Short Introduction to

(Classical) Electromagnetic Theory

(.. and applications to accelerators)

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(http://cern.ch/Werner.Herr/CAS2017_Chavannes/em.pdf)

Reading Material

- **J.D. Jackson,** *Classical Electrodynamics* **(Wiley, 1998 ..)**
- **R.P. Feynman,** *Feynman lectures on Physics*, **Vol2.**
- **A. Wolski,** *Theory of electromagnetic fields*, **Proc. CAS: "RF for accelerators", CERN-2011-007.**
- **J. Slater, N. Frank,** *Electromagnetism*, **(McGraw-Hill, 1947, and Dover Books, 1970)**

Variables, units and conventions

Maxwell's equations relate Electric and Magnetic fields from charge and current distributions (SI units).

$\vec{E}$	=	electric field [V/m]
$\vec{H}$	=	magnetic field [A/m]
$\vec{D}$	=	electric displacement [C/m^2]
$\vec{B}$	=	magnetic flux density [T]
q	=	electric charge [C]
ρ	=	electric charge density [C/m^3]
$\vec{j}$	=	current density [A/m^2]
μ_0	=	permeability of vacuum, 4 $\pi \cdot 10^{-7}$ [H/m or N/A^2]
ϵ_0	=	permittivity of vacuum, 8.854 $\cdot 10^{-12}$ [F/m]
c	=	speed of light, 2.99792458 $\cdot 10^8$ [m/s]

Basic concept: forces experienced by charged objects

They are conveniently described by abstract models, i.e. vector fields and potentials:

- **Electric phenomena:** $\vec{E}$ **and** $\vec{D}$
- **Magnetic phenomena:** $\vec{H}$ **and** $\vec{B}$
- **Electric and Magnetic potentials:** Φ **and** $\vec{A}$

➡ **Electrodynamics: need vectors to describe phenomena**

➡ **Need to know how to calculate with vectors**

Note: electromagnetism is not understood without relativity ...

Reminder: mathematics used here

- **Addition to previous lecture (R.S.)**
- **A bit specific to these lectures**
- **Not all details are strictly needed to understand, but required for calculations and later lectures**

- **Illustrations and examples ...**

We had: Vector Products (sometimes cross product)

Define a vector product for (usual) vectors like: $\vec{a} \times \vec{b}$,

$$\vec{a} = (x_a, y_a, z_a) \qquad \vec{b} = (x_b, y_b, z_b)$$

$$\vec{a} \times \vec{b} = (x_a, y_a, z_a) \times (x_b, y_b, z_b)$$
$$= (\underbrace{y_a \cdot z_b - z_a \cdot y_b}_{x_{ab}}, \quad \underbrace{z_a \cdot x_b - x_a \cdot z_b}_{y_{ab}}, \quad \underbrace{x_a \cdot y_b - y_a \cdot x_b}_{z_{ab}})$$

This product of two vectors is a "vector", not a scalar (number)

Example:

$$(-2, 2, 1) \times (2, 4, 3) = (2, 8, -12)$$

Need also Scalar Products

Define a scalar product for (usual) vectors like: $\vec{a} \cdot \vec{b}$,

$$\vec{a} \;=\; (x_a, y_a, z_a) \qquad\qquad \vec{b} \;=\; (x_b, y_b, z_b)$$

$$\vec{a} \cdot \vec{b} \;=\; (x_a, y_a, z_a) \;\cdot\; (x_b, y_b, z_b) = (x_a \cdot x_b \;+ y_a \cdot y_b \;+\; z_a \cdot z_b)$$

This product of two vectors is a <u>scalar</u> (number) not a vector.

Example:

$$(-2, 2, 1) \;\cdot\; (2, 4, 3) \;=\; -2 \cdot 2 + 2 \cdot 4 + 1 \cdot 3 \;=\; 7$$

Vector calculus - a new vector ...

We can define a special vector ∇ :

$$\nabla \overset{def}{=} (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

It is called the "gradient" and invokes "partial derivatives".
It can operate on a scalar function $\phi(x, y, z)$:

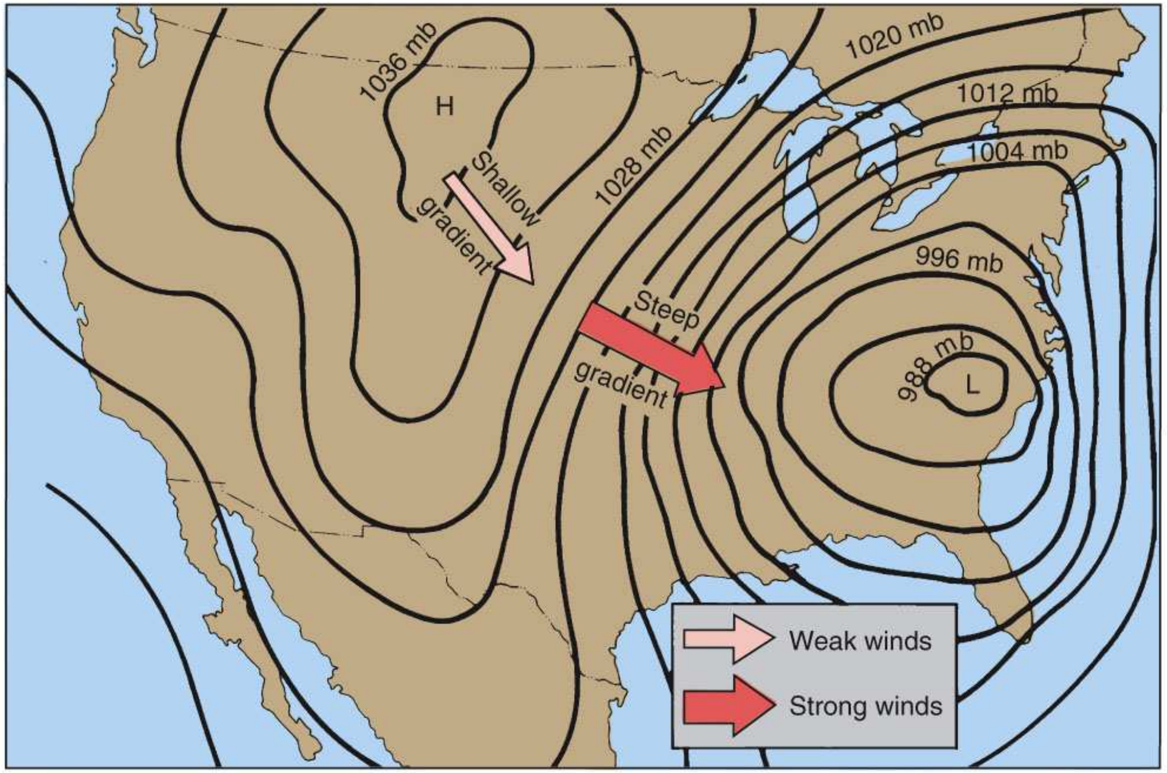
$$\nabla\phi \overset{def}{=} (\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}) = \vec{G} = (G_x, G_y, G_z)$$

and we get a vector $\vec{G}$. It is a kind of "slope" (steepness ..) in the 3 directions.

Example: $\phi(x, y, z) = 0.1x^2 - 0.2 \cdot x \cdot y + z^2$

→ $\nabla\phi = \vec{G} = (G_x, G_y, G_z) = (0.2x - 0.2y, \ -0.2x, \ 2z)$

Gradient (slope) of a scalar field (see Rende's lectures)



Lines of atmospheric pressure (isobars)

Gradient is large (steep) where lines are close (fast change of pressure)

∇ also operates on vector fields ...

Can be treated like an "ordinary" vector. Two operations of ∇ have special names:

Divergence (scalar product of ∇ with a vector):

$$\mathrm{div}(\vec{F}) \stackrel{def}{=} \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Physical significance: "amount of density", (see later)

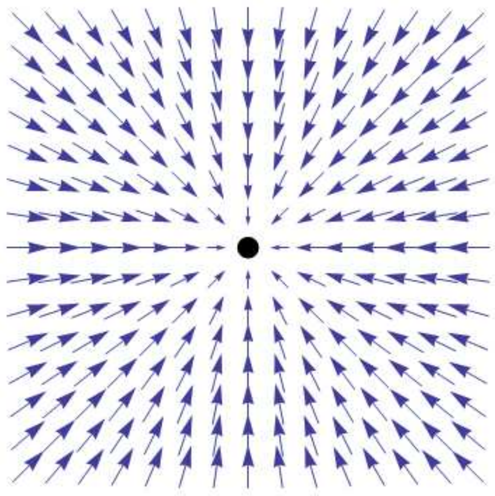
Curl (vector product of ∇ with a vector):

$$\mathrm{curl}(\vec{F}) \stackrel{def}{=} \nabla \times \vec{F} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right)$$

Physical significance: "amount of circulation", (see later)

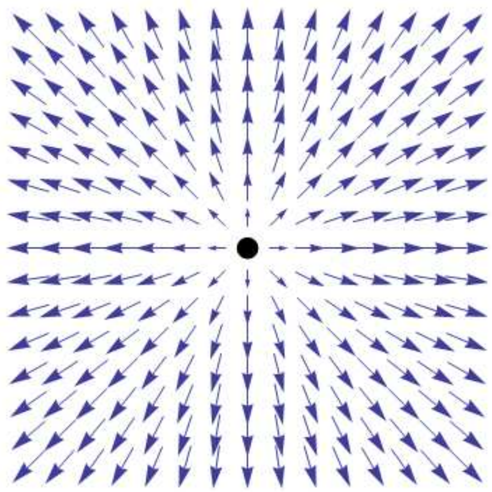
Meaning of Divergence of fields ...

Field lines seen from some origin:



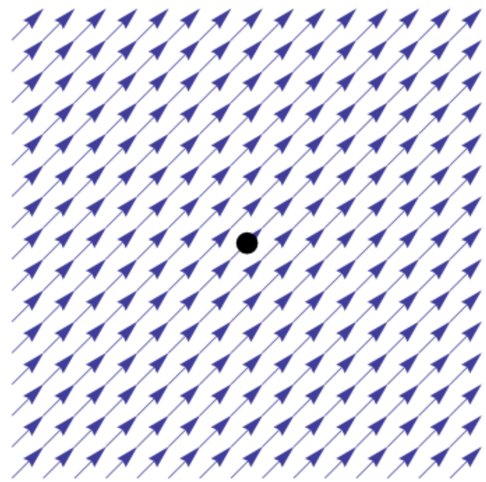
$\nabla \vec{F} < 0$

(sink)



$\nabla \vec{F} > 0$

(source)



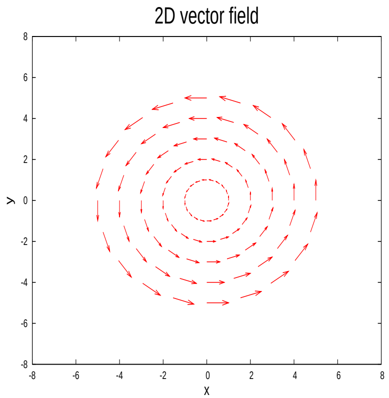
$\nabla \vec{F} = 0$

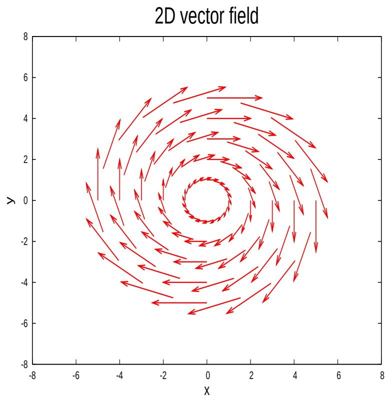
(fluid)

The divergence (scalar, a single number) characterizes what comes from (or goes to) the origin

Meaning of Curl of fields ...

Field lines follow a curve (e.g. circle)





Two examples for fields in $x - y$ plane::

$$\vec{F}_1 = (-0.2y, +0.2x, 0)$$

$$\nabla \times \vec{F}_1 \;=\; \text{curl}\vec{F}_1 \;=\; (0, 0, +0.4)$$

$$\vec{F}_2 = (+0.5y, -0.5x, 0)$$

$$\nabla \times \vec{F}_2 \;=\; \text{curl}\vec{F}_2 \;=\; (0, 0, -1.0)$$

Vectors in z-direction, perpendicular to $x - y$ plane

Values characterize "strength" and "direction" of circulation

To remember: ...

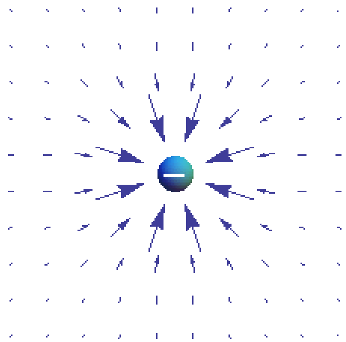
Not really rigorous, but:

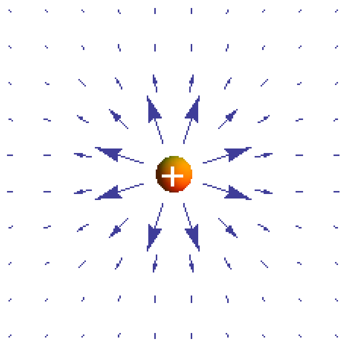
- DIV **measures what is coming out (or going in), integral is called the FLUX**

- $CURL$ **measures what is circulating, integral is called the CIRCULATION**

In general: a closed surface or closed line "measures" what is happening inside ...

Electric fields from charges





(negative charges) **(positive charges)**

Assume fields from a positive or negative charge q

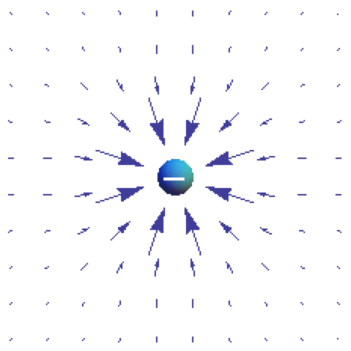
Electric field $\vec{E}$ **is written as (Coulomb law):**

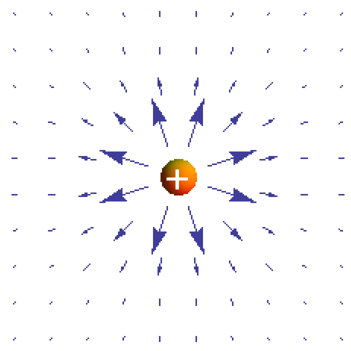
$$\vec{E} \; = \; \frac{\pm q}{4\pi\epsilon_0} \cdot \frac{\vec{r}}{|r|^3}$$

with:

$$\vec{r} \; = \; (x, y, z), \qquad |r| = \sqrt{x^2 + y^2 + z^2}$$

Applying Divergence and charges ..





We can do the (non-trivial) computation of the divergence:

$$\text{div}\vec{E} \;=\; \nabla\vec{E} \;=\; \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} \;=\; \frac{\rho}{\epsilon_0}$$

(negative charges)

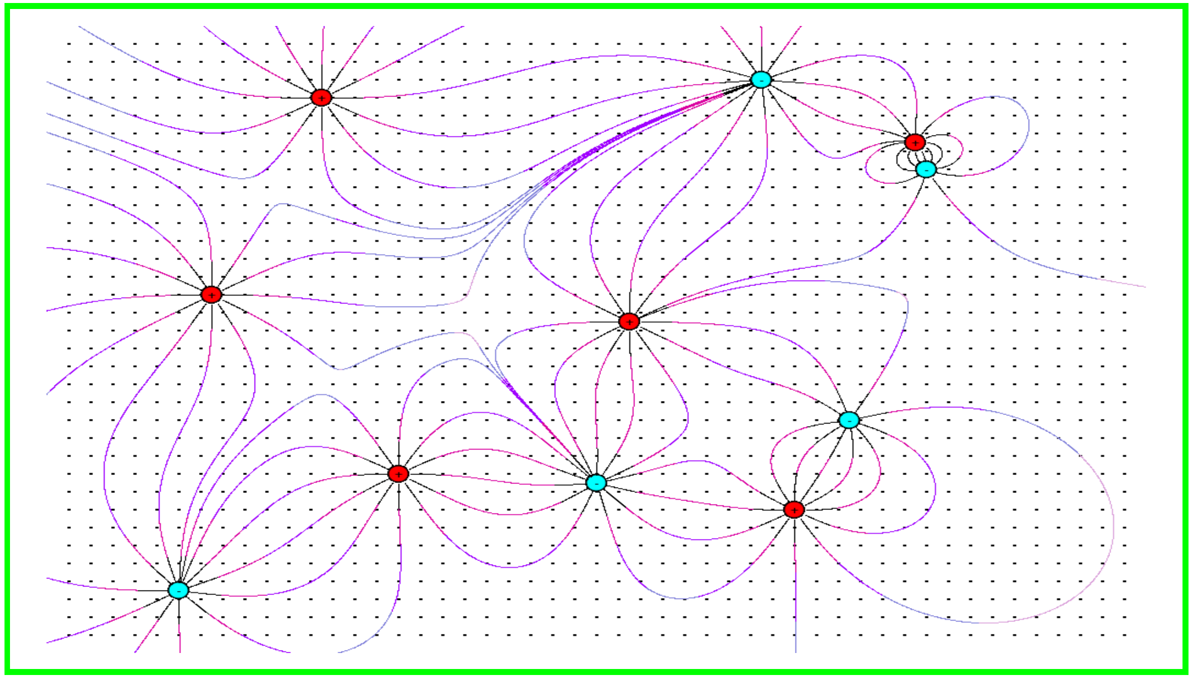
$$\nabla \cdot \vec{E} < 0$$

(positive charges)

$$\nabla \cdot \vec{E} > 0$$

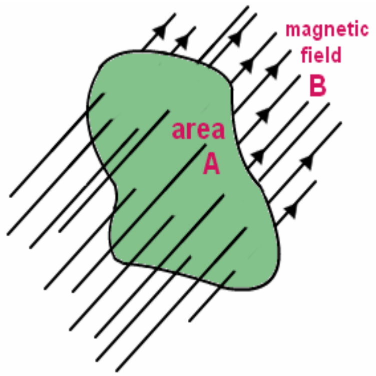
Divergence related to charge density ρ generating the field $\vec{E}$

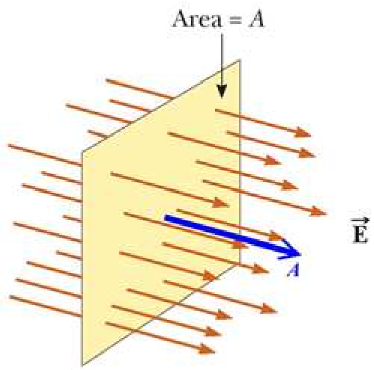
Combined electric field from several charges



Exact calculation of field lines from the superposition of (positive and negative) charges

Electric and Magnetic flux





$$\Omega = \int\int_A \vec{B} \cdot d\vec{A} \qquad \Phi = \int\int_A \vec{E} \cdot d\vec{A}$$

Integrate (count) field vectors through an oriented area (or surface)

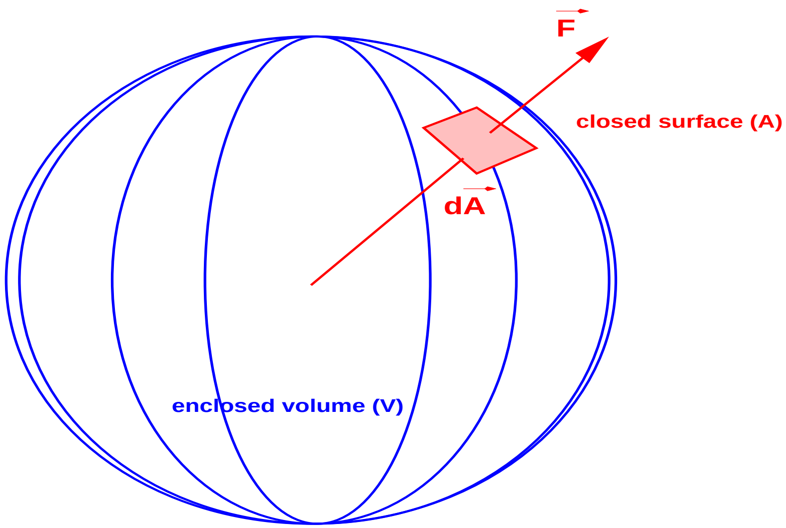
"Measures" the strength of the fields

Gives flux of electric and magnetic fields

Gauss' Theorem:

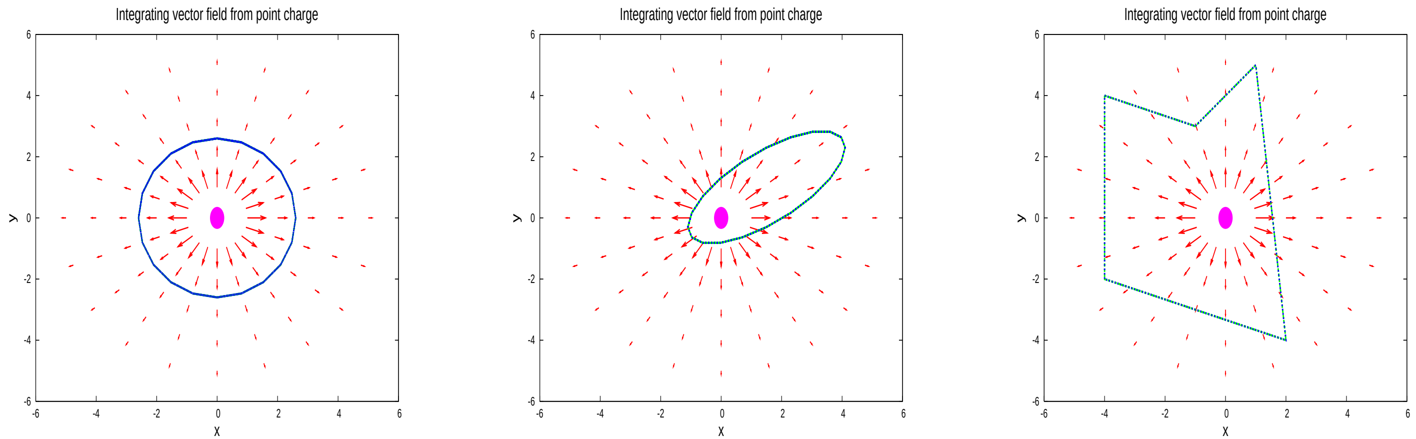
Integral through a closed surface (flux) is integral of divergence in the enclosed volume

$$\int\int_A \vec{F} \cdot d\vec{A} = \int\int\int_V \nabla \cdot \vec{F} \cdot dV$$



Relates flux to divergence ($\vec{F}$ can be $\vec{E}$ or $\vec{B}$)

Integrating fields from charges (2D !) ..



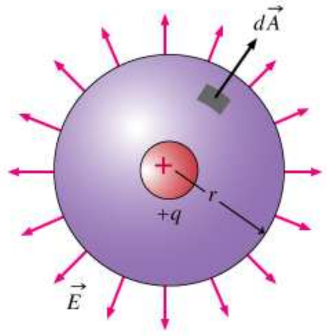
- **To compute the electric flux, add field lines through the surface:** $\int\int_A \vec{E} \cdot d\vec{A}$

- **Put any <u>closed</u> surface around charges (sphere, box, ...).**
  If all charges are enclosed: independent of shape !

→ **If <u>positive</u>: total net charge enclosed <u>positive</u>**

→ **If <u>negative</u>: total net charge enclosed <u>negative</u>**

More formal: Gauss's Theorem (Maxwell's first equation ...)



$$\int\int_A \vec{E} \cdot d\vec{A} = \int\int\int_V \nabla \vec{E} \cdot dV = \frac{q}{\epsilon_0}$$

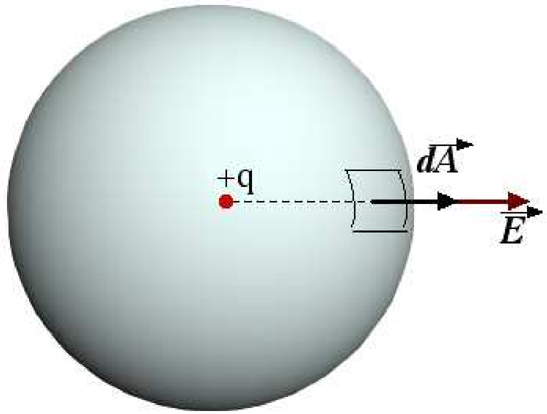
$$\int\int\int_V \frac{\rho}{\epsilon_0} \cdot dV = \frac{q}{\epsilon_0}$$

Flux of electric field $\vec{E}$ through a closed surface proportional to net electric charge q enclosed in the region (Gauss's Theorem).

Written with charge density ρ we get Maxwell's first equation:

$$\text{div}\vec{E} = \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Example: field from a charge q



A charge q generates a field $\vec{E}$ according to:

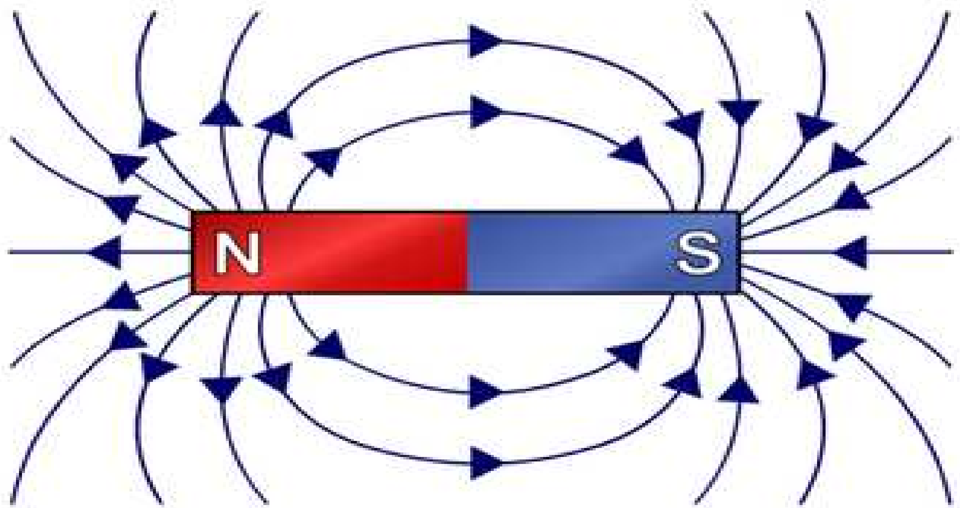
$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

Enclose it by a sphere: $\vec{E} = const.$ on a sphere (area is $4\pi \cdot r^2$):

$$\int\int_{sphere} \vec{E} \cdot d\vec{A} = \frac{q}{4\pi\epsilon_0}\int\int_{sphere}\frac{dA}{r^2} = \frac{q}{\epsilon_0}$$

Surface integral through sphere A is charge inside the sphere

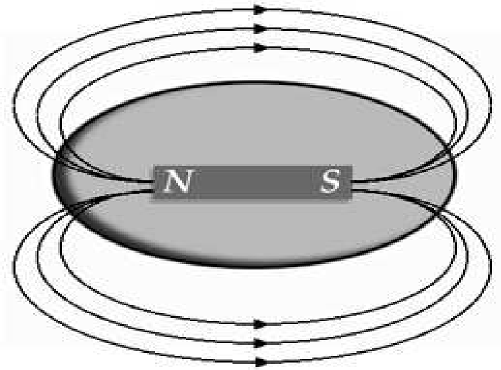
Divergence of magnetic fields



Definitions

- **Magnetic field lines from North to South**
- **Q: which is the direction of the earth magnetic field lines ?**

Maxwell's second equation ...



$$\int\int_A \vec{B}\ \ d\vec{A} = \int\int\int_V \nabla\vec{B}\ \ dV\ \ =\ \ 0$$

$$\nabla\vec{B} = 0$$

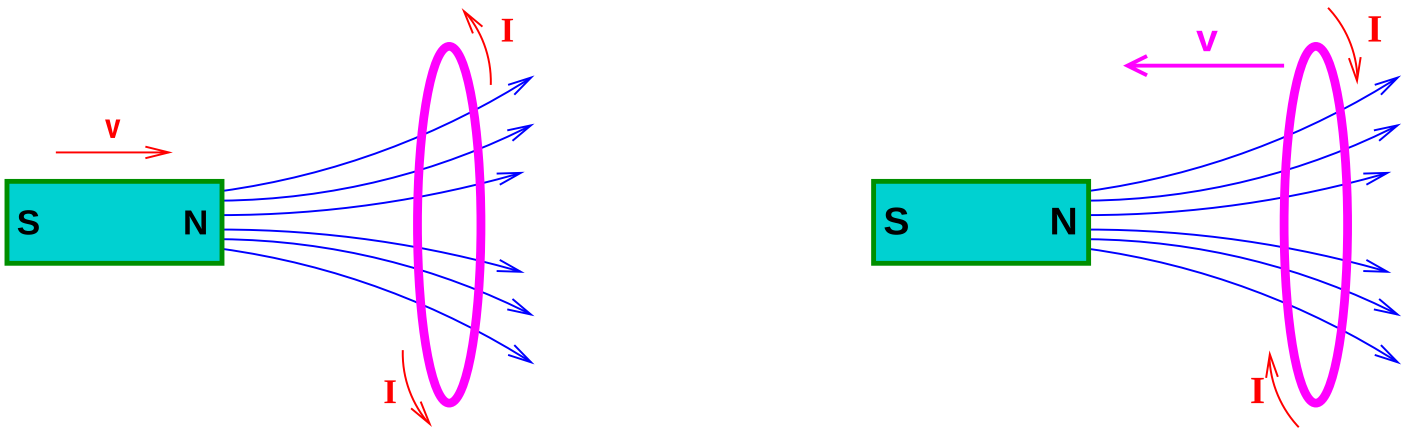
Closed field lines of magnetic flux density ($\vec{B}$): What goes out ANY closed surface also goes in, Maxwell's second equation:

$$\nabla\vec{B} = \mu_0\nabla\vec{H} = 0$$

➡ **Physical significance: no Magnetic Monopoles**

Maxwell's third equation - Faradays law...

Relative movement of a magnet and a conducting coil:



- **Changing magnetic flux through area of a coil introduces electric current I**
- **Can be changed by moving magnet or coil**

Maxwell's third equation ...

A changing flux Ω through an area A produces circular electric field $\vec{E}$, i.e. a current I (Faraday)

$$-\frac{\partial \Omega}{\partial t} = \frac{\partial}{\partial t} \underbrace{\int_A \vec{B} d\vec{A}}_{flux\ \Omega} = \oint_C \vec{E} \cdot d\vec{r}$$

- **Flux can be changed by:**

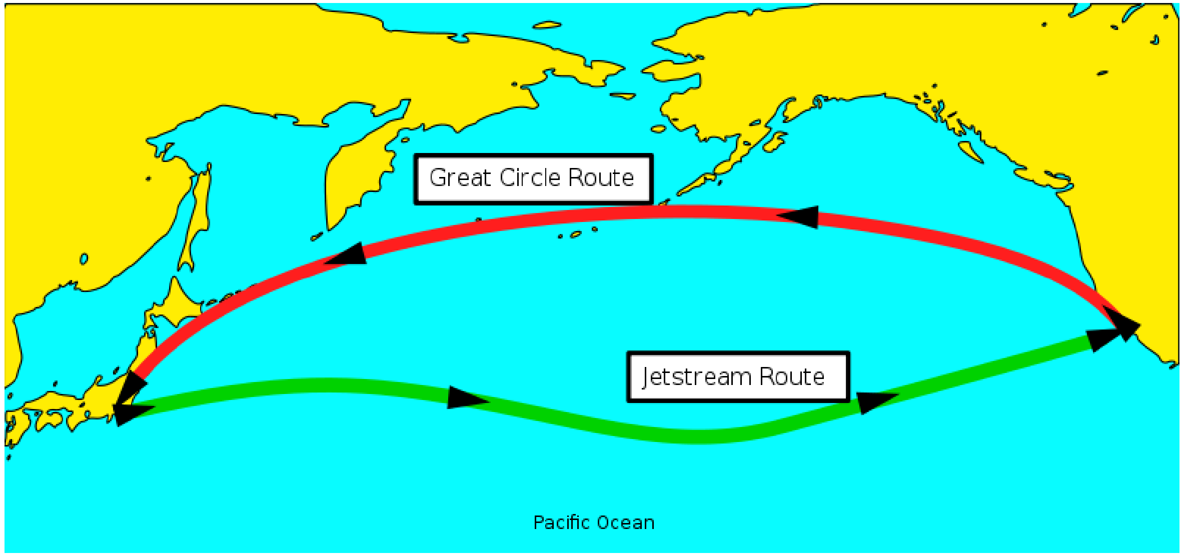
  - **Change of magnetic field $\vec{B}$ with time t (e.g. transformers)**

  - **Change of area A with time t (e.g. dynamos)**

$$\oint_C \vec{E} \cdot d\vec{r}$$ is a <u>line integral</u>

  - **Summed up charges along the closed wire, i.e. a current I**

A word on line integrals ...



Line integrals: integrate field vectors along a line C:

$$\rightarrow \quad \int_C \vec{F} \cdot d\vec{r}$$

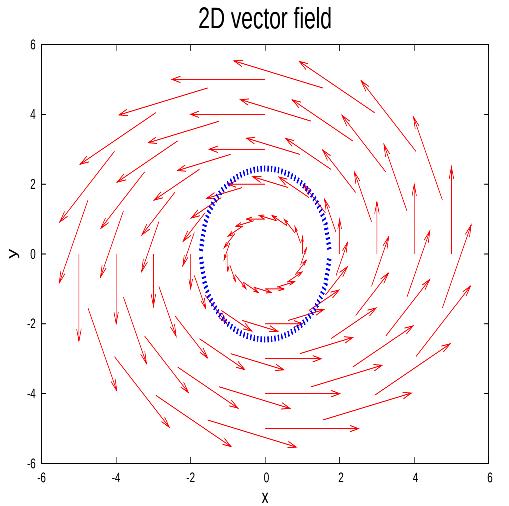
"sum up" vectors (length) in <u>direction</u> of line C

(e.g. work performed along a path ...)

Stokes' Theorem:

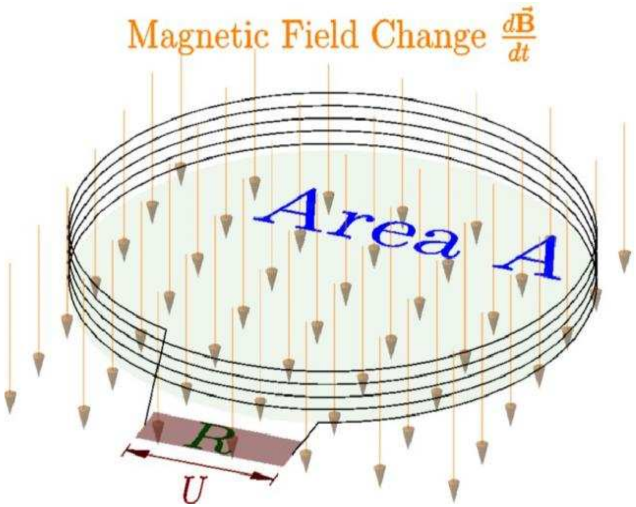
Integral along a closed line is integral of curl in the enclosed area

$$\oint_C \vec{F} \cdot d\vec{r} = \int\int_A \nabla \times \vec{F} \cdot d\vec{A}$$



Relates line integral to curl

Formally: Maxwell's third equation ...



$$-\int_A \frac{\partial \vec{B}}{\partial t} d\vec{A} = \underbrace{\int_A \nabla \times \vec{E}\ d\vec{A} = \oint_C \vec{E} \cdot d\vec{r}}_{Stoke's\ formula}$$

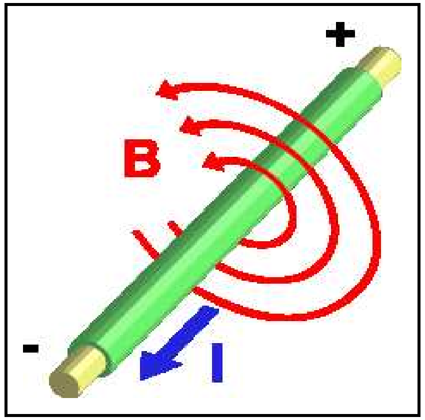
$$\underbrace{-\int_A \frac{\partial \vec{B}}{\partial t} d\vec{A} = \int_A \nabla \times \vec{E}\ d\vec{A}}_{same\ differentials} = \oint_C \vec{E} \cdot d\vec{r}$$

Changing magnetic field through an area induces electric field in coil around the area (Faraday)

$$-\frac{\partial \vec{B}}{\partial t} = \nabla \times \vec{E}$$

Maxwell's fourth equation (part 1) ...

From Ampere's law, for example current density $\vec{j}$:



$$\int\int_A \nabla \times \vec{B} \; d\vec{A} \; = \; \oint_C \vec{B} \cdot d\vec{r} \; = \; \mu_0 \vec{I}$$

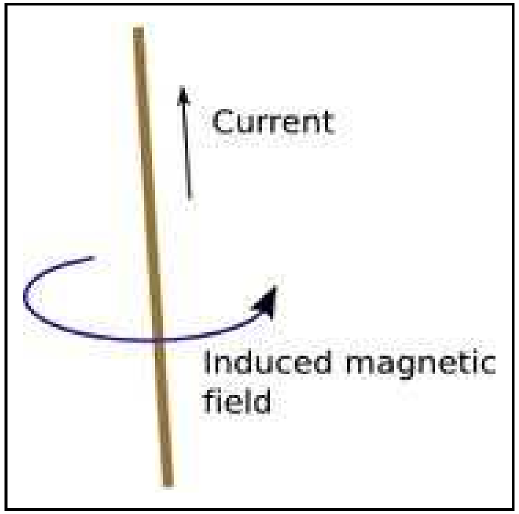
<u>Static</u> electric current induces circular magnetic field

Using the same argument as before:

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{j}}$$

Maxwell's fourth equation - application

For a static electric current I in a single wire we get Biot-Savart law (we have used Stoke's theorem and area of a circle $A = r^2 \cdot \pi$, we can easily do the integral):



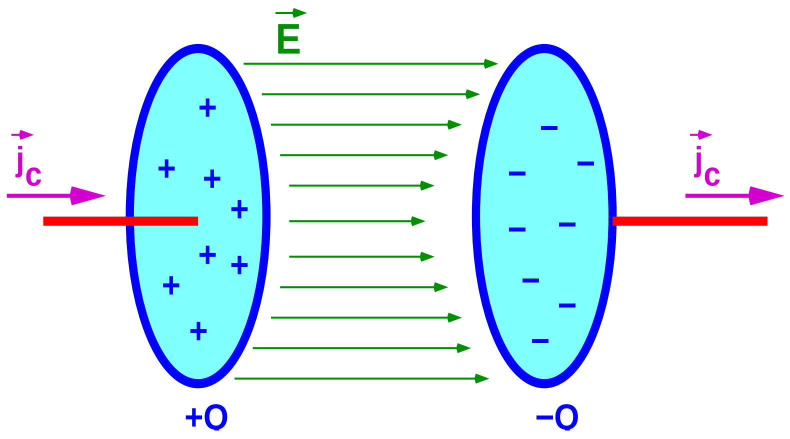
$$\vec{B} = \frac{\mu_0}{4\pi} \oint \vec{I} \cdot \frac{\vec{r} \cdot d\vec{r}}{r^3}$$

$$\vec{B} = \frac{\mu_0}{2\pi} \frac{\vec{I}}{r}$$

For magnetic field calculations in electromagnets

Maxwell's fourth equation (part 2)...

Charging capacitor: Current enters left plate - leaves from right plate, produces a change of flux (current changes with time until fully charged)



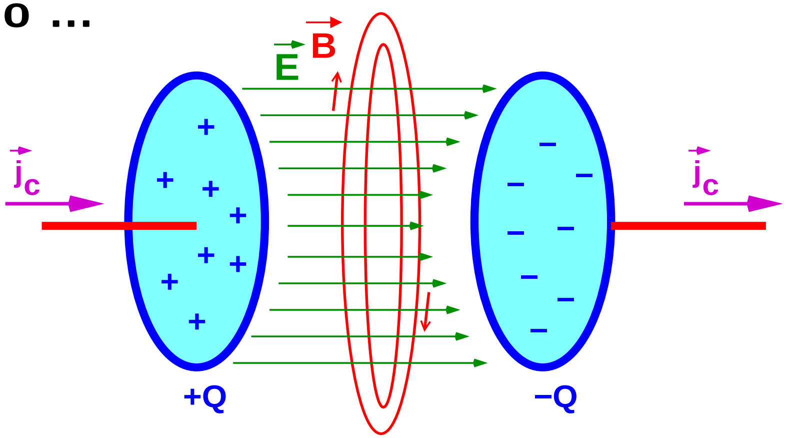
$$\vec{j}_c = \frac{d\Phi}{dt}$$

Not a current from <u>moving charges</u>

But a current from <u>time varying electric fields</u>

→ **Defining a Displacement Current** $\vec{j}_d = \vec{j}_c$

Displacement current I_d produces magnetic field, just like "actual currents" do ...



$$\vec{j}_c = \frac{d\Phi}{dt}$$

➡ **Time varying electric field induces circular magnetic field (using the current density $\vec{j}_c$**

$$\nabla \times \vec{B} = \mu_0 \vec{j}_c = \epsilon_0 \mu_0 \frac{d\vec{E}}{dt}$$

Maxwell's complete fourth equation ...

Magnetic fields $\vec{B}$ can be generated by two ways:

$\nabla \times \vec{B} = \mu_0 \vec{j}$ **(electrical current)**

$\nabla \times \vec{B} = \mu_0 \vec{j}_d = \epsilon_0 \mu_0 \dfrac{\partial \vec{E}}{\partial t}$ **(changing electric field)**

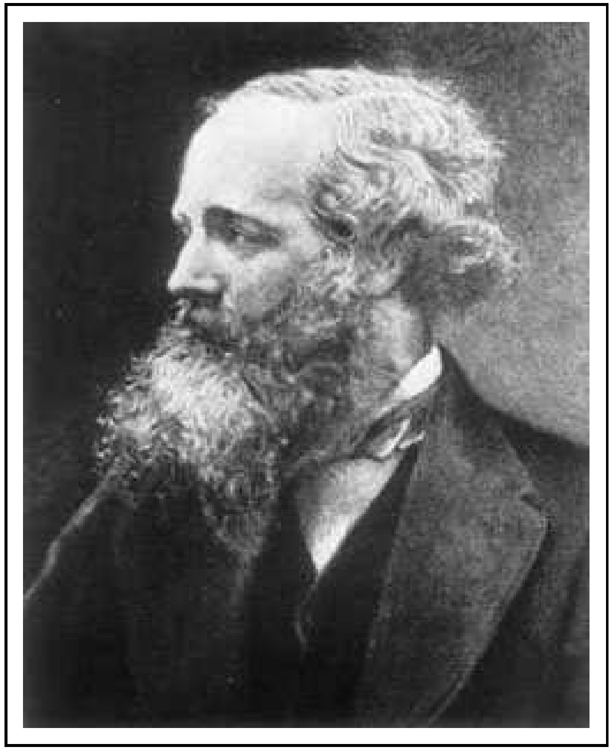
or putting them together:

$$\nabla \times \vec{B} = \mu_0 (\vec{j} + \vec{j}_d) = \mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

or in integral form:

$$\int_A \nabla \times \vec{B} \cdot d\vec{A} = \int_A \left(\mu_0 \vec{j} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A} \ = \ \mu_0 I \ + \ \epsilon_0 \mu_0 \frac{d\Phi}{dt}$$

Summary: Maxwell's Equations



$$\int_A \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}$$

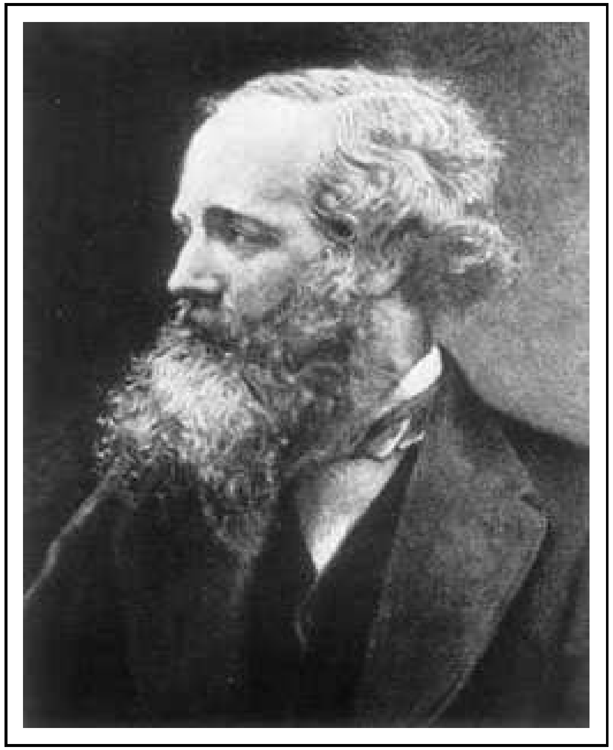
$$\int_A \vec{B} \cdot d\vec{A} = 0$$

$$\oint_C \vec{E} \cdot d\vec{r} = -\int_A \left(\frac{d\vec{B}}{dt} \right) \cdot d\vec{A}$$

$$\oint_C \vec{B} \cdot d\vec{r} = \int_A \left(\mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt} \right) \cdot d\vec{A}$$

Written in Integral form

Summary: Maxwell's Equations



$$\nabla \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

Written in Differential form

Something on potentials (needed in lecture on Relativity):

We have introduced (scalar) potential ϕ and (vector) potential $\vec{A}$

Since div $\vec{B} = 0$, **we can write $\vec{B}$ using a (vector) potential $\vec{A}$:**

$$\vec{B} \;=\; \vec{\nabla} \times \vec{A} \;=\; \text{curl } \vec{A}$$

combining Maxwell(I) + Maxwell(III):

$$\vec{E} \;=\; -\vec{\nabla}\phi \;-\; \frac{d\vec{A}}{dt}$$

➡ **Fields are derivatives of scalar and vector potentials $\Phi(x, y, z)$ and $\vec{A}(x, y, z)$**

(<u>absolute values</u> of potentials Φ and $\vec{A}$ can <u>not</u> be measured ..)

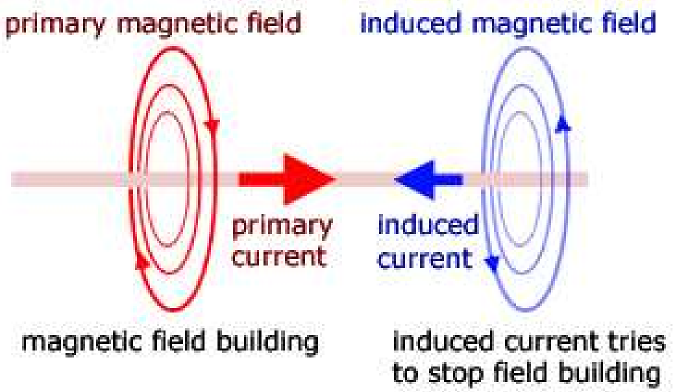
Maxwell's Equations - compact

1. **Electric fields $\vec{E}$ are generated by charges and proportional to total charge**
2. **Magnetic monopoles do <u>not</u> exist**
3. **Changing magnetic <u>flux</u> generates circular electric fields/currents**

4.1 Changing electric <u>flux</u> generates circular magnetic fields

4.2 Static electric current generates circular magnetic fields

Powering and self-induction



- Primary magnetic flux $\vec{B}$ changes with changing current

➡ Induces an electric field, resulting in a current and induced magnetic field $\vec{B}_i$

➡ Induced current will oppose change of primary current

➡ We want to change a current to ramp a magnet ...

Have to overcome this counteraction, applying a sufficient Voltage

Ramp rate defines required Voltage:

$$U = -L\frac{\partial I}{\partial t}$$

Inductance L in Henry (H)

Example:

- **Required ramp rate: 10 A/s**
- **With L = 15.1 H per powering sector**

→ **Required Voltage is $\approx$ 150 V**

Applications of Maxwell's Equations

- **Lorentz force, motion in EM fields**
  - **Motion in electric fields**
  - **Motion in magnetic fields**
- **EM waves (in vacuum and in material)**
- **Boundary conditions**
- **EM waves in cavities and wave guides**

Lorentz force on charged particles

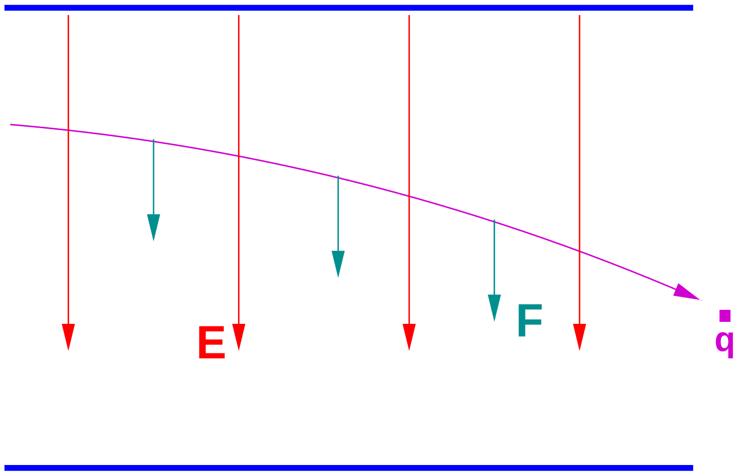
Moving ($\vec{v}$) charged (q) particles in electric ($\vec{E}$) and magnetic ($\vec{B}$) fields experience a force $\vec{f}$ like (Lorentz force):

$$\vec{f} \;=\; q \cdot (\vec{E} \;+\; \vec{v} \;\times\; \vec{B})$$

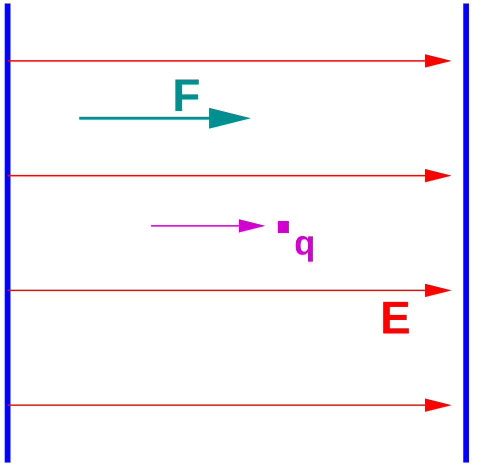
for the equation of motion we get (using Newton's law and relativistic γ);

$$\frac{d}{dt}(m_0 \gamma \vec{v}) \;=\; \vec{f} \;=\; q \cdot (\vec{E} \;+\; \vec{v} \;\times\; \vec{B})$$

Motion in electric fields



$\vec{v} \perp \vec{E}$



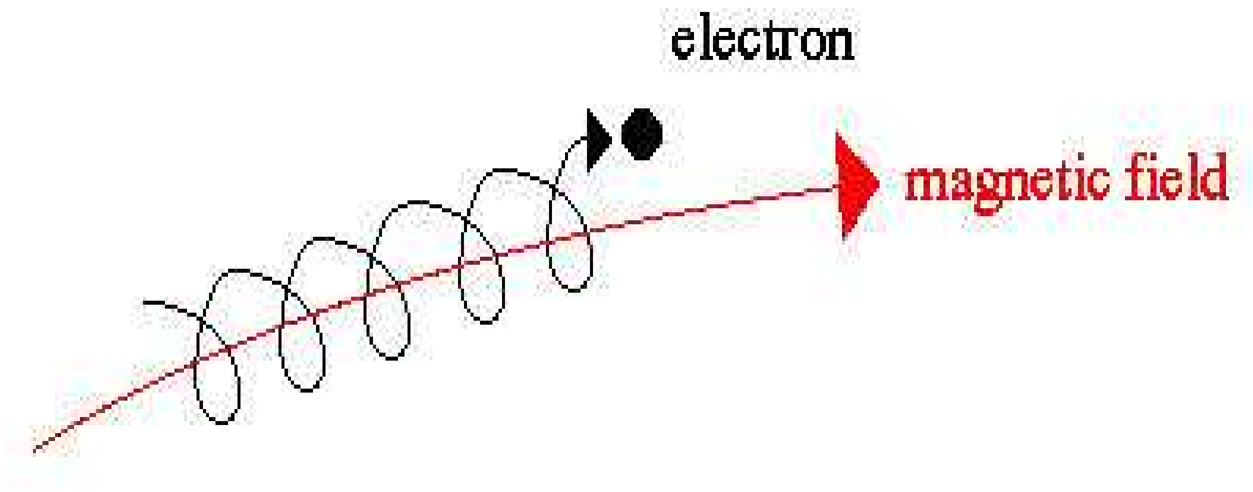
$\vec{v} \parallel \vec{E}$

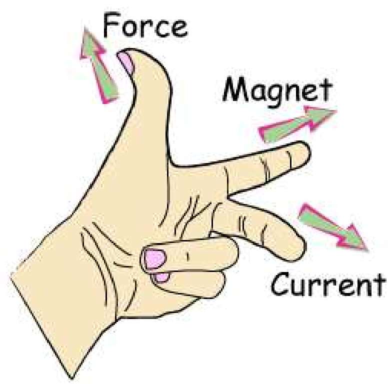
Assume no magnetic field:

$$\frac{d}{dt}(m_0 \gamma \vec{v}) = \vec{f} = q \cdot \vec{E}$$

Force always in direction of field $\vec{E}$, also for particles at rest.

Motion in magnetic fields





Assume first no electric field:

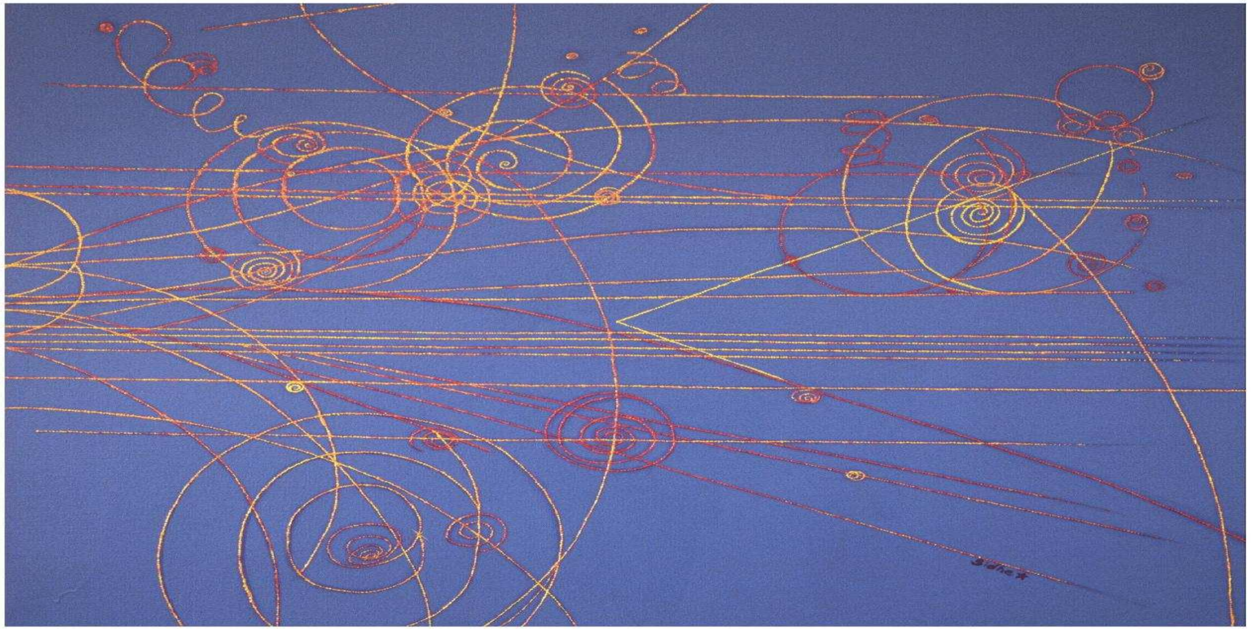
$$\frac{d}{dt}(m_0\gamma\vec{v}) \;=\; \vec{f} \;=\; q\cdot\vec{v}\times\vec{B}$$

Force is perpendicular to both, $\vec{v}$ **and** $\vec{B}$

No forces on particles at rest (because there is <u>no</u> field !)

Particles will spiral around the magnetic field lines ...

Motion in magnetic fields



- **Magnetic field perpendicular to motion**
- **Bending radius depends on momentum**
- **Bending radius depends on charge**

Practical units:

$$B[T] \cdot \rho[m] \quad = \quad \frac{p[eV]}{c[m/s]}$$

Example LHC:

B = 8.33 T, p = 7000 GeV/c ➡ ρ **= 2804 m**

Energy in electric and magnetic fields

Energy density in electric field:

$$U_E = \frac{\text{energy}}{\text{volume}} = \frac{1}{2}\epsilon_0 E^2$$

Energy density in magnetic field:

$$U_B = \frac{\text{energy}}{\text{volume}} = \frac{1}{2}\frac{B^2}{\mu_0}$$

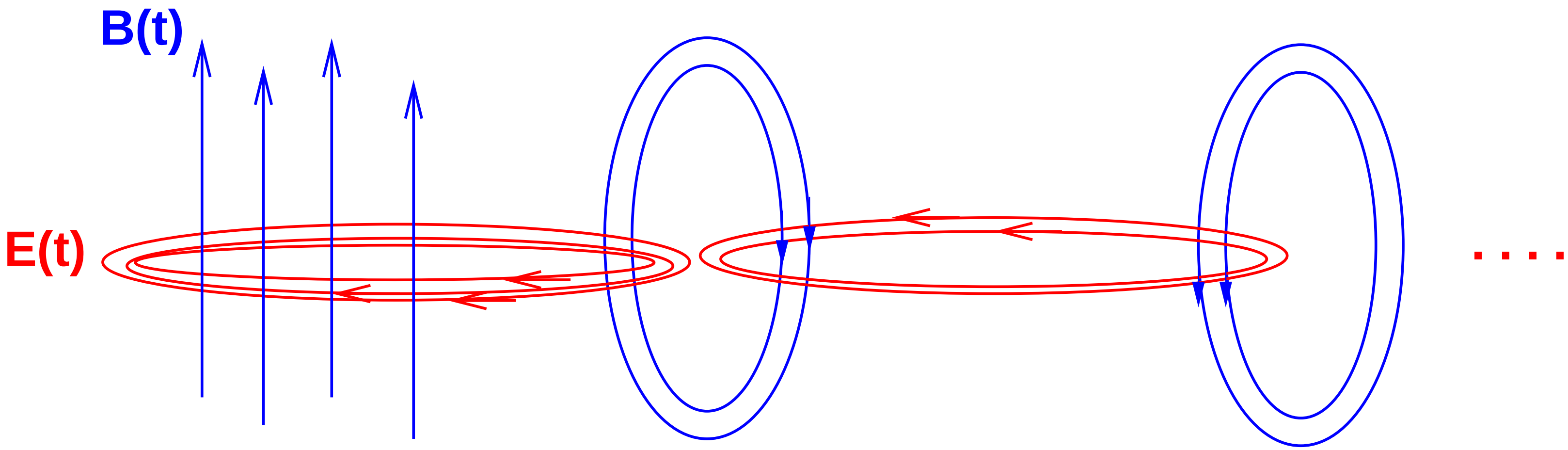
Example: B **= 5·10^{-5} T**

→ $U_B \approx$ **1 mJ/m^3**

Use of static fields (some examples, incomplete)

- Magnetic fields
  - Bending magnets
  - Focusing magnets (quadrupoles)
  - Correction magnets (sextupoles, octupoles, orbit correctors, ..)
- Electric fields
  - Electrostatic separators (beam separation in particle-antiparticle colliders)
  - Very low energy machines
- What about non-static, time-varying fields ?

Time Varying Fields - (Maxwell 1864)



Time varying magnetic fields produce circular electric fields

Time varying electric fields produce circular magnetic fields

➡ Can produce self-sustaining, propagating fields (i.e. waves)

➡ Rather naive picture (see "Relativity") but useful

Electromagnetic waves in vacuum

Vacuum: only fields, no charges ($\rho = 0$), no current ($j = 0$) ...

From: $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$$\begin{aligned} \Longrightarrow \quad \nabla \times (\nabla \times \vec{E}) &= -\nabla \times (\tfrac{\partial \vec{B}}{\partial t}) \\ \Longrightarrow \quad -(\nabla^2 \vec{E}) &= -\tfrac{\partial}{\partial t}(\nabla \times \vec{B}) \\ \Longrightarrow \quad -(\nabla^2 \vec{E}) &= -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \end{aligned}$$

$$\nabla^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{E}}{\partial t^2}$$

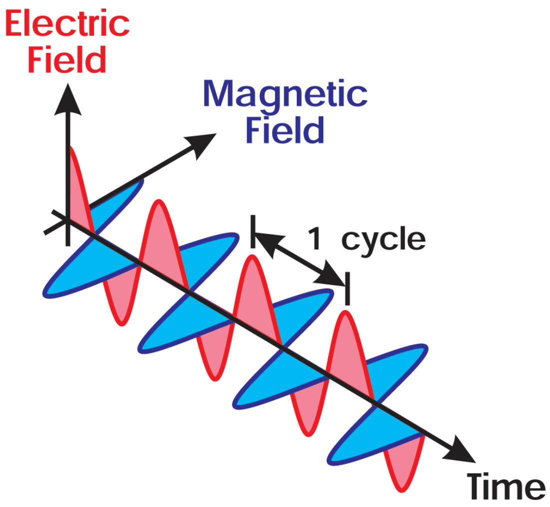
Similar expression for the magnetic field:

$$\nabla^2 \vec{B} = \frac{1}{c^2} \frac{\partial^2 \vec{B}}{\partial t^2} = \mu_0 \cdot \epsilon_0 \cdot \frac{\partial^2 \vec{B}}{\partial t^2}$$

➡ **Equation for a plane wave with velocity:** $c = \dfrac{1}{\sqrt{\mu_0 \cdot \epsilon_0}}$

Electromagnetic waves

$$\vec{E} = \vec{E_0} e^{i(\omega t - \vec{k}\cdot\vec{x})}$$
$$\vec{B} = \vec{B_0} e^{i(\omega t - \vec{k}\cdot\vec{x})}$$



Important quantities :

$|\vec{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{c}$ $\quad$ **(propagation vector)**

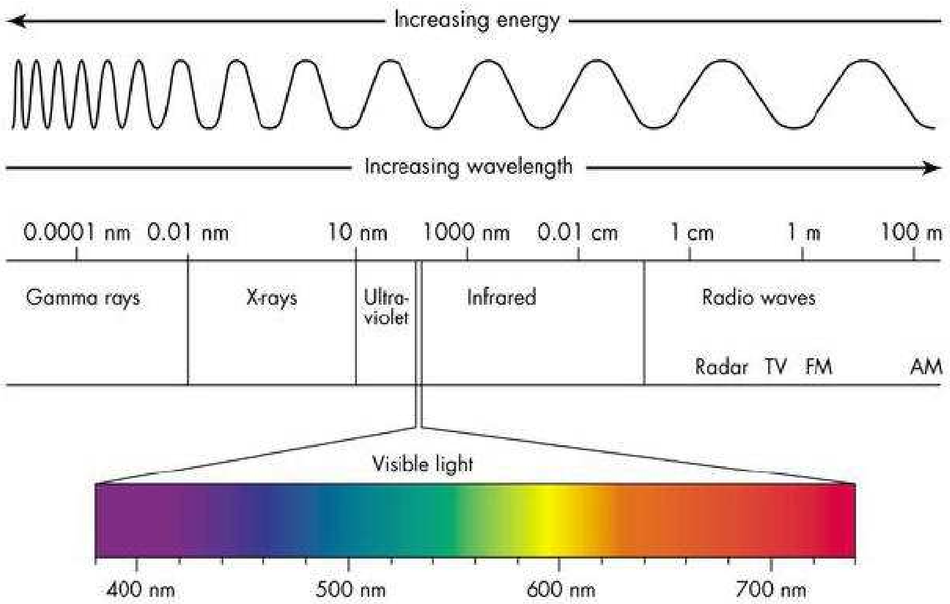
$\lambda =$ **(wave length, 1 cycle)**

$\omega =$ **(frequency** $\cdot\, 2\pi$**)**

Magnetic and electric fields are transverse to direction of propagation:

$\vec{E} \perp \vec{B} \perp \vec{k}$

Spectrum of Electromagnetic waves



Example: yellow light ➡ $\approx\ 5 \cdot 10^{14}$ **Hz (i.e.** $\approx$ **2 eV !)**

gamma rays ➡ $\leq\ 3 \cdot 10^{21}$ **Hz (i.e.** $\leq$ **12 MeV !)**

LEP (SR) ➡ $\leq\ 2 \cdot 10^{20}$ **Hz (i.e.** $\approx$ **0.8 MeV !)**

Waves hitting material

Need to look at the behaviour of electromagnetic fields at <u>boundaries between different materials</u> (air-glass, air-water, vacuum-metal, ...).

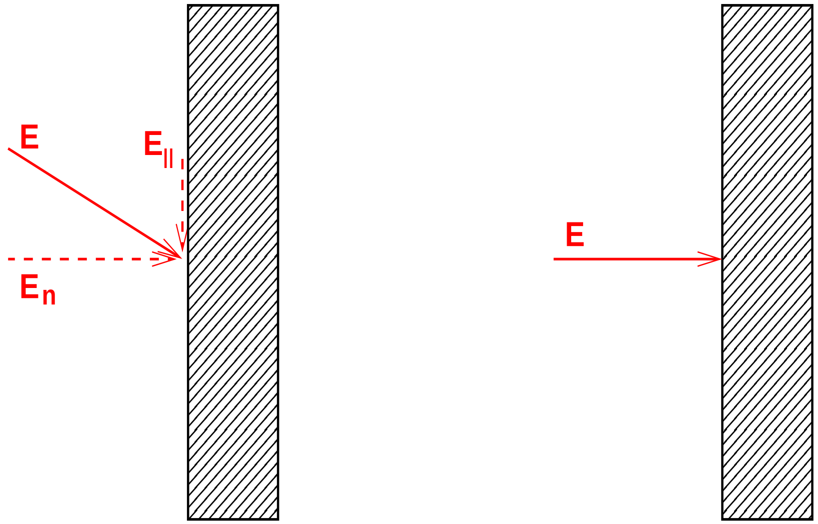
Important for highly conductive materials, e.g.:

- **RF systems**
- **Wave guides**
- **Impedance calculations**

Can be derived from Maxwell's equations, here only the results !

Boundary conditions: air and conductor

A simple case ($\vec{E}$-fields on a conducting surface):



- **Field parallel to surface $E_{\parallel}$ cannot exist (it would move charges and we get a surface current):** $E_{\parallel} = 0$
- **Only field normal to surface E_n is possible**
- **All conditions for $\vec{E}, \vec{D}, \vec{H}, \vec{B}$ can be derived from Maxwell's equations (see bibliography, e.g. R.P.Feynman or J.D.Jackson)**

Summary: Boundary conditions for fields

Electromagnetic fields at boundaries between different materials with different permittivity and permeability ($\epsilon^a, \epsilon^b, \mu^a, \mu^b$).

The requirements for the components are (summary of the results, not derived here !):

- $(E^a_{\|} \;=\; E^b_{\|}), (E^a_n \;\neq\; E^b_n)$
- $(D^a_{\|} \;\neq\; D^b_{\|}), (D^a_n \;=\; D^b_n)$
- $(H^a_{\|} \;=\; H^b_{\|}), (H^a_n \;\neq\; H^b_n)$
- $(B^a_{\|} \;\neq\; B^b_{\|}), (B^a_n \;=\; B^b_n)$

Conditions are used to compute reflection, refraction and refraction index n.

Extreme case: ideal conductor

For an ideal conductor (i.e. no resistance) we must have:

$$\vec{E}_{||} = 0, \quad \vec{B}_n = 0$$

otherwise the surface current becomes infinite

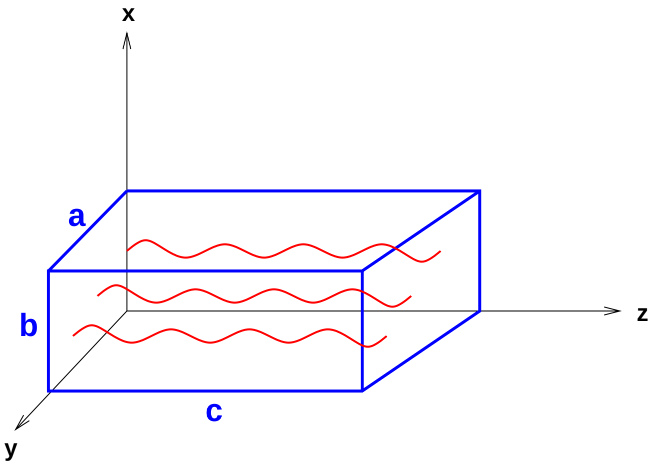
This implies:

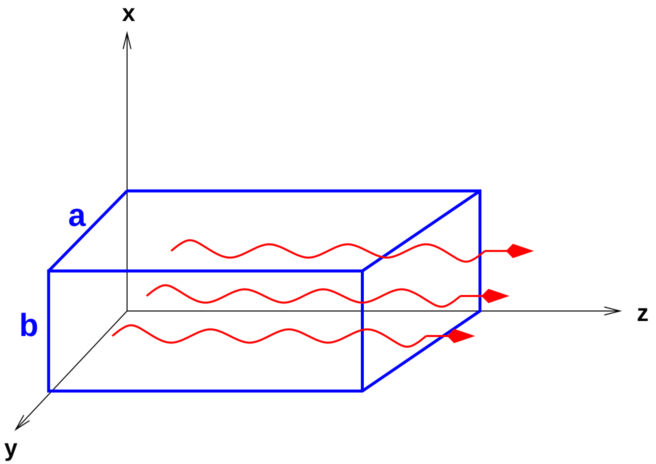
- **All energy of an electromagnetic wave is reflected from the surface of an ideal conductor.**
- **Fields at any point in the conductor are zero.**
- **Only some fieldpatterns are allowed in waveguides and RF cavities**

A very nice lecture in R.P.Feynman, Vol. II

Examples: cavities and wave guides

Rectangular, conducting cavities and wave guides (schematic) with dimensions $a \times b \times c$ and $a \times b$:





- **RF cavity, fields can persist and be stored (reflection !)**
- **Plane waves can propagate along wave guides, here in z-direction**

(here just the basics, many details in "RF Systems" by Frank Tecker)

Fields in RF cavities

Assume a rectangular RF cavity ($a,\ b,\ c$), ideal conductor.

Without derivations, the components of the fields are:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

$$E_z = E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_x = \frac{i}{\omega}(E_{y0}k_z - E_{z0}k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_y = \frac{i}{\omega}(E_{z0}k_x - E_{x0}k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot cos(k_z z) \cdot e^{-i\omega t}$$

$$B_z = \frac{i}{\omega}(E_{x0}k_y - E_{y0}k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot sin(k_z z) \cdot e^{-i\omega t}$$

Consequences for RF cavities

Field must be zero at conductor boundary, only possible under the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

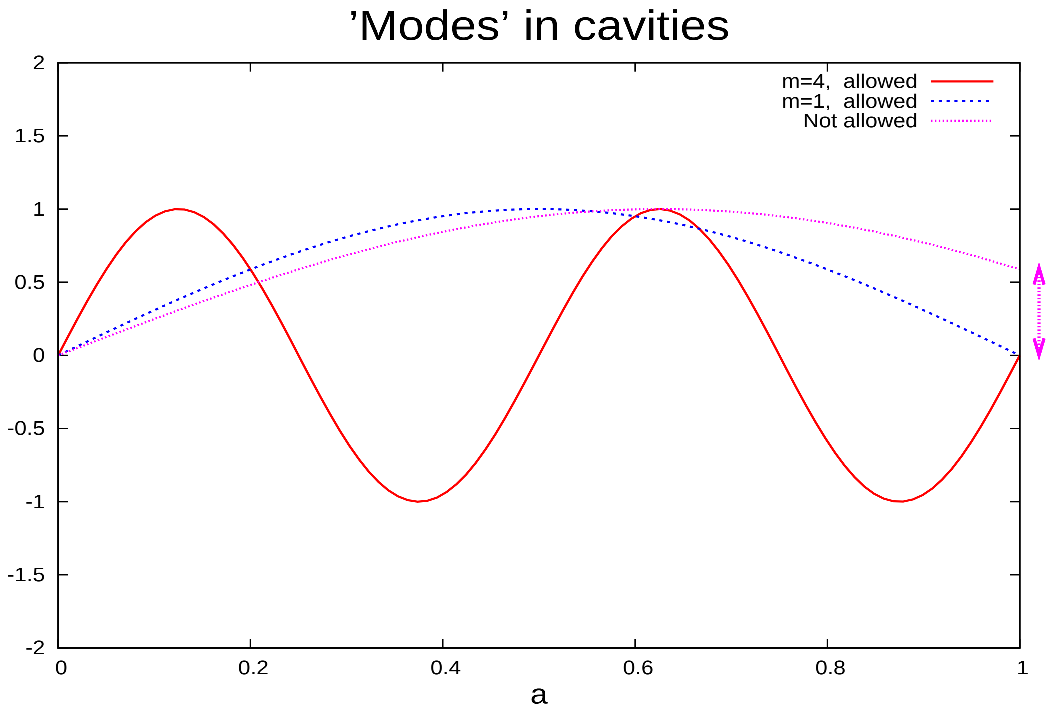
and for k_x, k_y, k_z **we can write:**

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b}, \qquad k_z = \frac{m_z \pi}{c},$$

The integer numbers m_x, m_y, m_z **are called mode numbers, important for shape of cavity !**

It means that a half wave length $\lambda/2$ **must always fit exactly the size of the cavity.**

Allowed modes - electric fields



- **Only modes which 'fit' into the cavity are allowed**
- $\frac{\lambda}{2} = \frac{a}{4}$, $\frac{\lambda}{2} = \frac{a}{1}$, $\frac{\lambda}{2} = \frac{a}{0.8}$
- **No electric field at boundaries, wave must have "nodes" at the boundaries**

Similar considerations lead to (propagating) solutions in (rectangular) wave guides:

$$E_x = E_{x0} \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$
$$E_y = E_{y0} \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$
$$E_z = i \cdot E_{z0} \cdot sin(k_x x) \cdot sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$

$$B_x = \frac{1}{\omega}(E_{y0} k_z - E_{z0} k_y) \cdot sin(k_x x) \cdot cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$
$$B_y = \frac{1}{\omega}(E_{z0} k_x - E_{x0} k_z) \cdot cos(k_x x) \cdot sin(k_y y) \cdot e^{i(k_z z - \omega t)}$$
$$B_z = \frac{1}{i \cdot \omega}(E_{x0} k_y - E_{y0} k_x) \cdot cos(k_x x) \cdot cos(k_y y) \cdot e^{i(k_z z - \omega t)}$$

This part is new : $e^{i(k_z z)}$

Consequences for wave guides

Similar considerations as for cavities, no field at boundary.
We must satisfy again the condition:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2}$$

This leads to modes like (no boundaries in direction of propagation z):

$$k_x = \frac{m_x \pi}{a}, \qquad k_y = \frac{m_y \pi}{b},$$

The numbers m_x, m_y are called mode numbers for planar waves in wave guides !

Re-writing the condition as:

$$k_z^2 = \frac{\omega^2}{c^2} - k_x^2 - k_y^2 \quad \Rightarrow \quad k_z = \sqrt{\frac{\omega^2}{c^2} - k_x^2 - k_y^2}$$

Propagation without losses requires k_z to be real, i.e.:

$$\frac{\omega^2}{c^2} > k_x^2 + k_y^2 = (\frac{m_x \pi}{a})^2 + (\frac{m_y \pi}{b})^2$$

which defines a cut-off frequency ω_c. For lowest order mode:

$$\omega_c = \frac{\pi \cdot c}{a}$$

- **Above cut-off frequency: propagation without loss**
- **At cut-off frequency: standing wave**
- **Below cut-off frequency: attenuated wave (means it does not "really fit" and k is complex).**

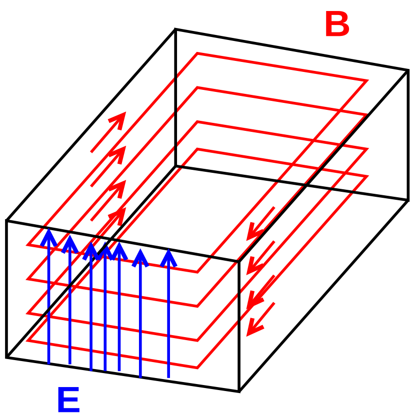
Classification of modes:

Transverse electric modes (TE): $E_z = 0 \quad H_z \neq 0$

Transverse magnetic modes (TM): $E_z \neq 0 \quad H_z = 0$

Transverse electric-magnetic modes (TEM): $E_z = 0 \quad H_z = 0$

(Not all of them can be used for acceleration ... !)



Note (here a TE mode) :

Electric field lines end at boundaries

Magnetic field lines appear as "loops"

Other case: finite conductivity

Starting from Maxwell equation:

$$\nabla \times \vec{B} = \mu \vec{j} \; + \; \mu\epsilon \frac{d\vec{E}}{dt} \; = \; \underbrace{\overbrace{\sigma \cdot \vec{E}}^{\vec{j}}}_{Ohm's\ law} + \mu\epsilon \frac{d\vec{E}}{dt}$$

Wave equations:

$$\vec{E} \; = \; \vec{E_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \qquad \vec{B} \; = \; \vec{B_0} e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

We want to know k, applying the calculus to the wave equations we have:

$$k^2 \; = \; \frac{\omega^2}{c^2} \; - \; \underbrace{i\omega\sigma\mu}_{new}$$

Consequence → Skin Depth

Electromagnetic waves can now penetrate into the conductor !

For a good conductor $\sigma \gg \omega\epsilon$:

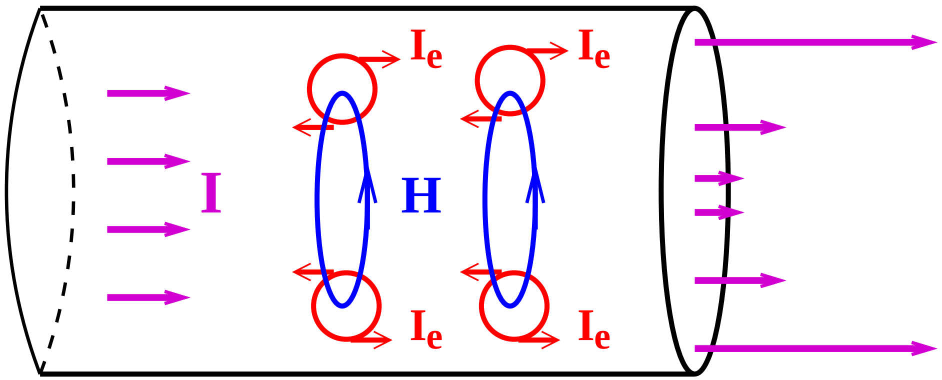
$$k^2 \approx -i\omega\mu\sigma \quad \rightarrow \quad k \approx \sqrt{\frac{\omega\mu\sigma}{2}}(1+i) = \frac{1}{\delta}(1+i)$$

$$\boxed{\delta = \sqrt{\frac{2}{\omega\mu\sigma}}}$$

is the Skin Depth

- **High frequency currents "avoid" penetrating into a conductor, flow near the surface**
- **Penetration depth small for large conductivity**

"Explanation" - inside a conductor (very schematic)



eddy currents from changing $\vec{H}$-field: $\nabla \times \vec{E} \;=\; \mu_0 \frac{d\vec{H}}{dt}$

Cancel current flow in the centre of the conductor $\mathbf{I} - \mathbf{I_e}$

Enforce current flow near the "skin" (surface) $\mathbf{I} + \mathbf{I_e}$

Q: Why are high frequency cables thin ??

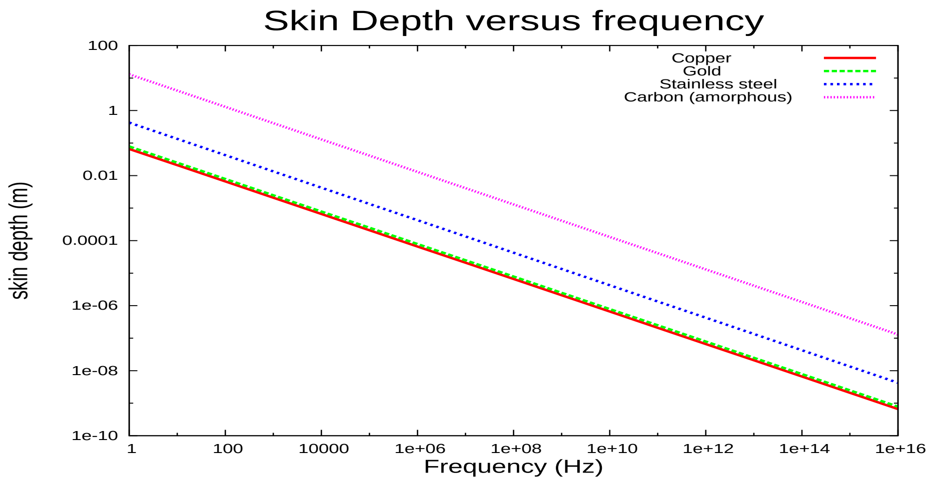
Attenuated waves - penetration depth

- **Waves incident on conducting material are attenuated**
- **Is basically Skin depth, (attenuation to 1/e)**

Wave form:

$$e^{i(kz-\omega t)} \;=\; e^{i((1+i)z/\delta-\omega t)} \;=\; e^{\frac{-z}{\delta}} \cdot e^{i(\frac{z}{\delta}-\omega t)}$$

Examples and applications



- **Skin depth Copper ($\sigma \approx 6 \cdot 10^7$ S/m):**

  1 GHz: $\delta \approx$ 2.1 μm, 50 Hz: $\delta \approx$ 10 mm

  (there is an easy way to waste your money ...)

- **Penetration depth Seawater ($\sigma \approx 4$ S/m):**

  to get $\delta \approx$ 25 - 30 m you need ➡ $\approx$ 76 Hz

  inefficient ($10^{-5} - 10^{-6}$) and very low bandwidth (0.03 bps)

- Review of basics and Maxwell's equations
- Lorentz force and motion of particles in electromagnetic fields
- Electromagnetic waves in vacuum
- Electromagnetic waves in conducting media
  - Waves in RF cavities
  - Waves in wave guides
  - Important concepts:
    mode numbers, cut-off frequency, skin depth

WE DID IT ...

But still a few problems to sort out →

- BACKUP SLIDES -

Interlude and Warning !!

Maxwell's equation can be written in other forms.

Often used: cgs (Gaussian) units instead of SI units, example:

Starting from (SI):

$$\nabla \cdot \vec{E} \;=\; \frac{\rho}{\epsilon_0}$$

we would use:

$$\vec{E}_{cgs} \;=\; \frac{1}{c} \;\cdot \vec{E}_{SI} \qquad \text{and} \qquad \epsilon_0 \;=\; \frac{1}{4\pi \cdot c}$$

and arrive at (cgs):

$$\nabla \cdot \vec{E} \;=\; 4\pi \cdot \rho$$

Beware: there are more different units giving: $\nabla \cdot \vec{E} \;=\; \rho$

Electromagnetic fields in material

In vacuum:

$$\vec{D} = \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_0 \cdot \vec{H}$$

In a material:

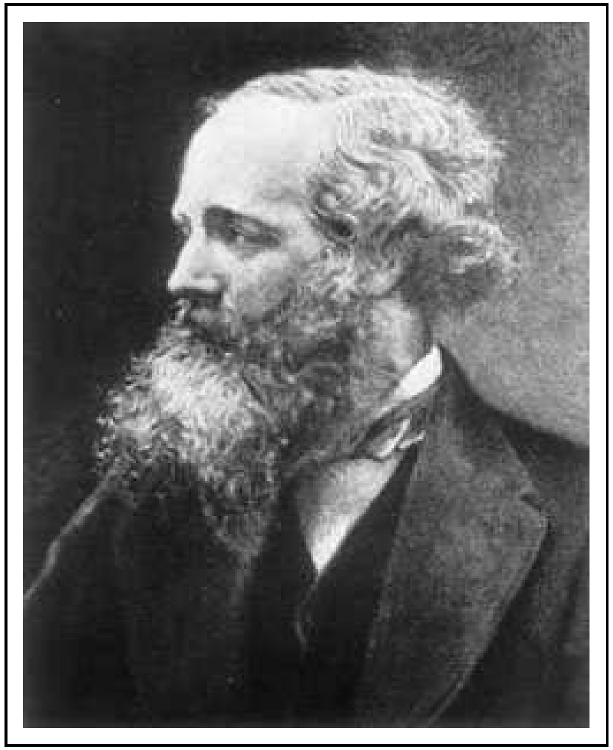
$$\vec{D} = \epsilon_r \cdot \epsilon_0 \cdot \vec{E}, \qquad \vec{B} = \mu_r \cdot \mu_0 \cdot \vec{H}$$

ϵ_r **is <u>relative</u> permittivity** $\approx \quad [1 - 10^5]$

μ_r **is <u>relative</u> permeability** $\approx \quad [0(!) - 10^6]$

Origin: polarization and Magnetization

Once more: Maxwell's Equations



$$\nabla \vec{D} = \rho$$
$$\nabla \vec{B} = 0$$
$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt}$$
$$\nabla \times \vec{H} = \vec{j} + \frac{d\vec{D}}{dt}$$

Re-factored in terms of the free current density $\vec{j}$ and free charge density ρ ($\mu_0 = 1, \epsilon_0 = 1$):

Some popular confusion ..

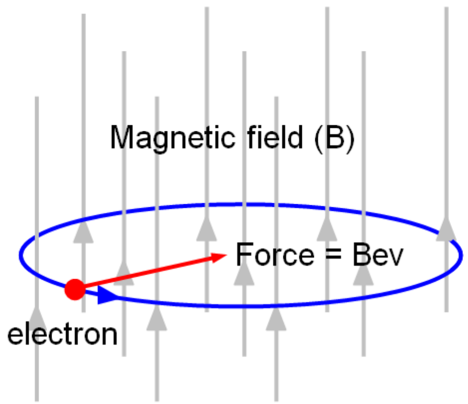
V.F.A.Q: why this strange mixture of $\vec{E}, \vec{D}, \vec{B}, \vec{H}$??

Materials respond to an applied electric E field and an applied magnetic B field by producing their own internal charge and current distributions, contributing to E and B. Therefore H and D fields are used to re-factor Maxwell's equations in terms of the free current density $\vec{j}$ and free charge density ρ:

$$\begin{aligned}\vec{H} &= \frac{\vec{B}}{\mu_0} - \vec{M} \\ \vec{D} &= \epsilon_0 \vec{E} + \vec{P}\end{aligned}$$

$\vec{M}$ **and** $\vec{P}$ **are** $Magnetization$ **and** $Polarisation$ **in material**

Is that the full truth ?



If we have a circular E-field along the circle of radius R ?

→ **should get acceleration !**

Remember Maxwell's third equation:

$$\oint_C \vec{E} \cdot d\vec{r} \quad = \quad -\frac{d}{dt}\int_A \vec{B} \cdot d\vec{A}$$

$$\rightarrow 2\pi R E_\theta \quad = \quad -\frac{d\Phi}{dt}$$

Motion in magnetic fields

- This is the principle of a Betatron
  - Time varying magnetic field creates circular electric field !
  - Time varying magnetic field deflects the charge !

For a constant radius we need:

$$-\frac{m \cdot v^2}{R} = e \cdot v \cdot B \quad \rightarrow \quad B = -\frac{p}{e \cdot R}$$

$$\frac{\partial}{\partial t} B(r,t) = -\frac{1}{e \cdot R}\frac{dp}{dt}$$

$$\rightarrow \quad B(r,t) = \frac{1}{2}\frac{1}{\pi R^2}\int\int B dS$$

B-field on orbit must be half the average over the circle → Betatron condition

Other case: finite conductivity

Assume conductor with finite conductivity ($\sigma_c = \rho_c^{-1}$) , waves will penetrate into surface. Order of the skin depth is:

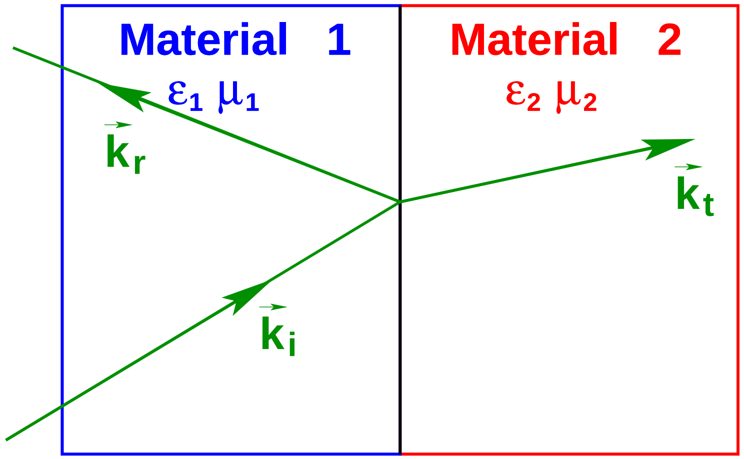
$$\delta_s = \sqrt{\frac{2\rho_c}{\mu\omega}}$$

i.e. depend on resistivity, permeability and frequency of the waves (ω).
We can get the surface impedance as:

$$Z = \sqrt{\frac{\mu}{\epsilon}} = \frac{\mu\omega}{k}$$

the latter follows from our definition of k and speed of light.
Since the wave vector k is complex, the impedance is also complex. We get a phase shift between electric and magnetic field.

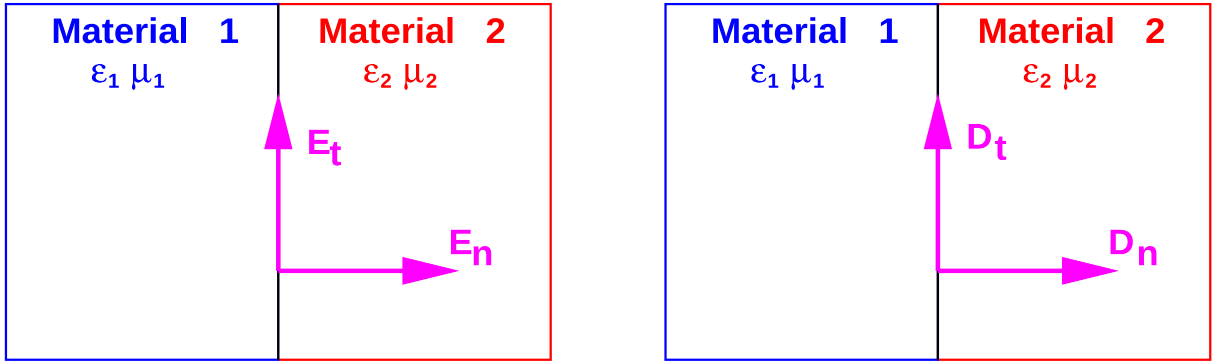
Boundary conditions for fields



What happens when an incident wave ($\vec{K}_i$) encounters a boundary between two different media ?

- **Part of the wave will be reflected ($\vec{K}_r$), part is transmitted ($\vec{K}_t$)**
- **What happens to the electric and magnetic fields ?**

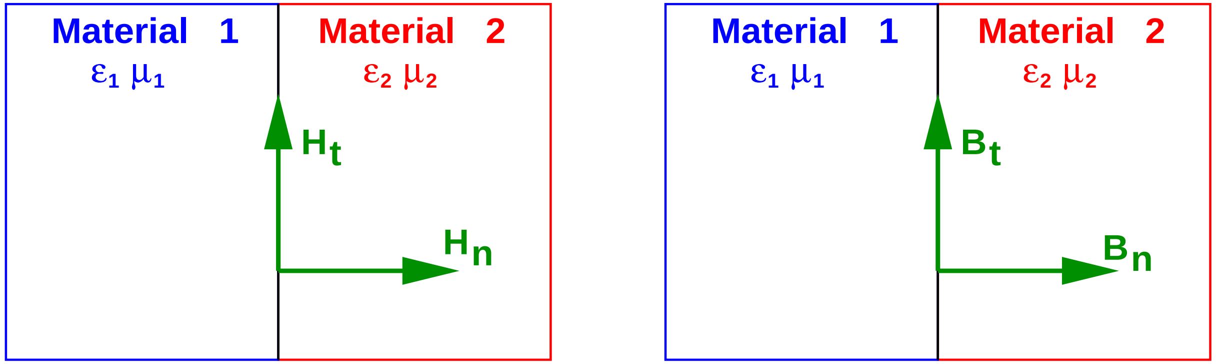
Boundary conditions for fields



Assuming no surface charges:

- **tangential $\vec{E}$-field constant across boundary** $(E_{1t} = E_{2t})$
- **normal $\vec{D}$-field constant across boundary** $(D_{1n} = D_{2n})$

Boundary conditions for fields



Assuming no surface currents:

- **tangential $\vec{H}$-field constant across boundary** $(H_{1t} = H_{2t})$
- **normal $\vec{B}$-field constant across boundary** $(B_{1n} = B_{2n})$