Transverse Beam Dynamics II

II) The State of the Art in High Energy Machines:

The Theory of Synchrotrons:
- Linear Beam Optics
- The Beam as Particle Ensemble
- Emittance and Beta-Function
- Colliding Beams & Luminosity

"... how does it work ?"
"...does it ?"
Astronomer Hill:

differential equation for motions with periodic focusing properties
„Hill‘s equation“

Example: particle motion with periodic coefficient

equation of motion: \[ x''(s) - k(s)x(s) = 0 \]

restoring force \( \neq \text{const} \),
\[ k(s) = \text{depending on the position } s \]
\[ k(s+L) = k(s), \text{ periodic function} \]

we expect a kind of quasi harmonic oscillation: amplitude & phase will depend on the position \( s \) in the ring.
7.) The Beta Function

„it is convenient to see“ ... after some beer ... general solution of Mr Hill can be written in the form:

Ansatz:

\[ x(s) = \sqrt{\epsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \]

\( \epsilon, \Phi = \) integration constants determined by initial conditions

\( \beta(s) \) periodic function given by focusing properties of the lattice ↔ quadrupoles

\[ \beta(s + L) = \beta(s) \]

\( \epsilon \) beam emittance = wozilycity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.

Scientifiquely spoken: area covered in transverse x, x’ phase space ... and it is constant !!!

\( \Psi(s) = \) „phase advance“ of the oscillation between point „0“ and „s“ in the lattice.

For one complete revolution: number of oscillations per turn „Tune“

\[ Q_y = \frac{1}{2\pi} \int \frac{ds}{\beta(s)} \]
Amplitude of a particle trajectory:

\[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \varphi) \]

Maximum size of a particle amplitude

\[ \hat{x}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \]

\( \beta \) determines the beam size ( ... the envelope of all particle trajectories at a given position “s” in the storage ring.

It reflects the periodicity of the magnet structure.
8.) Beam Emittance and Phase Space Ellipse

General solution of Hill equation

\[ (1) \quad x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos(\psi(s) + \phi) \]

\[ (2) \quad x'(s) = -\frac{\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \left\{ \alpha(s) \cos(\psi(s) + \phi) + \sin(\psi(s) + \phi) \right\} \]

From (1) we get

\[ \cos(\psi(s) + \phi) = \frac{x(s)}{\sqrt{\varepsilon} \sqrt{\beta(s)}} \]

\[ \alpha(s) = \frac{-1}{2} \beta'(s) \]

\[ \gamma(s) = \frac{1 + \alpha(s)^2}{\beta(s)} \]

Insert into (2) and solve for \( \varepsilon \)

\[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \]

* \( \varepsilon \) is a constant of the motion … it is independent of "s"

* Parametric representation of an ellipse in the \( x x' \) space

* Shape and orientation of ellipse are given by \( \alpha, \beta, \gamma \)
**Phase Space Ellipse**

Particel trajectory: \[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos \{ \psi(s) + \phi \} \]

Max. Amplitude: \[ \hat{x}(s) = \sqrt{\varepsilon \beta} \]

... put \( \hat{x}(s) \) into \[ \varepsilon = \gamma(s) x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'^2(s) \]

and solve for \( x' \)

\[ \varepsilon = \gamma \cdot \varepsilon \beta + 2\alpha \sqrt{\varepsilon \beta} \cdot x' + \beta x'^2 \]

\[ x' = -\alpha \cdot \sqrt{\varepsilon / \beta} \]

\[ \star \text{A high } \beta \text{-function means a large beam size and a small beam divergence.} \]

... et vice versa !!!

\[ \star \text{In the middle of a quadrupole } \beta = \text{maximum,} \]

\[ \alpha = \text{zero} \]

\[ x' = 0 \]

... and the ellipse is flat
**Beam Emittance and Phase Space Ellipse**

\[ x(s) = \sqrt{\epsilon} * \sqrt{\beta(s)} * \cos(\psi(s) + \varphi) \]

\[ \epsilon = \gamma(s) * x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)x'(s)^2 \]

**Liouville:** in reasonable storage rings
area in phase space is constant.

\[ A = \pi\epsilon = \text{const} \]

\( \epsilon \) beam emittance = wozilicity of the particle ensemble, intrinsic beam parameter, cannot be changed by the foc. properties.
Scientifiquely spoken: area covered in transverse \( x, x' \) phase space ... and it is constant !!!
Particle Tracking in a Storage Ring

Calculate $x$, $x'$ for each linear accelerator element according to matrix formalism

plot $x$, $x'$ as a function of "s"
... and now the ellipse:

Note for each turn $x, x'$ at a given position "s_1" and plot in the phase space diagram.
Emittance of the Particle Ensemble:
**Emittance of the Particle Ensemble:**

\[ x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cdot \cos(\Psi(s) + \phi) \]

\[ \dot{x}(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \]

*Gauß Particle Distribution:*

\[ \rho(x) = \frac{N \cdot e^{-\frac{1}{2} \sigma_x^2}}{\sqrt{2\pi}\sigma_x} \]

*particle at distance 1 \( \sigma \) from centre \( \leftrightarrow 68.3\% \) of all beam particles*

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**LHC:**

\( \beta = 180 m \)

\( \varepsilon = 5 \times 10^{-10} m \text{ rad} \)

\[ \sigma = \sqrt{\varepsilon \cdot \beta} = \sqrt{5 \times 10^{-10} m \cdot 180 m} = 0.3 mm \]

*aperture requirements: \( r_0 = 12 \times \sigma \)*
The „not so ideal“ World
Lattice Design in Particle Accelerators

1952: Courant, Livingston, Snyder:
Theory of strong focusing in particle beams
Recapitulation: ...the story with the matrices !!!

**Equation of Motion:**

\[ x'' + K x = 0 \quad K = \frac{1}{\rho^2} - k \quad \text{... hor. plane:} \]

\[ K = k \quad \text{... vert. Plane:} \]

**Solution of Trajectory Equations**

\[ \begin{pmatrix} x \\ x' \end{pmatrix}_{s_1} = M \ast \begin{pmatrix} x \\ x' \end{pmatrix}_{s_0} \]

\[ M_{drift} = \begin{pmatrix} 1 & I \\ 0 & 1 \end{pmatrix} \]

\[ M_{focus} = \begin{pmatrix} \cos(\sqrt{K} I) & \frac{1}{\sqrt{K}} \sin(\sqrt{K} I) \\ -\sqrt{K} \sin(\sqrt{K} I) & \cos(\sqrt{K} I) \end{pmatrix} \]

\[ M_{defocus} = \begin{pmatrix} \cosh(\sqrt{K} I) & \frac{1}{\sqrt{K}} \sinh(\sqrt{K} I) \\ \sqrt{K} \sinh(\sqrt{K} I) & \cosh(\sqrt{K} I) \end{pmatrix} \]

\[ M_{total} = M_{QF} \ast M_D \ast M_B \ast M_D \ast M_{QD} \ast M_D \ast \ldots \]
Geometry of the ring: \[ B \rho = \frac{p}{e} \]

\( p \) = momentum of the particle,
\( \rho \) = curvature radius

\( B\rho \) = beam rigidity

Circular Orbit: bending angle of one dipole

\[ \alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho} = \frac{Bdl}{B\rho} \]

The angle run out in one revolution must be \( 2\pi \), so for a full circle

\[ \alpha = \int \frac{Bdl}{B\rho} = 2\pi \]

\[ \int Bdl = 2\pi \frac{p}{q} \]

… defines the integrated dipole field around the machine.
Example LHC:

7000 GeV Proton storage ring
dipole magnets $N = 1232$
\[ l = 15 \text{ m} \]
$q = +1 \text{ e}$

\[
\int B \, dl \approx N \, l \, B = 2\pi \frac{p}{e}
\]

\[
B \approx \frac{2\pi \times 7000 \times 10^6 \text{ eV}}{1232 \times 15 \times 3 \times 10^8 \frac{m}{s} \cdot e} = 8.3 \text{ Tesla}
\]
10.) **Transfer Matrix M**  

... yes we had the topic already

**general solution of Hill’s equation**

\[
x(s) = \sqrt{\varepsilon} \sqrt{\beta(s)} \cos \{\psi(s) + \phi\}
\]

\[
x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta(s)}} \left[ \alpha(s) \cos \{\psi(s) + \phi\} + \sin \{\psi(s) + \phi\} \right]
\]

remember the trigonometrical gymnastics: \(\sin(a + b) = \ldots\) etc

\[
x(s) = \sqrt{\varepsilon} \sqrt{\beta_s} \left( \cos \psi_s \cos \phi - \sin \psi_s \sin \phi \right)
\]

\[
x'(s) = \frac{-\sqrt{\varepsilon}}{\sqrt{\beta_s}} \left[ \alpha_s \cos \psi_s \cos \phi - \alpha_s \sin \psi_s \sin \phi + \sin \psi_s \cos \phi + \cos \psi_s \sin \phi \right]
\]

**starting at point** \(s(0) = s_0\), **where we put** \(\Psi(0) = 0\)

\[
\cos \phi = \frac{x_0}{\sqrt{\varepsilon \beta_0}} \quad , \quad \sin \phi = -\frac{1}{\sqrt{\varepsilon}} \left( x'_0 \sqrt{\beta_0} + \frac{\alpha_0 x_0}{\sqrt{\beta_0}} \right)
\]

**inserting above ...**
\[ x(s) = \sqrt{\frac{\beta_s}{\beta_0}} \left\{ \cos \psi_s + \alpha_0 \sin \psi_s \right\} x_0 + \sqrt{\beta_s \beta_0} \sin \psi_s \right\} x_0' \]

\[ x'(s) = \frac{1}{\sqrt{\beta_s \beta_0}} \left\{ (\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s \right\} x_0 + \sqrt{\frac{\beta_0}{\beta_s}} \left\{ \cos \psi_s - \alpha_s \sin \psi_s \right\} x_0' \]

which can be expressed ... for convenience ... in matrix form

\[
\begin{pmatrix} x \\ x' \end{pmatrix}_s = M \begin{pmatrix} x \\ x' \end{pmatrix}_0
\]

\[
M = \begin{pmatrix} \sqrt{\frac{\beta_s}{\beta_0}} (\cos \psi_s + \alpha_0 \sin \psi_s) & \sqrt{\beta_s \beta_0} \sin \psi_s \\ (\alpha_0 - \alpha_s) \cos \psi_s - (1 + \alpha_0 \alpha_s) \sin \psi_s & \sqrt{\frac{\beta_0}{\beta_s}} (\cos \psi_s - \alpha_s \sin \psi_s) \end{pmatrix}
\]

* we can calculate the single particle trajectories between two locations in the ring, if we know the \( \alpha \beta \gamma \) at these positions.
* and nothing but the \( \alpha \beta \gamma \) at these positions.
* \( \cdots \) !
LHC: Lattice Design
the ARC 90° FoDo in both planes

**equipped with additional corrector coils**

- **MB**: main dipole
- **MQ**: main quadrupole
- **MQT**: Trim quadrupole
- **MQS**: Skew trim quadrupole
- **MO**: Lattice octupole (Landau damping)
- **MSCB**: Skew sextupole
- **MCDO**: Spool piece 8 / 10 pole
- **BPM**: Beam position monitor + diagnostics
A magnet structure consisting of focusing and defocusing quadrupole lenses in alternating order with nothing in.

(Nothing = elements that can be neglected on first sight: drift, bending magnets, RF structures ... and especially experiments...)

Starting point for the calculation: in the middle of a focusing quadrupole
Phase advance per cell $\mu = 45^\circ$,
$\rightarrow$ calculate the twiss parameters for a periodic solution
11.) Insertions
\[ \beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0} \]

At the end of a long symmetric drift space the beta function reaches its maximum value in the complete lattice.
-> here we get the largest beam dimension.

-> keep l as small as possible

7 sigma beam size inside a mini beta quadrupole
... clearly there is another problem.

... unfortunately ... in general high energy detectors that are installed in that drift spaces are a little bit bigger than a few centimeters ...
The Mini-β Insertion & Luminosity:

The production rate of events is determined by the cross section $\Sigma_{\text{react}}$ and a parameter $L$ that is given by the design of the accelerator: 

$$R = L \cdot \Sigma_{\text{react}} \approx 10^{-12} \, b \cdot 25 \cdot \frac{1}{10^{-15} \, b} = \text{some} \, 1000 \, H$$

The luminosity is a storage ring quality parameter and depends on beam size ($\beta$!!) and stored current

$$L = \frac{1}{4\pi e^2 f_0 \cdot b} \cdot \frac{I_1 \cdot I_2}{\sigma_x \cdot \sigma_y}$$

remember: $1b = 10^{-24} \, \text{cm}^2$
11.) Luminosity

Example: Luminosity run at LHC

\[ \beta_{x,y} = 0.55 \text{ m} \]
\[ \epsilon_{x,y} = 5 \times 10^{-10} \text{ rad m} \]
\[ \sigma_{x,y} = 17 \mu\text{m} \]
\[ f_0 = 11.245 \text{ kHz} \]
\[ n_b = 2808 \]
\[ I_p = 584 \text{ mA} \]

\[ L = \frac{1}{4 \pi e^2 f_0 n_b} \frac{I_{p1} I_{p2}}{\sigma_x \sigma_y} \]

\[ L = 1.0 \times 10^{34} \frac{1}{cm^2 s} \]
Mini-β Insertions: Betafunctions

A mini-β insertion is always a kind of special symmetric drift space. 
⇒ greetings from Liouville

the smaller the beam size
the larger the beam divergence
Mini-\(\beta\) Insertions: some guide lines

* calculate the periodic solution in the arc

* introduce the drift space needed for the insertion device (detector ...)

* put a quadrupole doublet (triplet ?) as close as possible

* introduce additional quadrupole lenses to match the beam parameters to the values at the beginning of the arc structure

parameters to be optimised & matched to the periodic solution: \(\alpha_x, \beta_x, D_x, D'_x\) \(\alpha_y, \beta_y, Q_x, Q_y\)

8 individually powered quad magnets are needed to match the insertion (... at least)
The LHC Insertions

**Inner Triplet**
- IP1
- TAS
- Q1  Q2  Q3
- D1  (1.38 T)

**Separation/Recombination**
- TAN
- D2  (3.8 T)
- Q4
- Q5
- Q6
- Q7

**Matching Quadrupoles**
- 4.5 K
- 1.9 K

**ATLAS**
- R1

**Mini β optics**
- βx, βy

**Momentum offset**
- 12.850 to 13.705
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