Lattice Design in Particle Accelerators
Bernhard Holzer, CERN

1952: Courant, Livingston, Snyder:
Theory of strong focusing in particle beams

Lattice Design: "... how to build a storage ring"

High energy accelerators → circular machines
somewhere in the lattice we need a number of dipole magnets,
that are bending the design orbit to a closed ring

Geometry of the ring:
centrifugal force = Lorentz force

\[ e \cdot v \cdot B = \frac{mv^2}{\rho} \]
\[ \rightarrow e \cdot B = \frac{mv}{\rho} = \frac{p}{\rho} \]
\[ \rightarrow B \cdot \rho = \frac{p}{e} \]

\( p = \text{momentum of the particle,} \)
\( \rho = \text{curvature radius} \)

Example: heavy ion storage ring TSR
8 dipole magnets of equal bending strength
1.) Circular Orbit:

"... defining the geometry"

\[ \alpha = \frac{ds}{\rho} = \frac{dl}{\rho} \]

\[ \alpha = \frac{B^* \, dl}{B^* \, \rho} \]

The angle swept out in one revolution must be \(2\pi\), so

\[ \alpha = \frac{\int B \, dl}{B^* \, \rho} = 2\pi \quad \rightarrow \quad \int B \, dl = 2\pi \frac{P}{q} \]

... for a full circle

Nota bene: \[ \frac{N \, B}{B} = 10^{-8} \] is usually required !!
"Focusing forces --- single particle trajectories"

\[ y'' + K \cdot y = 0 \]

\[ K = -k + \frac{1}{\rho^2} \text{ hor. plane} \]

\[ K = k \text{ vert. plane} \]

\[
\begin{align*}
\text{dipole magnet} & \quad \frac{1}{\rho} - \frac{B}{p/q} \\
\text{quadrupole magnet} & \quad k - \frac{q}{p/q}
\end{align*}
\]

Example: HERA Ring:
- Bending radius: \( \rho = 580 \text{ m} \)
- Quadrupol Gradient: \( g = 110 \text{ T/m} \)
- \( k = 33.64 \times 10^3 / \text{m}^2 \)
- \( 1/\rho^2 = 2.97 \times 10^6 / \text{m}^2 \)

For estimates in large accelerators the weak focusing term \( 1/\rho^2 \) can in general be neglected.

Solution for a focusing magnet:
\[
\begin{align*}
y(s) &= y_0 \cdot \cos(\sqrt{K} \cdot s) + \frac{y_0'}{\sqrt{K}} \cdot \sin(\sqrt{K} \cdot s) \\
y'(s) &= -y_0 \cdot \sqrt{K} \cdot \sin(\sqrt{K} \cdot s) + y_0' \cdot \cos(\sqrt{K} \cdot s)
\end{align*}
\]

Or written more convenient in matrix form:
\[
\begin{pmatrix} y' \\ y'' \end{pmatrix} = M \cdot \begin{pmatrix} y \\ y' \end{pmatrix}
\]

Hor. focusing Quadrupole Magnet:
\[
M_{QF} = \begin{pmatrix} \cos(\sqrt{K} \cdot f) & \frac{1}{\sqrt{K}} \cdot \sinh(\sqrt{K} \cdot f) \\ -\sqrt{K} \sinh(\sqrt{K} \cdot f) & \cos(\sqrt{K} \cdot f) \end{pmatrix}
\]

Hor. defocusing Quadrupole Magnet:
\[
M_{QD} = \begin{pmatrix} \cosh(\sqrt{K} \cdot f) & \frac{1}{\sqrt{K}} \cdot \sinh(\sqrt{K} \cdot f) \\ \sqrt{K} \sinh(\sqrt{K} \cdot f) & \cosh(\sqrt{K} \cdot f) \end{pmatrix}
\]

Drift space:
\[
M_{\text{def}} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}
\]

\[ M_{\text{lattice}} = M_{QF1} \cdot M_{DA} \cdot M_{QD} \cdot M_{D1} \cdot M_{QF2} \cdots \]
2.) Reminder: Beam Dynamics Language

**Transfer Matrix M**

describes the transformation of amplitude \(x\) and angle \(x'\) through a number of lattice elements

\[ \begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0 \]

... and can be expressed by the optics parameters

\[
M = \begin{pmatrix}
\sqrt{\beta_s} \left( \cos \psi_s + \alpha_s \sin \psi_s \right) & \sqrt{\beta_s} \beta_0 \sin \psi_s \\
(\alpha_s - \alpha_s) \cos \psi_s - (1 + \alpha_s \alpha_s) \sin \psi_s & \sqrt{\beta_s} \beta_0 \sin \psi_s
\end{pmatrix}
\]

* we can calculate the single particle trajectories between two locations in the ring, if we know the \(\alpha \beta \gamma\) at these positions.
* and nothing but the \(\alpha \beta \gamma\) at these positions.
* ... !

---

**Periodic Lattices**

In the case of periodic lattices the transfer matrix can be expressed as a function of a set of periodic parameters \(\alpha, \beta, \gamma\)

\[
M(s) = \begin{pmatrix}
\cos \psi_{\text{period}} + \alpha_s \sin \psi_{\text{period}} & \beta_s \sin \psi_{\text{period}} \\
-\gamma_s \sin \psi_{\text{period}} & \cos \psi_{\text{period}} - \alpha_s \sin \psi_{\text{period}}
\end{pmatrix}
\]

\[ \Psi_{\text{period}} = \int_s ds \beta(s) \]

\(\psi = \text{phase advance per period}\)

For stability of the motion in periodic lattice structures it is required that

\[ |\text{trace}(M)| < 2 \]

In terms of these new periodic parameters the solution of the equation of motion is

\[
y(s) = \sqrt{\kappa} \sqrt{\beta(s)} \cos(\Phi(s) - \delta) \\
y'(s) = -\sqrt{\kappa} \left\{ \sin(\Phi(s) - \delta) + \alpha \cos(\Phi(s) - \delta) \right\}
\]
Transformation of $a, \beta, \gamma$

Consider two positions in the storage ring: $s_0, s$ since $\varepsilon = \text{const}$:

\[
\varepsilon = \beta x^2 + 2\alpha xx' + \gamma x'^2
\]

\[
\varepsilon = \beta_0 x_0^2 + 2\alpha_0 x_0 x_0' + \gamma_0 x_0'^2
\]

Express $x_0, x'_0$ as a function of $x, x'$.

\[
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_0 = M^{-1}
\begin{pmatrix}
  x \\
  x'
\end{pmatrix}_s
\]

\[
M^{-1} = \begin{pmatrix}
  m_{22} & -m_{12} \\
  -m_{21} & m_{11}
\end{pmatrix}
\]

Inserting into $\varepsilon = \beta x^2 + 2\alpha xx' + \gamma x'^2$

\[
\varepsilon = \beta_0 (m_{11}x' - m_{12}x)^2 + 2\alpha_0 (m_{22}x - m_{21}x')(m_{11}x' - m_{12}x) + \gamma_0 (m_{22}x - m_{12}x')^2
\]

Sort via $x, x'$ and compare the coefficients to get ....

The new parameters $a, \beta, \gamma$ can be transformed through the lattice via the lattice matrix elements defined above.

\[
\begin{pmatrix}
  \beta \\
  a \\
  \gamma
\end{pmatrix}_s = \begin{pmatrix}
  m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\
  -m_{12}m_{21} & m_{12}m_{21} + m_{22}m_{11} & -m_{12}m_{22} \\
  m_{22}^2 & -2m_{22}m_{21} & m_{21}^2
\end{pmatrix}
\begin{pmatrix}
  \beta \\
  a \\
  \gamma
\end{pmatrix}_0
\]

The optical parameters depend on the focusing properties of the lattice, 
... and can be optimised accordingly !!!

... and here starts the lattice design !!!
Most simple example: drift space

\[ M_{\text{drift}} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \]

particle coordinates

\[
\begin{pmatrix} x \\ x' \end{pmatrix}_f = \begin{pmatrix} 1 & \ell \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0
\]

\[ \rightarrow \begin{align*}
x(f) &= x_0 + \ell \cdot x'_0 \\
x'(f) &= x'_0
\end{align*} \]

transformation of twiss parameters:

\[
\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_f = \begin{pmatrix} 1 & -2 \ell & \ell^2 \\ 0 & 1 & -\ell \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_0
\]

\[ \beta(f) = \beta_0 - 2\ell \cdot \alpha_0 - \ell^2 \cdot \gamma_0 \]

Stability ...

\[ \text{trace}(M) = 1 + 1 = 2 \]

\[ \rightarrow \text{A periodic solution doesn’t exist in a lattice built exclusively out of drift spaces.} \]

Arc: regular (periodic) magnet structure:

- bending magnets \(\rightarrow\) define the energy of the ring
- main focusing & tune control, chromaticity correction
- multipoles for higher order corrections

Straight sections: drift spaces for injection, dispersion suppressors,
- low beta insertions, RF cavities, etc....
- ... and the high energy experiments if they cannot be avoided
3.) The FoDo-Lattice

A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in between.
(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)

Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell \( \mu = 45^\circ \),
→ calculate the twiss parameters for a periodic solution

### Output of the optics program:

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<th>Nr</th>
<th>Type</th>
<th>Length</th>
<th>Strength</th>
<th>( \beta_x )</th>
<th>( \alpha_x )</th>
<th>( \phi_x )</th>
<th>( \beta_z )</th>
<th>( \alpha_z )</th>
<th>( \phi_z )</th>
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<td>0,000</td>
<td>0,000</td>
<td>0,000</td>
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<td>0,000</td>
</tr>
<tr>
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<td>-0,541</td>
<td>11,228</td>
<td>1,514</td>
<td>0,004</td>
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<tr>
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<td>0,125</td>
<td>5,295</td>
<td>0,000</td>
<td>0,125</td>
</tr>
</tbody>
</table>

\( QX = 0,125 \)  \( QZ = 0,125 \)

\[ 0.125 \times 2\pi = 45^\circ \]
Can we understand what the optics code is doing?

matrices

\[ M_{gf} = \begin{pmatrix} \cos(K * l_q) & \frac{1}{\sqrt{K}} \sin(K * l_q) \\ -\sqrt{K} \sin(K * l_q) & \cos(K * l_q) \end{pmatrix}, \quad M_{def} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]

strength and length of the FoDo elements

\[ K = +/- 0.54102 \ m^{-2} \]
\[ l_q = 0.5 \ m \]
\[ l_d = 2.5 \ m \]

The matrix for the complete cell is obtained by multiplication of the element matrices:

\[ M_{Fodo} = M_{gf} * M_{ld} * M_{qd} * M_{ld} * M_{gf} \]

Putting the numbers in and multiplying out ...

\[ M_{Fodo} = \begin{pmatrix} 0.707 & 8.206 \\ -0.061 & 0.707 \end{pmatrix} \]

The transfer matrix for 1 period gives us all the information that we need!

1.) is the motion stable? \[ \text{trace}(M_{Fodo}) = 1.415 \rightarrow < 2 \]

2.) Phase advance per cell

\[ M(\epsilon) = \begin{pmatrix} \cos\psi_{cell} + \alpha \sin\psi_{cell} & \beta \sin\psi_{cell} \\ -\beta \sin\psi_{cell} & \cos\psi_{cell} - \alpha \sin\psi_{cell} \end{pmatrix} \]

\[ \psi_{cell} = \frac{1}{2} \text{trace}(M) = 0.707 \]

\[ \psi_{cell} = \cos\left(\frac{1}{2} \text{trace}(M)\right) = 45 \]

3.) hor \( \beta \)-function

\[ \beta = \frac{m_{12}}{\sin\psi_{cell}} = 11.611 \ m \]

4.) hor \( \alpha \)-function

\[ \alpha = \frac{m_{11} - \cos\psi_{cell}}{\sin\psi_{cell}} = 0 \]
Can we do a bit easier?
We can ... in thin lens approximation!

Matrix of a focusing quadrupole magnet:

\[ M_{QR} = \begin{pmatrix} \cos(\sqrt{K} \cdot f) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} \cdot f) \\ -\sqrt{K} \sin(\sqrt{K} \cdot f) & \cos(\sqrt{K} \cdot f) \end{pmatrix} \]

If the focal length \( f \) is much larger than the length of the quadrupole magnet,

\[ f = \frac{1}{k l_0} \gg l_0 \]

the transfer matrix can be approximated using:

\[ k l_q = \text{const}, \quad l_q \to 0 \]

\[ M = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \]

4.) FoDo in thin lens approximation

Calculate the matrix for a half cell, starting in the middle of a foc. quadrupole:

\[ M_{\text{half cell}} = M_{QR2} \cdot M_{ID} \cdot M_{QR2} \]

\[ M_{\text{half cell}} = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & l_0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \]

note: \( f \) denotes the focusing strength of half a quadrupole, so \( f = 2f \)

for the second half cell set \( f \to \frac{1}{f} \)

\[ M_{\text{half cell}} = \begin{pmatrix} 1 - \frac{l_0}{f} & l_0 \\ -\frac{l_0}{f} & 1 + \frac{l_0}{f} \end{pmatrix} \]
Now we know, that the phase advance is related to the transfer matrix by
\[
\cos \psi_{cell} = \frac{1}{2} \text{trace}(M) = \frac{1}{2} \left( 2 - \frac{4l_0^2}{f^2} \right) = 1 - \frac{2l_0^2}{f^2}
\]
After some beer and with a little bit of trigonometric gymnastics
\[
\cos(x) = \cos^2 \left( \frac{x}{2} \right) - \sin^2 \left( \frac{x}{2} \right) = 1 - 2 \sin^2 \left( \frac{x}{2} \right)
\]
}\[
M = \begin{pmatrix}
1 + \frac{l_0}{f} & l_0 \\
-\frac{l_0}{f^2} & 1
\end{pmatrix}
\begin{pmatrix}
1 + \frac{l_0}{f} & l_0 \\
-\frac{l_0}{f^2} & 1
\end{pmatrix}
\]
\[
M = \begin{pmatrix}
1 - \frac{2l_0^2}{f^2} & 2l_0 \left( 1 + \frac{l_0}{f} \right) \\
2 \left( \frac{l_0^2}{f} - \frac{l_0}{f^2} \right) & 1 - \frac{2l_0^2}{f^2}
\end{pmatrix}
\]
\[
L_{cell} = l_{00} + l_{10} + l_{11} + l_{20} = 0.5m + 2.5m + 0.5m + 2.5m = 6m
\]
\[
1/f = k^2 l_0 = 0.5m^2 \cdot 0.541 m^{-2} = 0.27 m^{-1}
\]
\[
\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f}
\]
\[
\sin(\psi_{cell}/2) = \frac{L_{cell}}{4f} = 0.405
\]
\[
\rightarrow \psi_{cell} = 47.8\degree
\]
\[
\rightarrow \beta = 11.4 m
\]
\[
\text{Remember:}
\text{Exact calculation yields:}
\]
\[
\rightarrow \psi_{cell} = 45\degree
\]
\[
\rightarrow \beta = 11.6 m
\]
Stability in a FoDo structure

\[
M_{Fools} = \begin{pmatrix}
1 - \frac{2l_0^2}{f^2} & 2l_0(1 + \frac{l_0}{f}) \\
2(l_0^2 - \frac{l_0}{f^2}) & 1 - 2\frac{l_0}{f^2}
\end{pmatrix}
\]

Stability requires:

\[
|\text{Trace}(M)| < 2
\]

\[
|\text{Trace}(M)| = 2 - \frac{4l_0^2}{f^2} < 2
\]

\[\rightarrow f > \frac{l_{cell}}{4}\]

For stability the focal length has to be larger than a quarter of the cell length...

don’t focus to strong!

---

Transformation Matrix in Terms of the Twiss Parameters

Transformation of the coordinate vector \((x, x')\) in a lattice

\[
\begin{pmatrix}
x(s) \\
x'(s)
\end{pmatrix} = M_{s_1, s_2} \begin{pmatrix}
x_0 \\
x'_0
\end{pmatrix}
\]

General solution of the equation of motion

\[
x(s) = \sqrt{\varepsilon} \beta(s) \cos(\psi(s) + \varphi)
\]

\[
x'(s) = \sqrt{\varepsilon} \beta(s) \left\{ \alpha(s) \cos(\psi(s) + \varphi) + \sin(\psi(s) + \varphi) \right\}
\]

Transformation of the coordinate vector \((x, x')\)
expressed as a function of the twiss parameters

\[
M_{1 \rightarrow 2} = \begin{pmatrix}
\sqrt{\beta_1} \beta_2 \cos\psi_{12} + \alpha_1 \sin\psi_{12} & \sqrt{\beta_1 \beta_2} \sin\psi_{12} \\
(\alpha_1 - \alpha_2) \cos\psi_{12} - (1 + \alpha_1 \alpha_2) \sin\psi_{12} & \sqrt{\beta_1} \beta_2 \cos\psi_{12} - \alpha_2 \sin\psi_{12}
\end{pmatrix}
\]
In the middle of a foc (defoc) quadrupole of the FoDo we always have $\alpha = 0$, and the half cell will lead us from $\beta_{\text{max}}$ to $\beta_{\text{min}}$

Transfer Matrix for half a FoDo cell:

$$M_{\text{half cell}} = \begin{pmatrix} 1 - \frac{t_p}{f} & t_p \\ -t_p/f^2 & 1 + t_p/f \end{pmatrix}$$

Compare to the twiss parameter form of $M$

$$M_{1 \rightarrow 2} = \begin{pmatrix} \frac{\beta_2}{\beta_1} (\cos \psi_{12} + \alpha \sin \psi_{12}) & \frac{\beta_2}{\beta_1} \sqrt{\beta_1 \beta_2} \sin \psi_{12} \\ (\alpha - \alpha_z) \cos \psi_{12} - (1 + \alpha \alpha_z) \sin \psi_{12} & \sqrt{\beta_1 \beta_2} (\cos \psi_{12} - \alpha \sin \psi_{12}) \end{pmatrix}$$

In the middle of a foc (defoc) quadrupole of the FoDo we always have $\alpha = 0$, and the half cell will lead us from $\beta_{\text{max}}$ to $\beta_{\text{min}}$

$$M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta}{\beta_0}} \cos \frac{\psi_{\text{cell}}}{2} & \sqrt{\frac{\beta}{\beta_0}} \frac{\beta}{\beta_1} \sin \frac{\psi_{\text{cell}}}{2} \\ -1 & \sqrt{\frac{\beta}{\beta_0}} \frac{\beta}{\beta_1} \cos \frac{\psi_{\text{cell}}}{2} \end{pmatrix}$$

Solving for $\beta_{\text{max}}$ and $\beta_{\text{min}}$ and remembering that ...

$$\frac{\sin \psi_{\text{cell}}}{2} \frac{t_p}{f} = L \frac{1}{f}$$

$$\begin{aligned}
\frac{m_{22}}{m_{11}} &= \frac{\hat{\beta}}{\beta} = 1 + \frac{t_p}{f} \frac{f}{L} = 1 + \sin \left( \frac{\psi_{\text{cell}}}{2} \right) \\
\frac{m_{21}}{m_{12}} &= \hat{\beta} \frac{f}{L} = \frac{t_p}{\sin \left( \frac{\psi_{\text{cell}}}{2} \right)}
\end{aligned}$$

The maximum and minimum values of the $\beta$-function are solely determined by the phase advance and the length of the cell.

Longer cells lead to larger $\beta$

**Typical shape of a proton bunch in a FoDo Cell**
5.) Beam dimension:

Optimisation of the FoDo Phase advance:

In both planes a gaussian particle distribution is assumed, given by the beam emittance $\varepsilon$ and the $\beta$-function

$$\sigma = \sqrt{\varepsilon \beta}$$

In general proton beams are "round" in the sense that

$$\varepsilon_x = \varepsilon_y$$

So for highest aperture we have to minimise the $\beta$-function in both planes:

$$r^2 = \varepsilon_x \beta_x + \varepsilon_y \beta_y$$

Optimising the FoDo phase advance

search for the phase advance $\mu$ that results in a minimum of the sum of the beta's

$$\tilde{\beta} + \overline{\beta} = \frac{2L}{\sin \psi_{cell}}$$

$$\frac{d}{d \psi_{cell}} \left( \frac{2L}{\sin \psi_{cell}} \right) = 0$$

$$\frac{L}{\sin^2 \psi_{cell}} \cos \psi_{cell} = 0 \quad \rightarrow \quad \psi_{cell} = 90^\circ$$
Electrons are different

\[ \text{electron beams are usually flat, } e_y \approx 2 - 10\% e_x, \]
\[ \Rightarrow \text{optimise only } \beta_{\text{neq}} \]

\[ \frac{d}{d\psi_{\text{cell}}}(\hat{\beta}) = \frac{\frac{L(1 + \sin \frac{\psi_{\text{cell}}}{2})}{\sin \psi_{\text{cell}}}}{d\psi_{\text{cell}}} = 0 \rightarrow \psi_{\text{cell}} = 76^\circ \]

red curve: \( \beta_{\text{neq}} \)
blue curve: \( \beta_{\text{max}} \)
as a function of the phase advance \( \psi \)

**Orbit distortions in a periodic lattice**

field error of a dipole/distorted quadrupole

\[ \delta (\text{mrad}) = \frac{ds}{\rho} = \frac{\int B ds}{p \cdot e} \]

the particle will follow a new closed trajectory, the distorted orbit:

\[ x(s) = \frac{\sqrt{\beta(s)}}{2 \sin(\psi)} \int \sqrt{\beta(s)} \frac{1}{\rho(s)} \cos[\beta(s) - \beta(s)] - M \text{Q}(s) ds \]

* the orbit amplitude will be large if the \( \beta \) function at the location of the kick is large
* \( \beta(s) \) indicates the sensitivity of the beam
* here orbit correctors should be placed in the lattice

* the orbit amplitude will be large at places where in the lattice \( \beta(s) \) is large
* here beam position monitors should be installed
**Orbit Correction and Beam Instrumentation in a storage ring**

![Image of the Elsa ring, Bonn](image)

**Resumé:**

1.) Dipole strength

\[ \int B ds = N^* R^* l_{eff} = \frac{2\pi}{q} \]

\( l_{eff} \) effective magnet length, \( N \) number of magnets

2.) Stability condition

\[ \text{Trace}(M) < 2 \]

for periodic structures within the lattice / at least for the transfer matrix of the complete circular machine

3.) Transfer matrix for periodic cell

\[ M(i) = \begin{pmatrix}
\cos\psi_{cell} + \alpha_i \sin\psi_{cell} & \beta_i \sin\psi_{cell} \\
-\gamma_i \sin\psi_{cell} & \cos\psi_{cell} - \alpha_i \sin\psi_{cell}
\end{pmatrix} \]

\( \alpha, \beta, \gamma \) depend on the position \( s \) in the ring, \( \mu \) (phase advance) is independent of \( s \)

4.) Thin lens approximation

\[ M_{qr} = \begin{pmatrix}
1 & 0 \\
1 & f_q \end{pmatrix} \]

\[ f_q = \frac{1}{k_q l_0} \ll l_0 \]

focal length of the quadrupole magnet \( f_q = 1/(k_q l_0) \gg l_0 \)
5.) Tune (rough estimate)

\[ \Psi_{\text{period}} = \int_0^L \frac{d\phi}{\beta(s)} \]

\[ Q = N * \frac{\Psi_{\text{period}}}{2\pi} = \frac{1}{2\pi} \int_0^L \frac{d\phi}{\beta(s)} = \frac{1}{2\pi} \frac{2\pi R}{\beta} = \frac{R}{\beta} \]

\( R, \beta \) average radius and \( \beta \)-function

6.) Phase advance per FoDo cell (thin lens approx)

\[ \sin \left( \frac{\Psi_{\text{cell}}}{2f_0} \right) = \frac{L_{\text{cell}}}{f_0 \beta} \]

\( L_{\text{cell}} \) length of the complete FoDo cell, \( f_0 \) focal length of the quadrupole, \( \mu \) phase advance per cell

7.) Stability in a FoDo cell (thin lens approx)

\[ f_0 > \frac{L_{\text{cell}}}{4} \]

8.) Beta functions in a FoDo cell (thin lens approx)

\[ \beta = \frac{1 + \sin \left( \frac{\Psi_{\text{cell}}}{2L_{\text{cell}}} \right)}{2 \sin \frac{\Psi_{\text{cell}}}{2L_{\text{cell}}}} \]

\[ \hat{\beta} = \frac{1 - \sin \left( \frac{\Psi_{\text{cell}}}{2L_{\text{cell}}} \right)}{2 \sin \frac{\Psi_{\text{cell}}}{2L_{\text{cell}}}} \]

\( L_{\text{cell}} \) length of the complete FoDo cell, \( \mu \) phase advance per cell