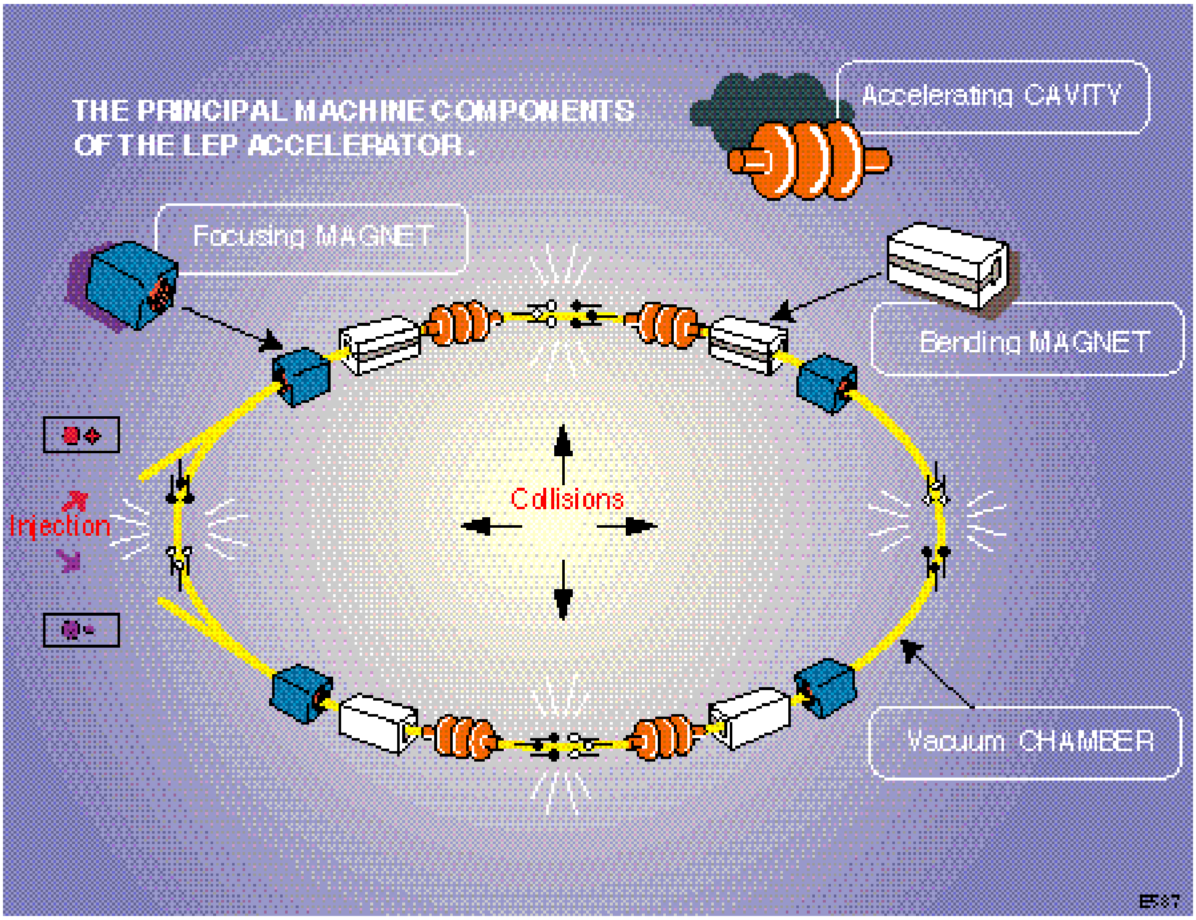
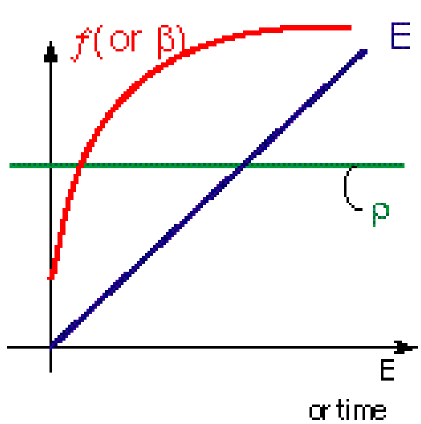
Transverse Dynamics

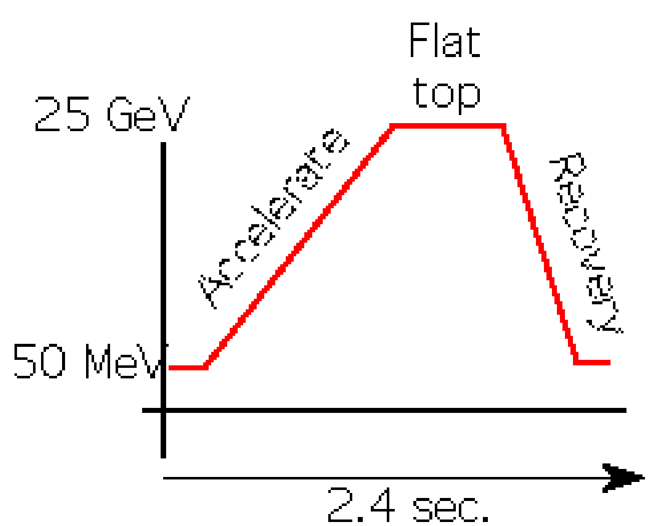
E. Wilson - CERN

- Components of a synchrotron
- Dipole Bending Magnet
- Magnetic rigidity
- Bending Magnet
- Weak focusing - gutter
- Transverse ellipse
- Fields and force in a quadrupole
- Strong focusing
- Equation of motion in transverse co-ordinates
- Twiss Matrix
- The lattice
- Dispersion
- Chromaticity

Components of a synchrotron

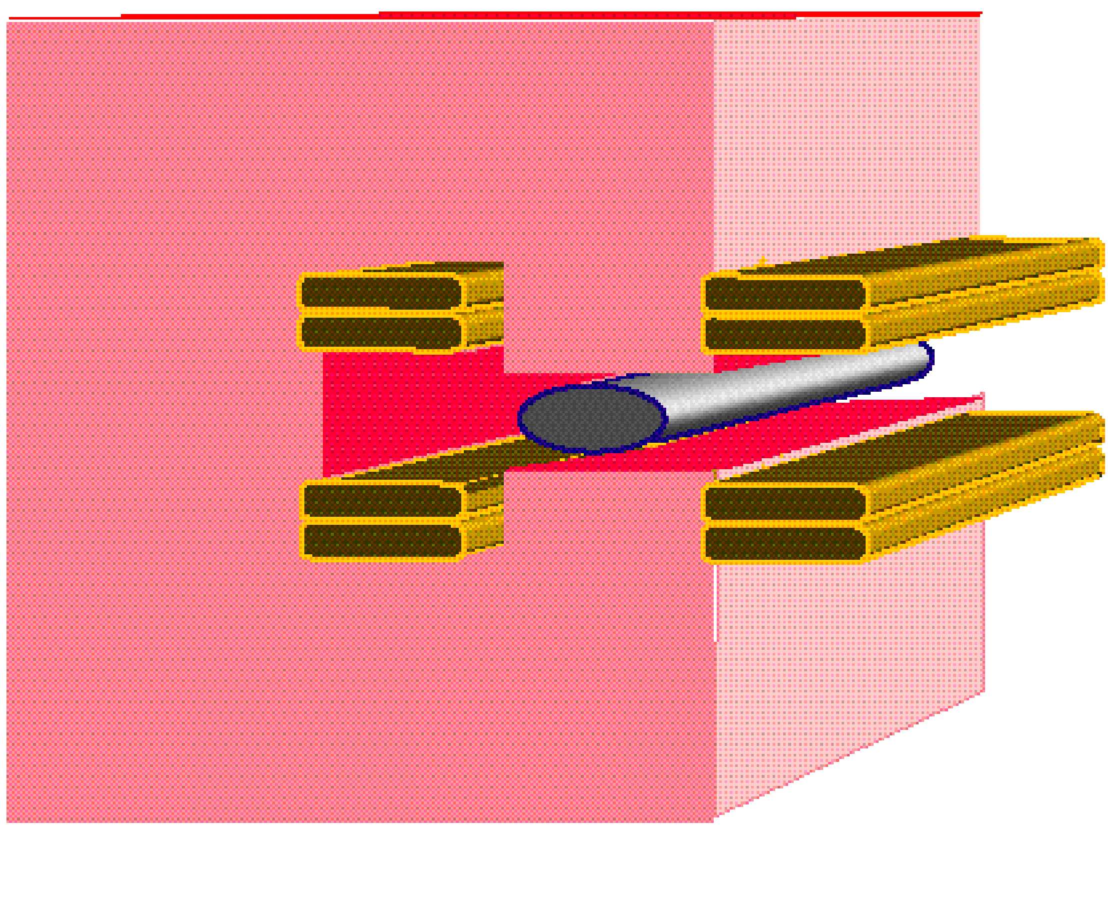




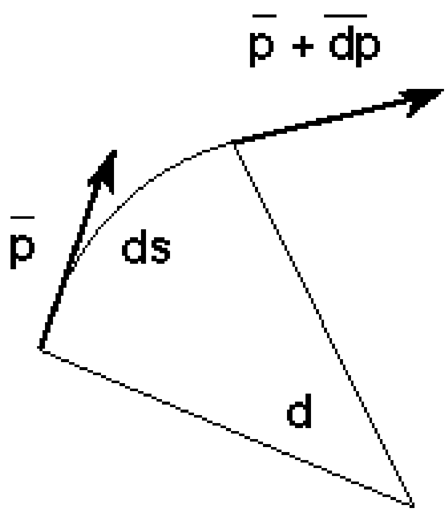


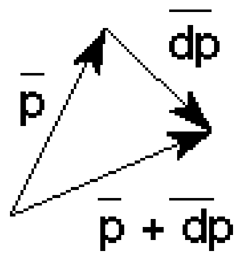
- RING.GIF, Fig. sans nom 1_PULSE, Annexe1C

Dipole Bending Magnet



Magnetic rigidity *(vectors)*





$$\frac{1}{\rho} = \frac{d\theta}{ds}$$

$$\frac{d\mathbf{p}}{dt} = |\mathbf{p}|\frac{d\theta}{dt} = |\mathbf{p}|\frac{d\theta}{ds}\frac{ds}{dt} = \frac{|\mathbf{p}|}{\rho}\frac{ds}{dt}$$

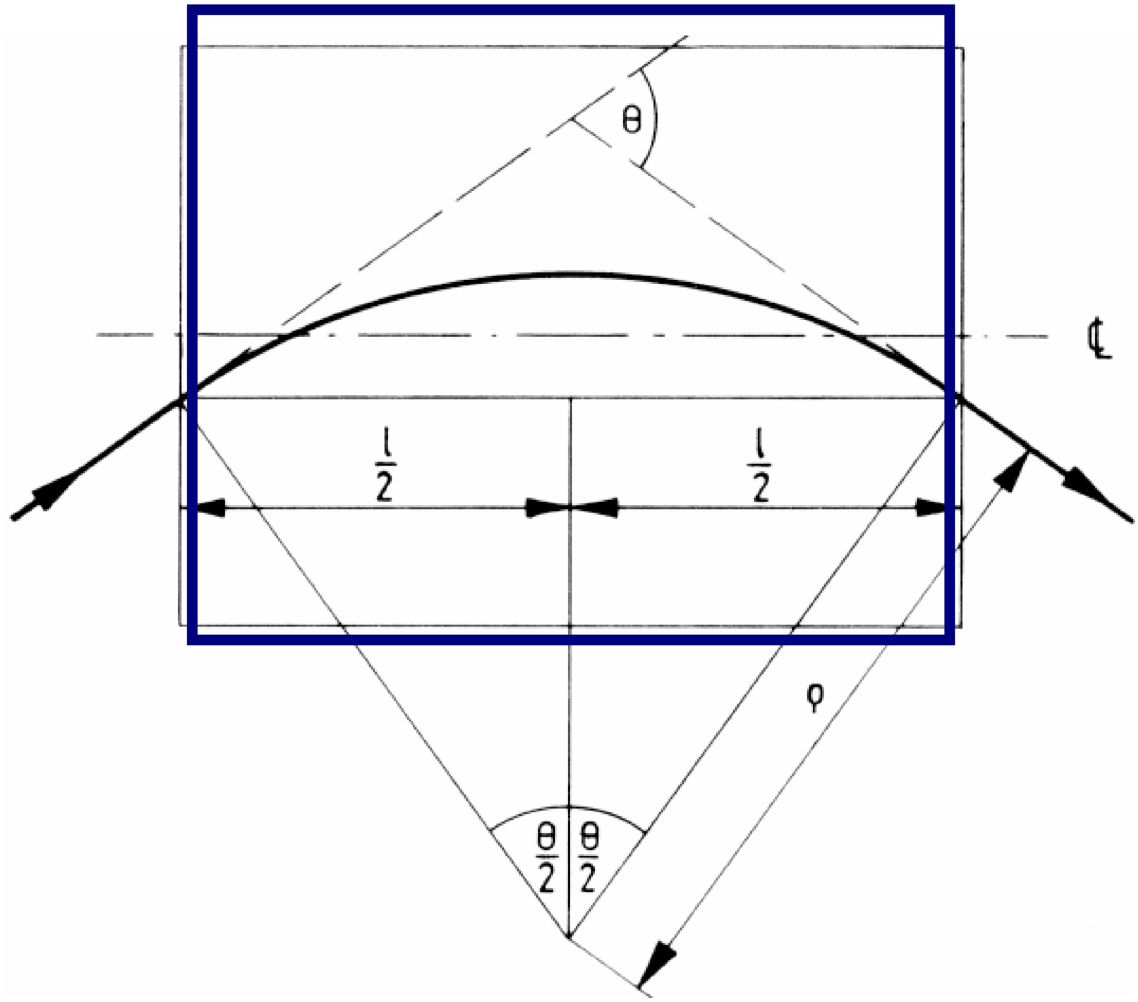
$$= e\mathbf{v} \times \mathbf{B} = e\frac{ds}{dt}B$$

$$(B\rho) = \frac{p}{e} = \frac{pc}{ec} = \frac{\boxed{\beta E}}{ec} = \frac{\beta\boxed{\gamma E_0}}{ec} = \frac{m_0 c}{e}(\beta\gamma)$$

$$(B\rho)[T.m] = \frac{pc[eV]}{c\left[m.s^{-1}\right]} = 3.3356(pc)[GeV]$$

Fig.Brho 4.8

Bending Magnet



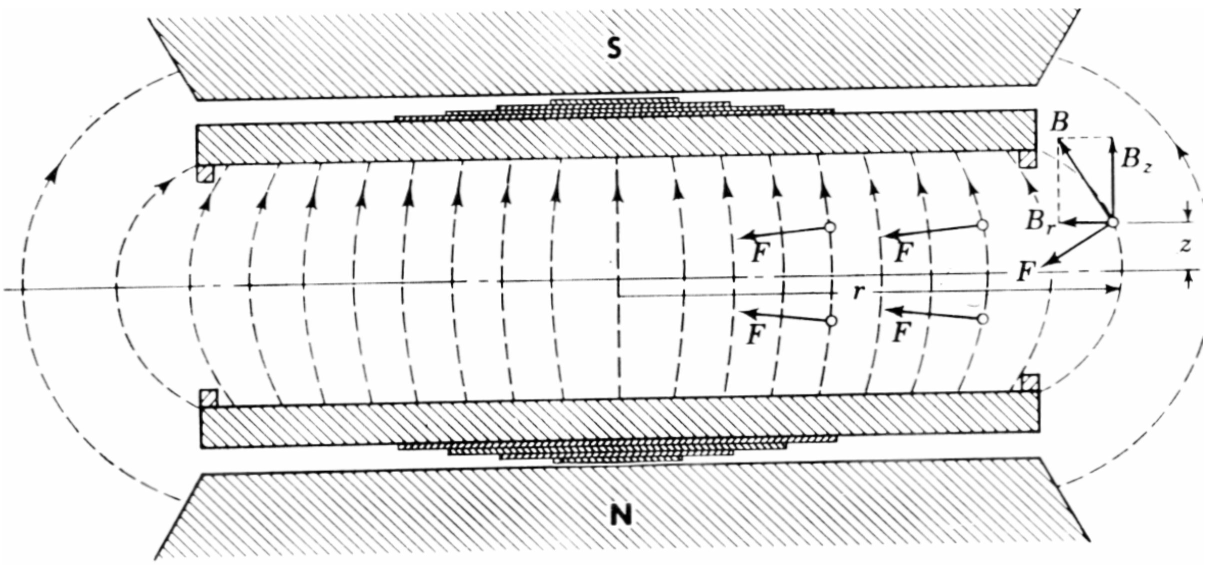
- **Effect of a uniform bending (dipole) field**

$$\sin\left(\theta/2\right) = \frac{1}{2\rho} = \frac{1\,B}{2(B\rho)}$$

- **If** $\theta << \pi/2$ **then** $\theta \approx \dfrac{1B}{(B\rho)}$

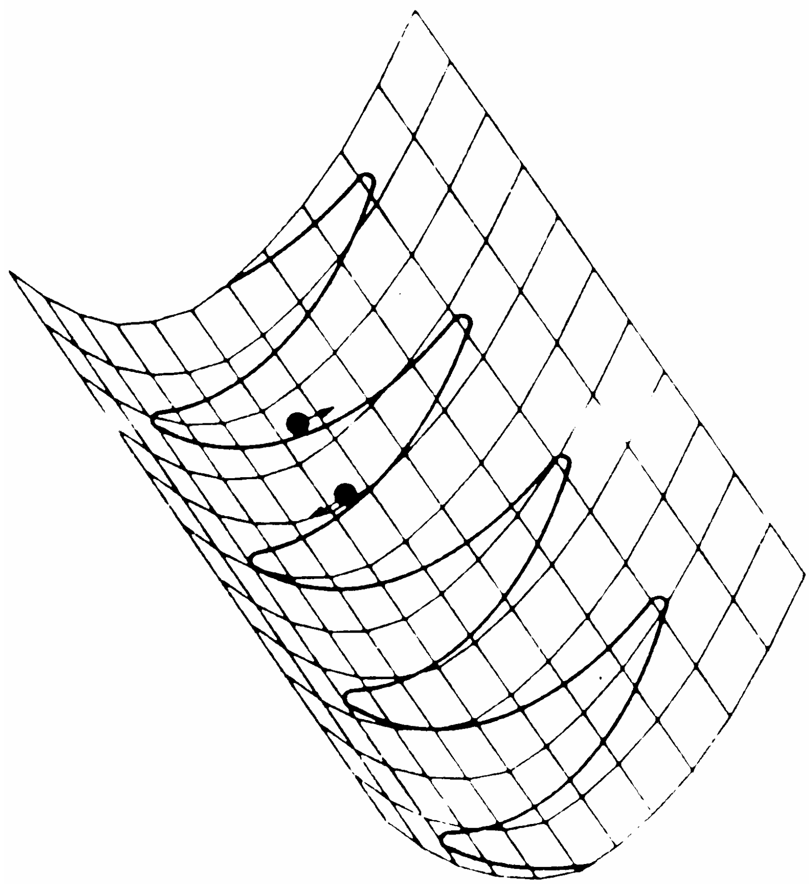
- **Sagitta** $\pm\dfrac{\rho}{2}\left(1-\cos\left(\theta/2\right)\right) \approx \pm\dfrac{\rho\theta^2}{16} \approx \dfrac{1\,\theta}{16}$

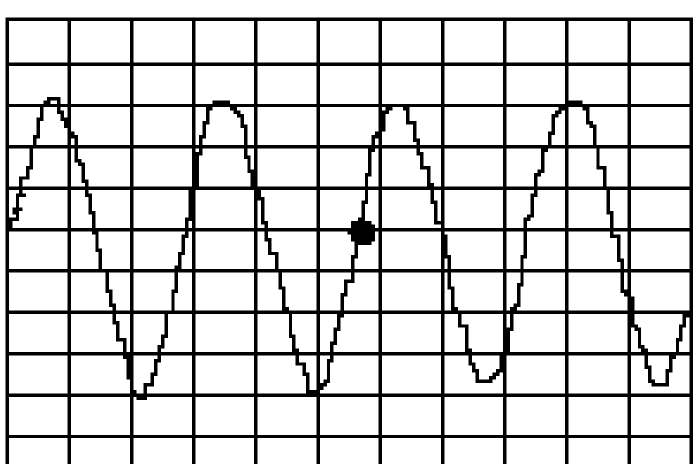
Vertical Focusing



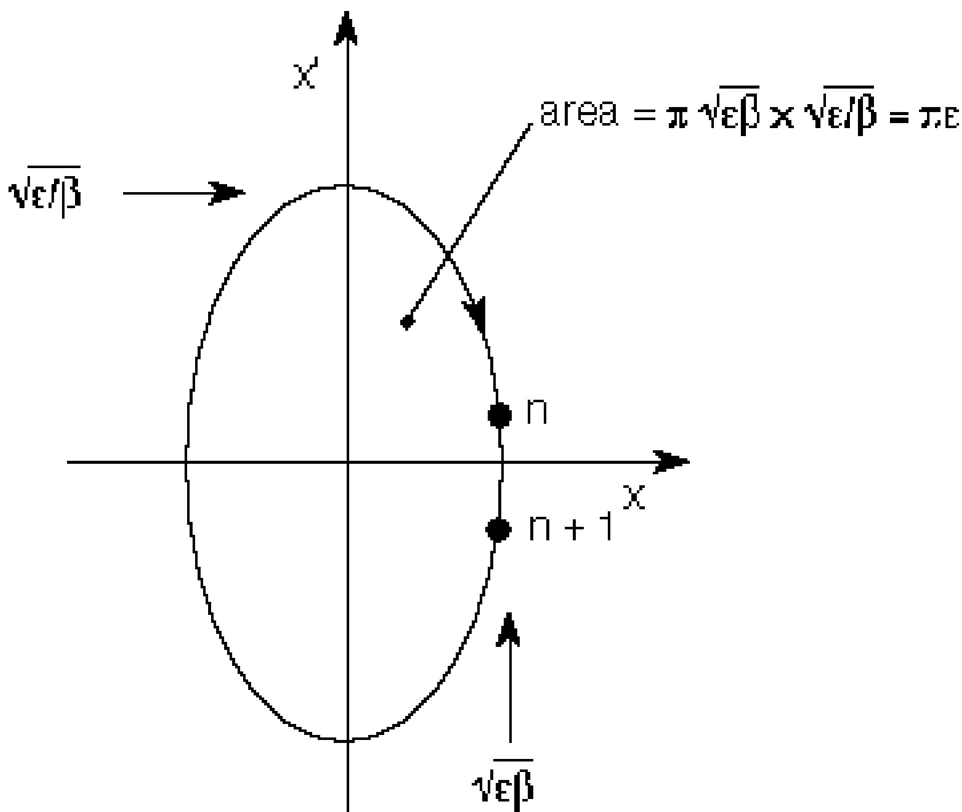
- People just got on with the job of building them.
- Then one day someone was experimenting
- Figure shows the principle of vertical focusing in a cyclotron
- In fact the shims did not do what they had been expected to do
- Nevertheless the cyclotron began to accelerate much higher currents

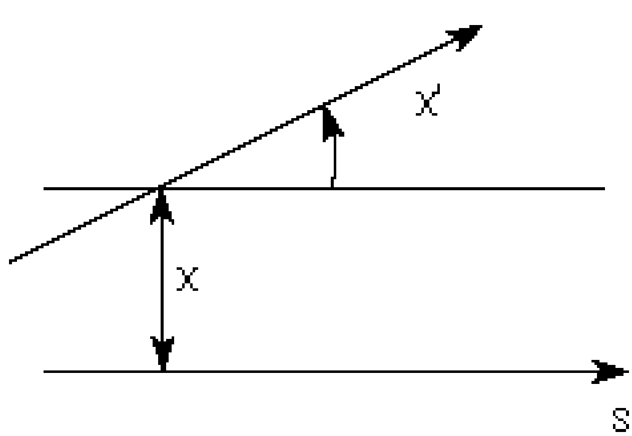
Gutter



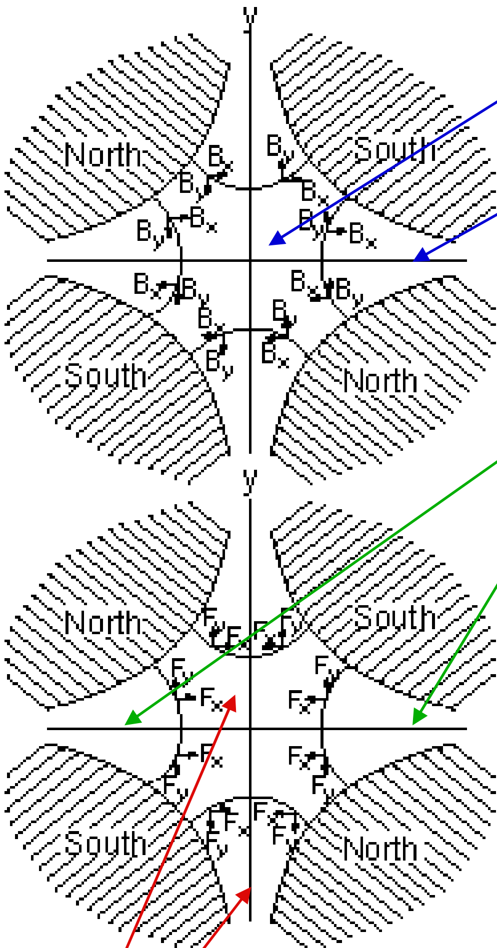


Transverse ellipse





Fields and force in a quadrupole



No field on the axis

Field strongest here

$$B \propto x$$

(hence is linear)

Force restores

Gradient $\frac{dB_z}{dx}$

Normalised:

$$k = -\frac{1}{(B\rho)}\frac{dB_z}{dx}$$

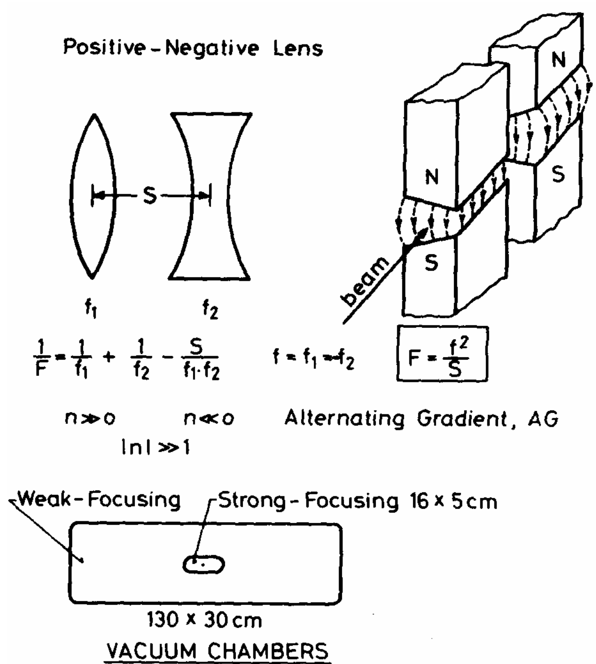
Defocuses in vertical plane

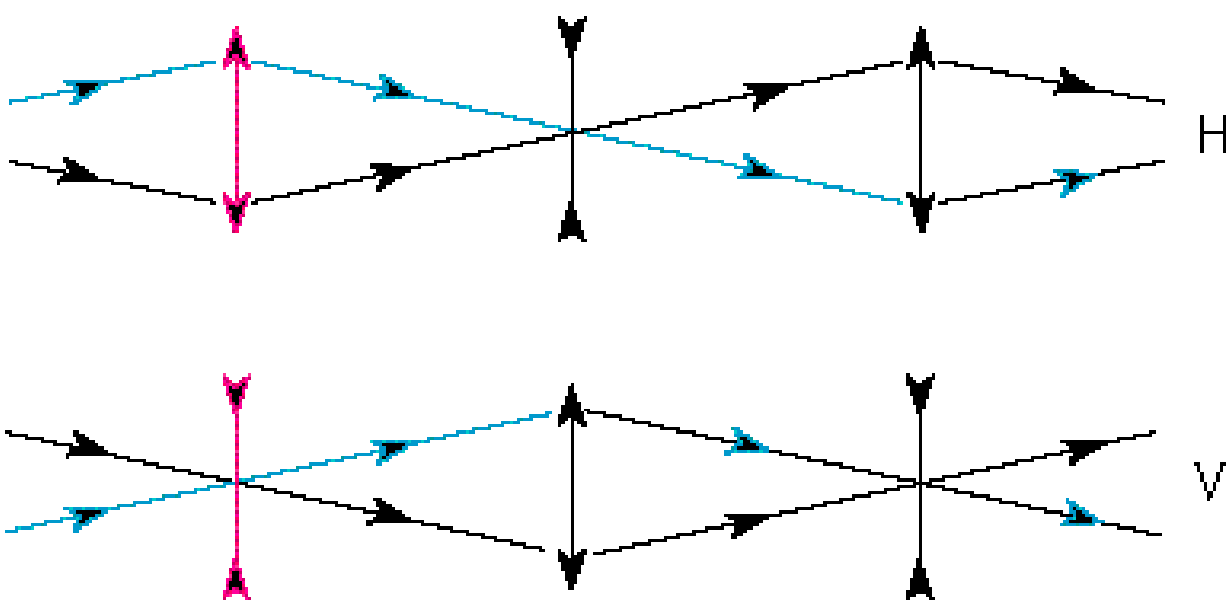
POWER OF LENS

$$lk = -\frac{l}{(B\rho)} \cdot \frac{\partial B_z}{\partial x} = \frac{1}{f}$$

SOLUTION IS TO ALTERNATE THE GRADIENTS OF A SERIES OF QUADS

Strong focusing





Equation of motion in transverse co-ordinates

- **Hill's equation (linear-periodic coefficients)**

$$\frac{d^2y}{ds^2} + k(s)y = 0$$

where $k = -\frac{1}{(B\rho)}\frac{dB_z}{dx}$ **at quadrupoles**

like restoring constant in harmonic motion

- **Solution (e.g. Horizontal plane)**

$$y = \sqrt{\beta(s)}\sqrt{\varepsilon}\sin\left[\phi(s) + \phi_0\right]$$

- **Condition** $$\varphi = \int \frac{ds}{\beta(s)}$$

- **Property of machine** $\sqrt{\beta(s)}$
- **Property of the particle (beam) ε**
- **Physical meaning (H or V planes)**

Envelope $\sqrt{\varepsilon\beta(s)}$

Maximum excursions

$$\hat{y} = \sqrt{\varepsilon\beta(s)}$$

$$\hat{y}' = \sqrt{\varepsilon / \beta(s)}$$

Twiss Matrix

- All such linear motion from points 1 to 2 can be described by a matrix like:

$$\begin{pmatrix} y(s_2) \\ y'(s_2) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} = \mathbf{M}_{12} \begin{pmatrix} y(s_1) \\ y'(s_1) \end{pmatrix} .$$

- We define the "Twiss" parameters:

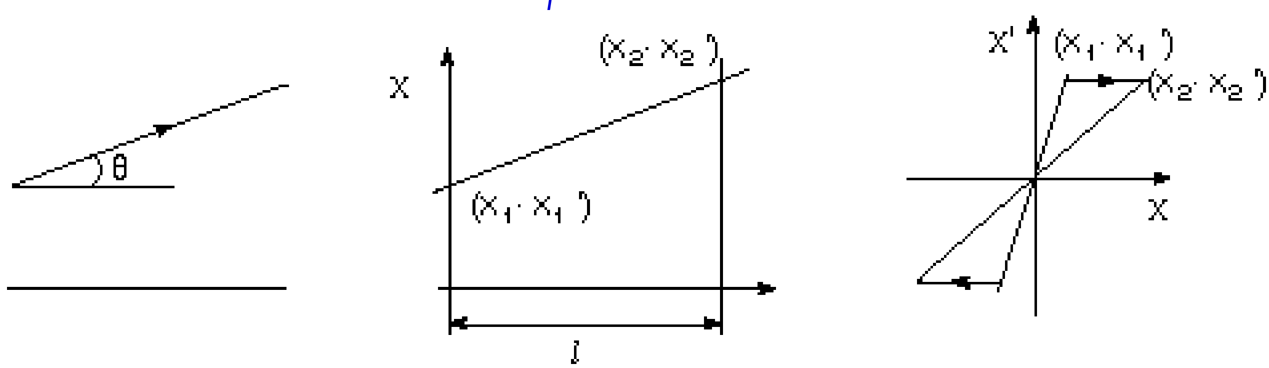
$$\beta = w^2 \ , \ \alpha = -\frac{1}{2}\beta' \ , \ \gamma = \frac{1+\alpha^2}{\beta}$$

- Giving the matrix for a ring (or period)

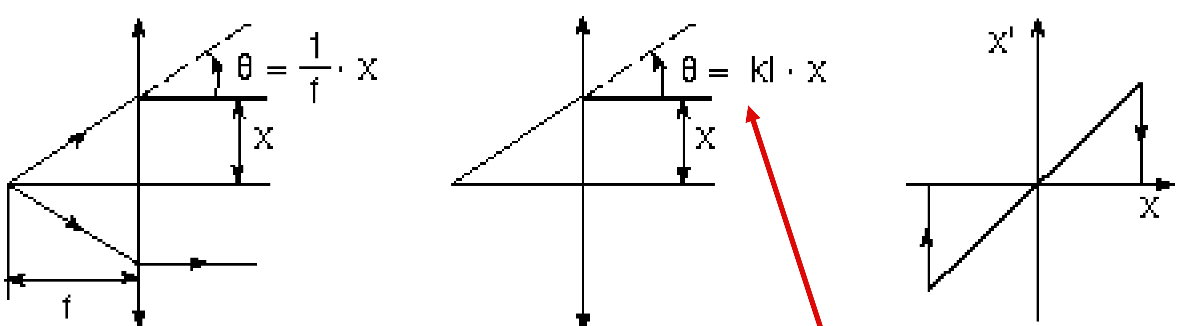
$$M = \begin{pmatrix} \cos\mu + \alpha\sin\mu \ , & \beta\sin\mu \\ -\gamma \ \sin\mu, & \cos\mu - \alpha \ \sin\mu \end{pmatrix}$$

Effect of a drift length and a quadrupole

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$



(a)



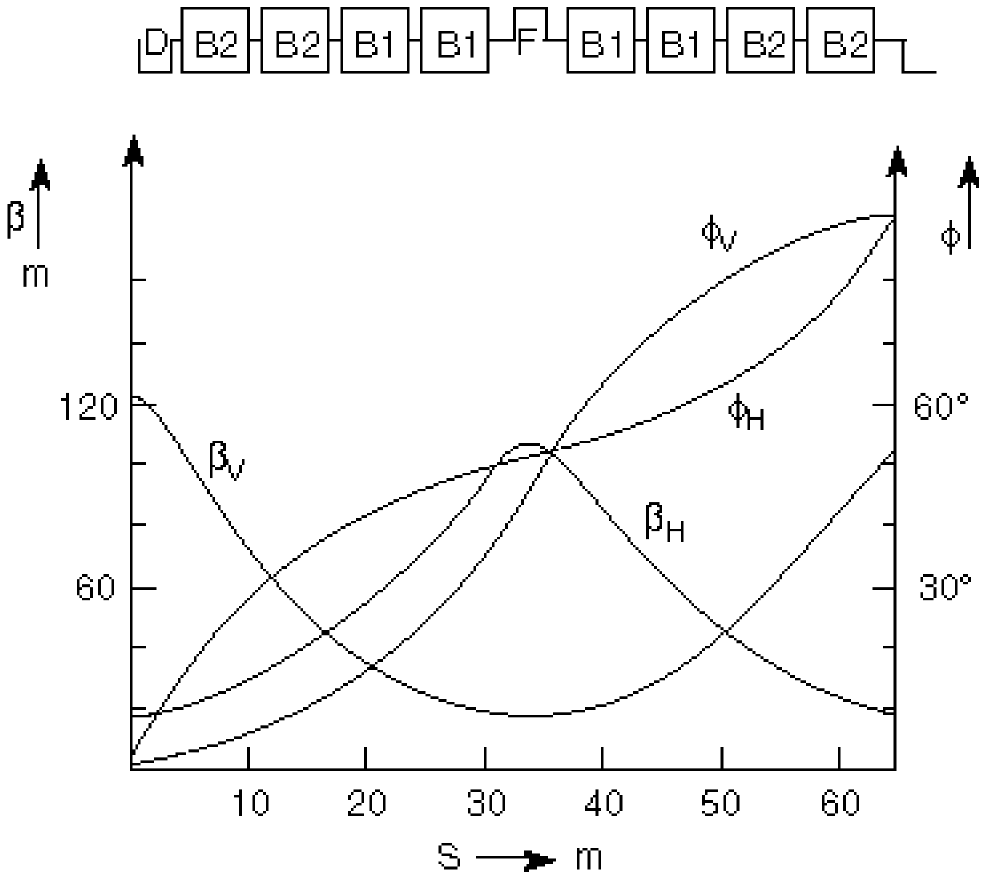
(b)

$$\theta = \frac{1}{f} \cdot x$$

$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1\ , & 0 \\ -1/f\ , & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

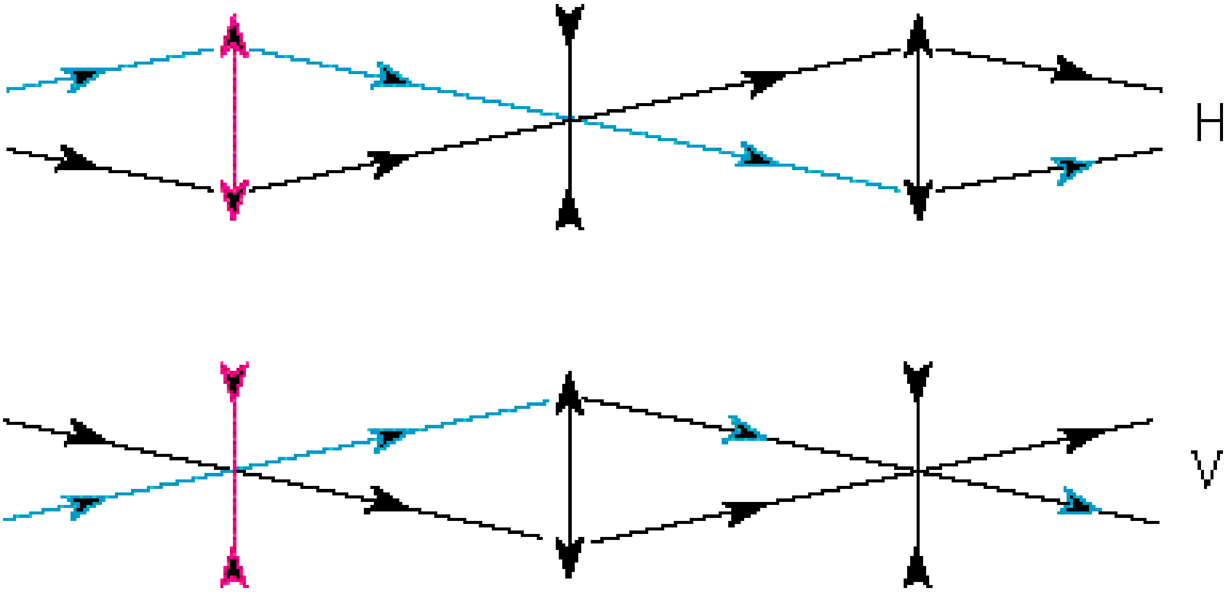
$$\begin{pmatrix} x_2 \\ x_2' \end{pmatrix} = \begin{pmatrix} 1\ , & 0 \\ -kl\ , & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_1' \end{pmatrix}$$

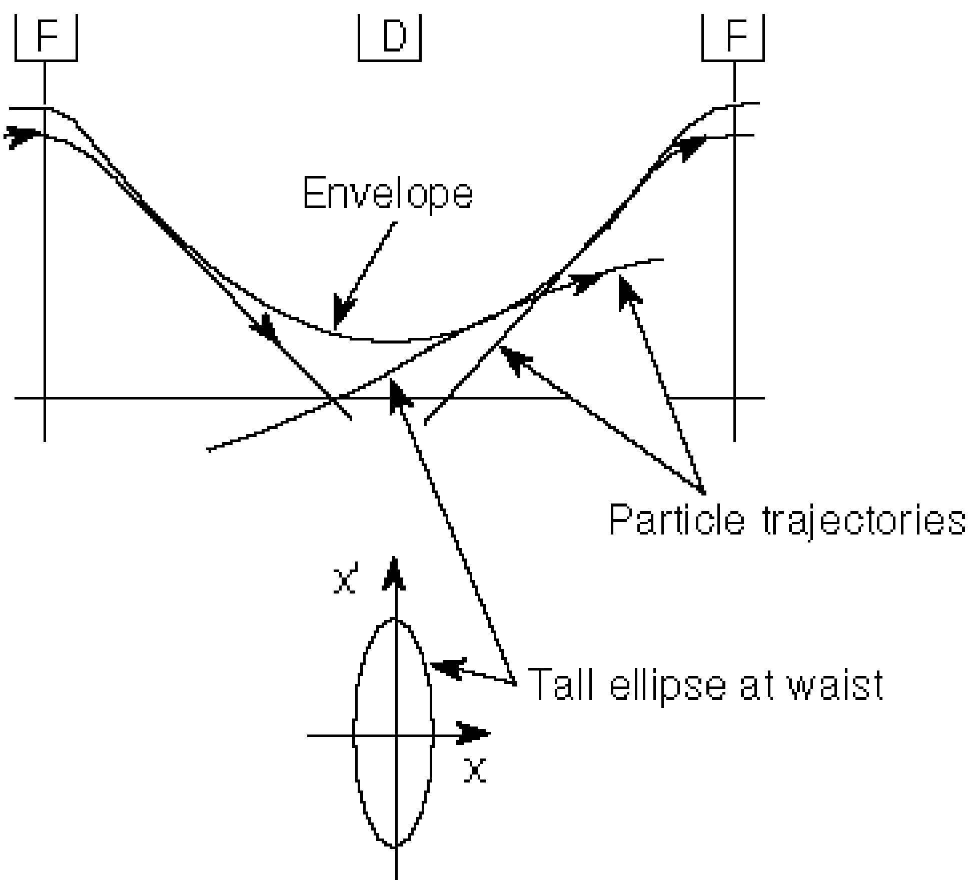
The lattice



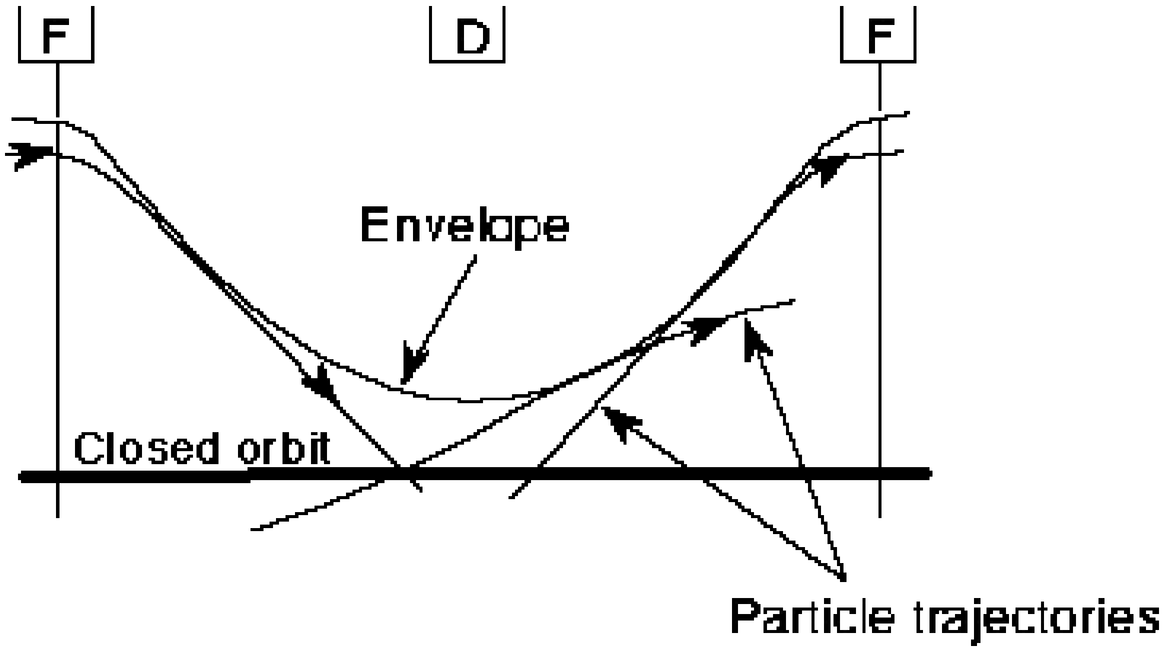
	LENGTH	ANGLE	K(V)	ALPHA(P)	BETA(H)	ALPHA(H)	MUH/2PI	BETA(V)	ALPHA(V)	MUV/2PI	AH/2	AV/2
01	3,085000	0,000000	-,015063	1,386440	104,884855	2,452160	,004571	19,011703	-,520345	,026571	65,715663	9,917560
2	,360000	0,000000	0,000000	1,374653	103,127966	2,428089	,005122	19,395014	-,544408	,029555	64,547813	10,017039
03	6,260000	,008445	0,000000	1,196124	75,348859	2,009521	,016433	28,828710	-,962519	,072198	64,004371	12,212911
4	,400000	0,000000	0,000000	1,186405	73,751941	1,982775	,017287	29,609417	-,989248	,074377	54,751341	12,376828
05	6,260000	,008445	0,000000	1,060742	51,548094	1,564207	,033474	44,610910	-1,407071	,101988	54,174091	15,192432
6	,390000	0,000000	0,000000	1,054559	50,338182	1,538130	,034692	45,718685	-1,433122	,103302	45,428681	15,379447
07	6,260000	,008445	0,000000	,981762	33,701223	1,119563	,058975	66,274961	-1,850527	,121441	44,905056	18,517478
8	,380000	0,000000	0,000000	,978948	32,860011	1,094184	,060793	67,691002	-1,875896	,122344	36,980337	18,713705
09	6,260000	,008445	0,000000	,959017	21,781569	,675586	,098381	93,787676	-2,292753	,134861	36,534921	22,028267
10	2,342700	0,000000	0,000000	,961460	18,983146	,518942	,116758	104,896272	-2,449038	,138621	30,069327	23,295624
11	3,085000	0,000000	,015037	1,034354	18,983068	-,518916	,143368	104,901620	2,447388	,143191	28,349412	23,716825
12	,350000	0,000000	0,000000	1,050730	19,354500	-,542318	,146275	103,196611	2,424067	,143726	28,638028	23,296218
13	6,260000	,008445	0,000000	1,370047	28,764399	-,960879	,189011	75,452122	2,007802	,165027	35,089639	23,106121
14	,380000	0,000000	0,000000	1,391035	29,504322	-,986287	,191088	73,935822	1,982463	,165836	35,546047	19,757412
15	6,260000	,008445	0,000000	1,763219	44,472640	-1,404847	,218731	51,724094	1,565610	,171975	43,750575	19,557880
16	,390000	0,000000	0,000000	1,788053	45,578591	-1,430924	,220109	50,513067	1,539589	,173189	44,298587	16,358398
17	6,260000	,008445	0,000000	2,213103	66,113699	-1,849484	,238298	33,849177	1,122280	,197377	53,470174	16,165762
18	,400000	0,000000	0,000000	2,241952	67,603985	-1,876229	,239281	32,962034	1,095579	,199283	54,079136	13,233307
19	6,260000	,008445	0,000000	2,719868	93,714254	-2,294790	,251780	21,859390	,677943	,336745	63,830251	13,058741
20	2,352700	0,000000	0,000000	2,909420	104,882261	-2,452099	,255558	19,038995	,520847	,355140	67,592709	10,634409
21	3,085000	0,000000	-,015063	2,946010	104,882266	2,452098	,260129	19,038106	-,520546	,361673	68,853088	9,924676
22	,360000	0,000000	0,000000	2,925443	103,125421	2,428027	,260680	19,421551	-,544579	,364653	67,665889	10,023890
23	6,260000	,008445	0,000000	2,594240	75,347037	2,009467	,271992	28,854181	-,962177	,327246	67,105194	12,218305
24	,400000	0,000000	0,000000	2,574765	73,750162	1,982722	,272846	29,634602	-,988874	,339424	57,546939	12,382087
25	6,260000	,008445	0,000000	2,296428	51,546933	1,564162	,289032	44,628208	-1,406185	,356957	56,950187	15,195377
26	,390000	0,000000	0,000000	2,280734	50,337057	1,538085	,290251	45,735180	-1,432204	,358331	47,899567	15,382238
27	6,260000	,008445	0,000000	2,055264	33,700612	1,119525	,314534	66,276862	-1,849098	,376466	47,356928	18,517744
28	,380000	0,000000	0,000000	2,043182	32,859428	1,094117	,316382	67,691805	-1,874435	,377369	39,127022	18,713817
29	6,260000	,008445	0,000000	1,870577	21,781395	,675557	,353941	93,766993	-2,290782	,389888	38,663082	22,025838
30	2,342700	0,000000	0,000000	1,815875	18,983101	,518917	,373318	104,865902	-2,446875	,393648	31,892336	23,292251
31	3,085000	0,000000	,015037	1,873603	18,983178	-,518943	,398928	104,862544	2,447912	,398220	30,027986	23,712598

Envelope and trajectories

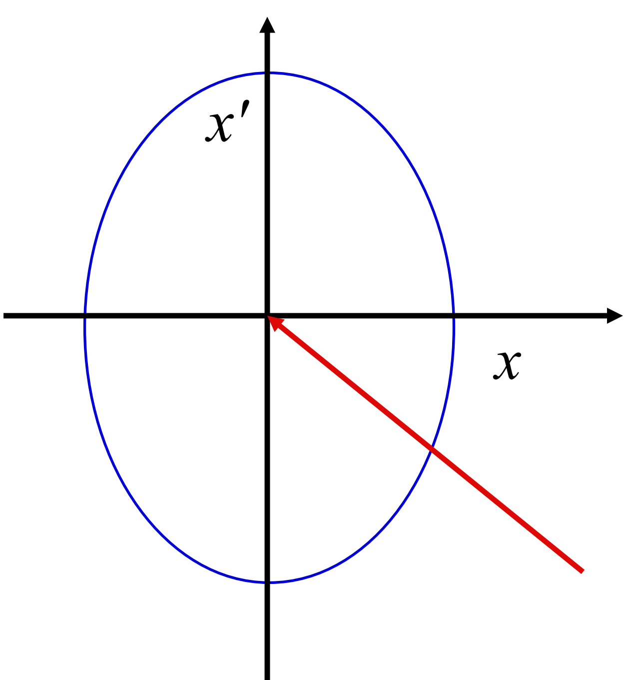




Closed orbit of an ideal machine

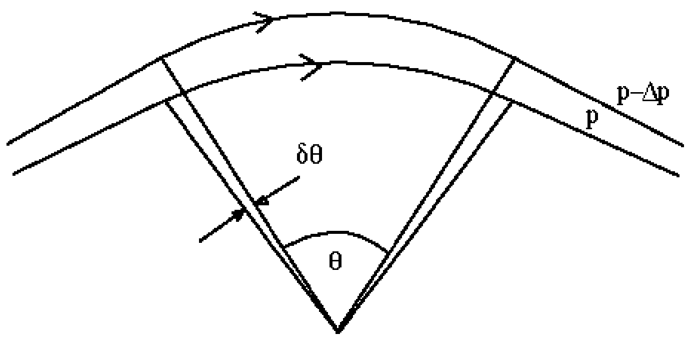


- In general particles executing betatron oscillations have a finite amplitude
- One particle will have zero amplitude and follows an orbit which closes on itself
- In an ideal machine this passes down the axis

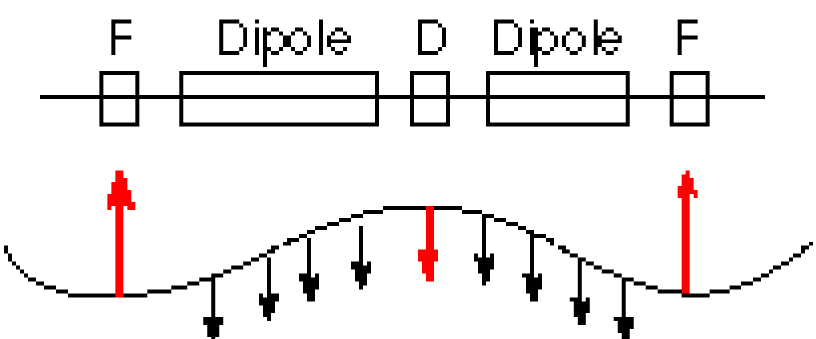


Closed orbit
Zero betatron amplitude

Dispersion



- Low momentum particle is bent more
- It should spiral inwards but:
- There is a displaced (inwards) closed orbit
- Closer to axis in the D's
- Extra (outward) force balances extra bends

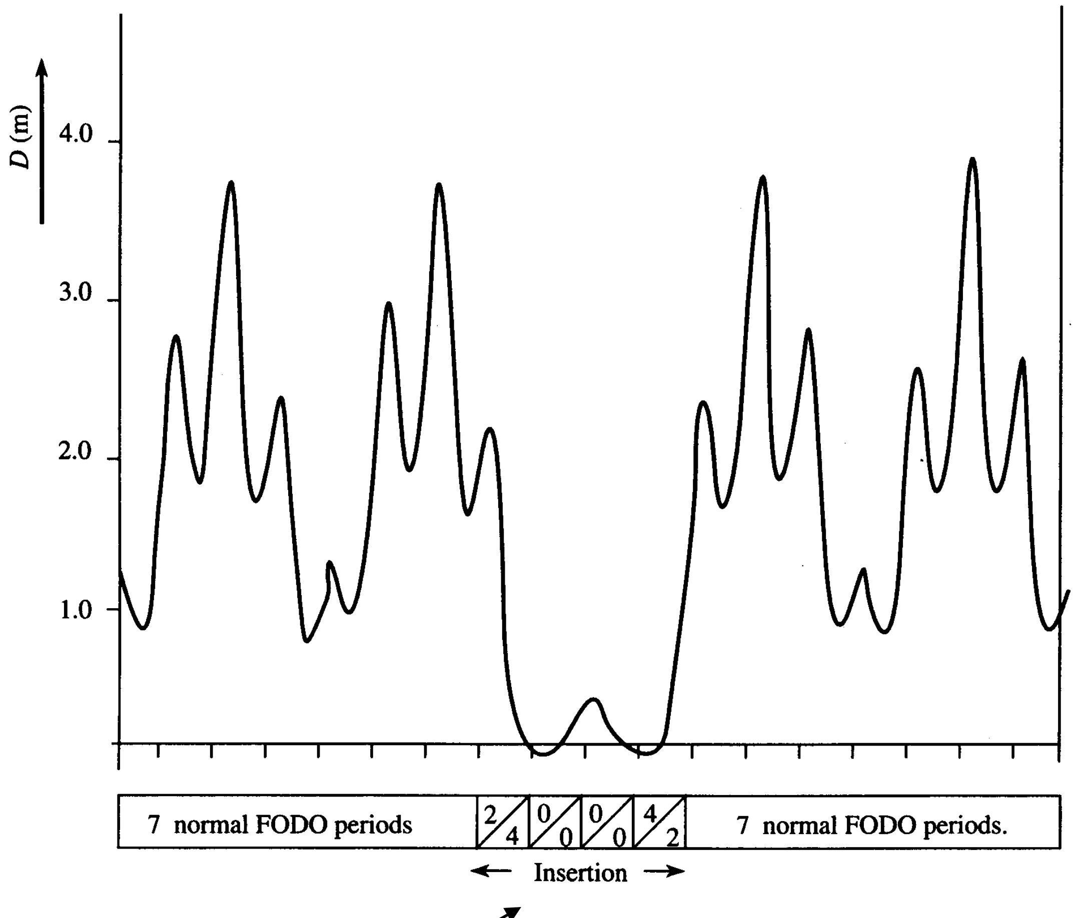


- ***D(s)*** is the "dispersion function"

$$x = D(s)\frac{\Delta p}{p}$$

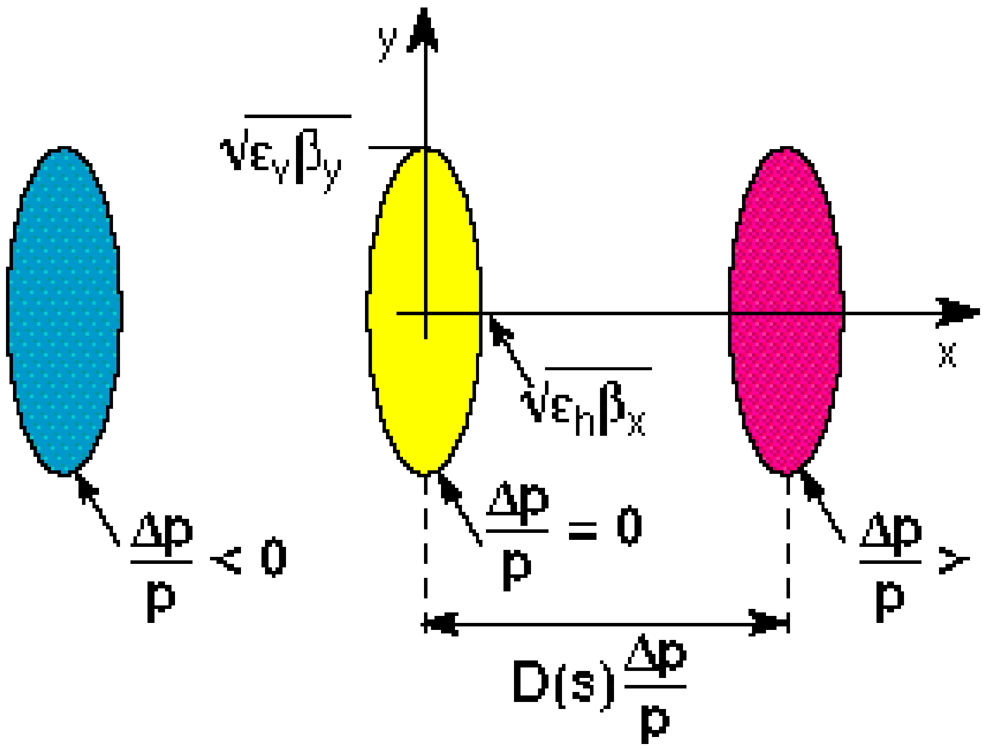
Fig. cas 1.7-7.1C

Dispersion in the SPS



- This is the long straight section where dipoles are omitted to leave room for other equipment - RF - Injection - Extraction, etc
- The pattern of missing dipoles in this region indicated by “0” is chosen to control the Fourier harmonics and make *D(s)* small
- It doesn’t matter that it is big elsewhere

Dispersed beam cross sections



- These are real cross-section of beam
- The central and extreme momenta are shown
- There is of course a continuum between
- The vacuum chamber width must accommodate the full spread
- Half height and half width are:

$$a_V = \sqrt{\beta_V \varepsilon_V} \ , \quad a_H = \sqrt{\beta_H \varepsilon_H} + D(s)\frac{\Delta p}{p} \ .$$

Physics of Chromaticity

- The Q is determined by the lattice quadrupoles whose strength is:

$$k = \frac{1}{(B\rho)}\frac{dB_z}{dx} \quad \propto \frac{1}{p}$$

- Differentiating:

$$\frac{\Delta k}{k} = -\frac{\Delta p}{p} \ .$$

- From gradient error analysis

$$\delta Q = \frac{1}{4\pi}\beta\delta(kl)$$

Giving by substitution

$$\Delta Q = \frac{1}{4\pi}\int \beta(s)\delta k(s)\ ds \ .$$

Q' is the chromaticity

- "Natural" chromaticity

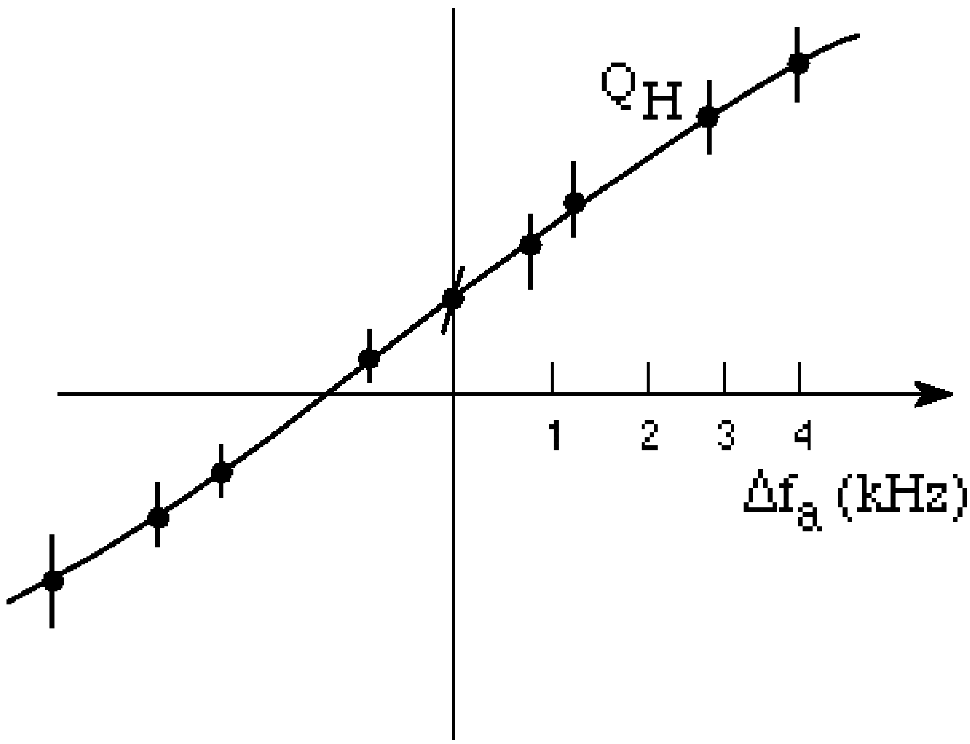
$$\Delta Q = \frac{1}{4\pi}\int \beta(s)\Delta k(s)ds = \left[\frac{-1}{4\pi}\int \beta(s)k(s)ds\right]\frac{\Delta p}{p} \ .$$

$$\Delta Q = Q'\frac{\Delta p}{p}$$

$$Q' = -\frac{1}{4\pi}\oint \beta(s)k(s)ds \approx -1.3Q$$

N.B. Old books say $\xi = \frac{p}{Q}\frac{dQ}{dp} = \frac{Q'}{Q}$

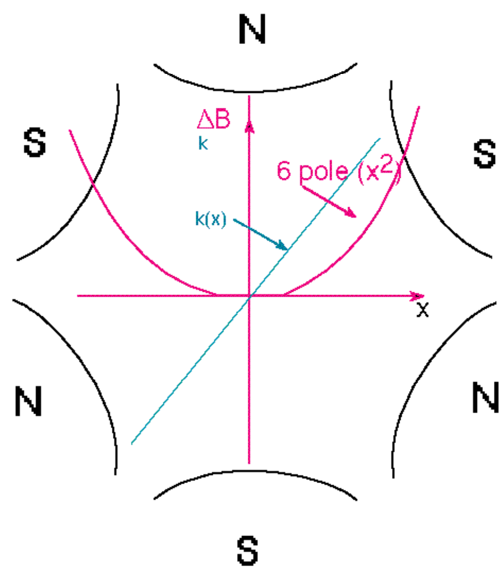
Measurement of Chromaticity



- We can steer the beam to a different mean radius and a different momentum by changing the rf frequency and measure Q
- Since $$\Delta f_a = f_a \eta \frac{\Delta p}{p} \qquad \Delta r = D_{av} \frac{\Delta p}{p}$$
- Hence $$\Delta Q = Q' \frac{\Delta p}{p}$$

$$\therefore Q' = f_a \eta \frac{dQ}{df_a}$$

Correction of Chromaticity



- **Parabolic field of a 6 pole is really a gradient which rises linearly with x**
- **If x is the product of momentum error and dispersion**

- **The effect of all this extra focusing cancels chromaticity**

$$\Delta k = \frac{B'' D}{(B\rho)} \frac{\Delta p}{p} \quad .$$

- **Because gradient is opposite in v plane we must have two sets of opposite polarity at F and D quads where betas are different**

$$\Delta Q = \left[\frac{1}{4\pi} \int \frac{B''(s)\beta(s)D(s)ds}{(B\rho)} \right] \frac{dp}{p} \quad .$$

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