

\[
M_{\text{quad}} = M_{\text{ideal}} \cdot M_s = \begin{pmatrix}
1 & 0 \\
\Delta k ds & 1
\end{pmatrix}
\begin{pmatrix}
\cos \psi_{\text{term}} + \alpha \sin \psi_{\text{term}} & \beta \sin \psi_{\text{term}} \\
-\gamma \sin \psi_{\text{term}} & \cos \psi_{\text{term}} - \alpha \sin \psi_{\text{term}}
\end{pmatrix}
\]

\[
M_{\text{quad}} = \begin{pmatrix}
\cos \psi_0 + \alpha \sin \psi_0 & \beta \sin \psi_0 \\
\Delta k ds (\cos \psi_s + \alpha \sin \psi_s) - \gamma \sin \psi_s & \Delta k ds \beta \sin \psi_s + \cos \psi_s - \alpha \sin \psi_0
\end{pmatrix}
\]

Rule for getting the tune

\[
\text{Trace}(M) = 2 \cos \psi_s - 2 \cos \psi_s + \Delta k ds \beta \sin \psi_0
\]
Quadrupole error $\rightarrow$ Tune Shift

$$\psi = \psi_0 + \Delta \psi$$

$$\cos(\psi_0 + \Delta \psi) = \cos \psi_0 + \frac{\Delta \psi}{2}$$

remember the old fashioned trigonometric stuff and assume that the error is small !!!

$$\cos \psi_0 \cos \Delta \psi - \sin \psi_0 \sin \Delta \psi = \cos \psi_0 + \frac{\Delta \psi}{2}$$

$$\Delta \psi = \frac{kds \beta}{2}$$

and referring to $Q$ instead of $\psi$:

$$\psi = 2\pi Q$$

$$\Delta Q = \int \frac{\Delta k(s) \beta(s)}{4\pi} ds$$

a quadrupole error leads to a shift of the tune:

$$\Delta Q = \int \frac{\Delta k \beta(s)}{4\pi} ds = \frac{\Delta k \beta(s)}{4\pi}$$

Example: measurement of $\beta$ in a storage ring:

tune spectrum
16.) Chromaticity: 

A Quadrupole Error for \( \Delta p / p \neq 0 \)

Influence of external fields on the beam: prop. to magn. field & prop. zu \( 1 / p \)

**dipole magnet** \( \alpha = \frac{\int B \, dl}{p / e} \)

\[ \Delta p(s) = D(s) \frac{\Delta p}{p} \]

**focusing lens** \( k = \frac{g}{p / e} \)

Particle having...
- to high energy
- to low energy
- ideal energy

---

**Chromaticity: \( Q' \)**

\[ k = \frac{g}{p / e} \quad p = p_0 + \Delta p \]

In case of a momentum spread:

\[ k = \frac{eg}{p_0 + \Delta p} = \frac{e}{p_0} \left(1 - \frac{\Delta p}{p_0}\right) g = k_0 + \Delta k \]

\[ \Delta k = -\frac{\Delta p}{p_0} k_0 \]

… which acts like a quadrupole error in the machine and leads to a tune spread:

\[ \Delta Q = -\frac{1}{4\pi} \frac{\Delta p}{p_0} k_0 \beta(s) ds \]

Definition of chromaticity:

\[ \Delta Q = Q' \frac{\Delta p}{p} ; \quad Q' = -\frac{1}{4\pi} \int k(s) \beta(s) ds \]
... what is wrong about Chromaticity:

Problem: chromaticity is generated by the lattice itself!!

Q' is a number indicating the size of the tune spot in the working diagram, Q' is always created if the beam is focussed
⇒ it is determined by the focusing strength k of all quadrupoles

\[ Q' = -\frac{1}{4\pi} \int k(s)\beta(s)ds \]

k = quadrupole strength
β = betafunction indicates the beam size … and even more the sensitivity of the beam to external fields

Example: LHC

\[ Q' = 250 \]
\[ \Delta p/p = +/- 0.2 \times 10^{-3} \]
\[ 4\ Q = 0.256 \ldots 0.36 \]

⇒Some particles get very close to resonances and are lost

in other words: the tune is not a point it is a pancake

Correction of Q':

1.) sort the particles acording to their momentum \[ x_p(s) = D(s)\frac{\Delta p}{p} \]

2.) apply a magnetic field that rises quadratically with x (sextupole field)

\[ B_x = \tilde{g}x^2 \]
\[ B_y = \frac{1}{2} \tilde{g}(x^2 - y^2) \]

\[ \frac{\partial B_x}{\partial z} = \frac{\partial B_y}{\partial x} = \tilde{g}x \]

linear rising „gradient“:

Sextupole Magnets:

normalised quadrupole strength:

\[ k_{\text{sext}} = \frac{\tilde{g}x}{p/e} = m_{\text{sext}}x \]

\[ k_{\text{sext}} = m_{\text{sext}}D\frac{\Delta p}{p} \]

corrected chromaticity:

\[ Q' = -\frac{1}{4\pi} \int [K(s) - mD(s)]\beta(s)ds \]
Chromaticity in the FoDo Lattice

\[ Q' = \frac{-1}{4\pi} \int k(s) \beta(s) \, ds \]

\[ \beta = \frac{(1 + \sin \frac{\psi_{\text{cell}}}{2}) L}{\sin \psi_{\text{cell}}} \]

\[ \beta = \frac{(1 - \sin \frac{\psi_{\text{cell}}}{2}) L}{\sin \psi_{\text{cell}}} \]

\[ Q' = \frac{-1}{4\pi} N \cdot \frac{\bar{\beta} - \bar{\beta}}{f_0} \]

\[ Q' = \frac{-1}{4\pi} N \cdot \frac{1}{f_0} \left[ \frac{L(1 + \sin \frac{\psi_{\text{cell}}}{2}) - L(1 - \sin \frac{\psi_{\text{cell}}}{2})}{\sin \mu} \right] \]

using some TLC transformations ... \( \xi \) can be expressed in a very simple form:

\[ Q' = \frac{-1}{4\pi} N \cdot \frac{1}{f_0} \left[ \frac{2L \sin \frac{\psi_{\text{cell}}}{2}}{\sin \psi_{\text{cell}}} \right] \]

\[ Q' = \frac{-1}{4\pi} N \cdot \frac{1}{f_0} \left[ \frac{L \sin \frac{\psi_{\text{cell}}}{2}}{\sin \psi_{\text{cell}} \cos \psi_{\text{cell}}} \right] \]

\[ Q' = \frac{-1}{4\pi} f_0 \left[ \tan \frac{\psi_{\text{cell}}}{2} \right] \]

\[ Q' = \frac{-1}{\pi} \tan \frac{\psi_{\text{cell}}}{2} \]

\[ \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} \]

\[ \sin \frac{\psi_{\text{cell}}}{2} = \frac{L}{4f_0} \]

\[ \frac{\psi_{\text{cell}}}{2} \]

contribution of one FoDo Cell to the chromaticity of the ring:
Chromaticity

\[ Q' = -\frac{1}{4\pi} \int K(s)\beta(s)ds \]

question: main contribution to \( \xi \) in a lattice … ?

**17.) Résumé:**

beam rigidity: 
\[ B \cdot \rho = \frac{F}{q} \]

bending strength of a dipole: 
\[ \frac{1}{\rho} \left[ m^{-1} \right] = \frac{0.2998 \cdot B_0(T)}{p(\text{GeV}/c)} \]

focusing strength of a quadrupole: 
\[ k \left[ m^{-2} \right] = \frac{0.2998 \cdot g}{p(\text{GeV}/c)} \]

focal length of a quadrupole: 
\[ f = \frac{1}{k \cdot l_q} \]

equation of motion: 
\[ x^* + K x = \frac{1}{\rho} \frac{\Delta p}{p} \]

matrix of a foc. quadrupole: 
\[ x_{12} = M \cdot x_{11} \]

\[ M = \begin{pmatrix} \cos \sqrt{|K|} & \frac{1}{\sqrt{|K|}} \sin \sqrt{|K|} \\ \frac{1}{\sqrt{|K|}} \sin \sqrt{|K|} & \cos \sqrt{|K|} \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \]
**Resume':**

beam emittance: \[ e = \frac{1}{\beta y} \]

beta function in a drift: \[ \beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \]

... and for \( \alpha = 0 \)

\[ \beta(s) = \beta_0 + \frac{s^2}{\beta_0} \]

particle trajectory for \( \Delta p/p \neq 0 \)

inhomogenous equation: \[ x'' + x(\frac{1}{p^2} - k) = \frac{\Delta p}{P_0} \frac{1}{p} \]

... and its solution:

\[ x(s) = x_\beta(s) + D(s) \frac{\Delta p}{p} \]

momentum compaction: \[ \frac{\delta l}{L} = \alpha_p \frac{\Delta p}{p} \]

\[ \alpha_p = \frac{2\pi}{L} \langle D \rangle \frac{\langle D \rangle}{R} \]

quadrupole error: \[ \Delta Q = \int \frac{\Delta K(s) \beta(s) ds}{4\pi} \]

chromaticity: \[ Q' = -\frac{1}{4\pi} \int K(s) \beta(s) ds \]

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