BEAM DIAGNOSTICS WITH SYNCHROTRON RADIATION

CAS 2003, Brunnen
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INTRODUCTION

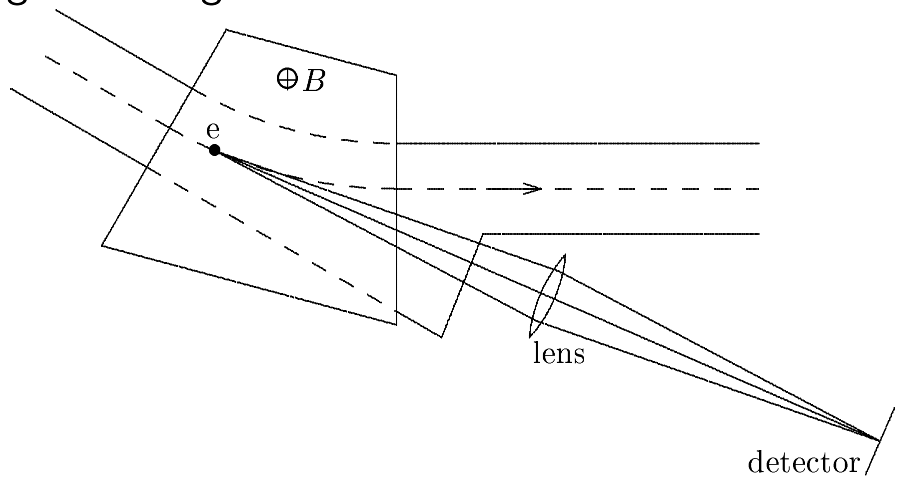
Synchrotron radiation is used to measure the properties of the electron beam. Forming an image and measuring the opening angle of the radiation gives the cross section and transverse momentum distribution of the electrons. The longitudinal bunch shape is directly obtained from the time structure of the radiation.

Bending magnets and undulators are used as radiation sources. The resolution is limited by diffraction due to the small opening angle. this can be improved by using short wavelength. However, for imaging visible light is used for practical reason.

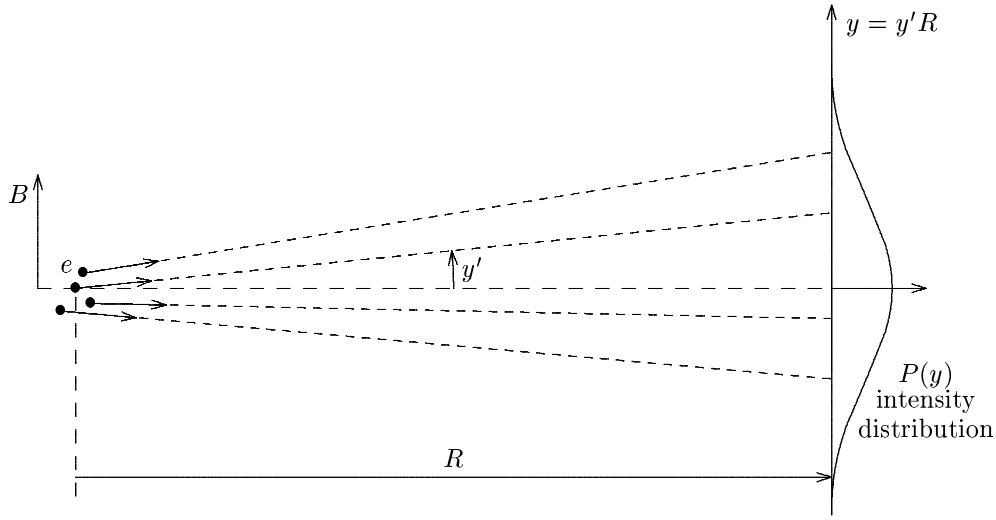
Here we concentrate on the basic physics of these measurements and not on their techniques.

Types of measurements

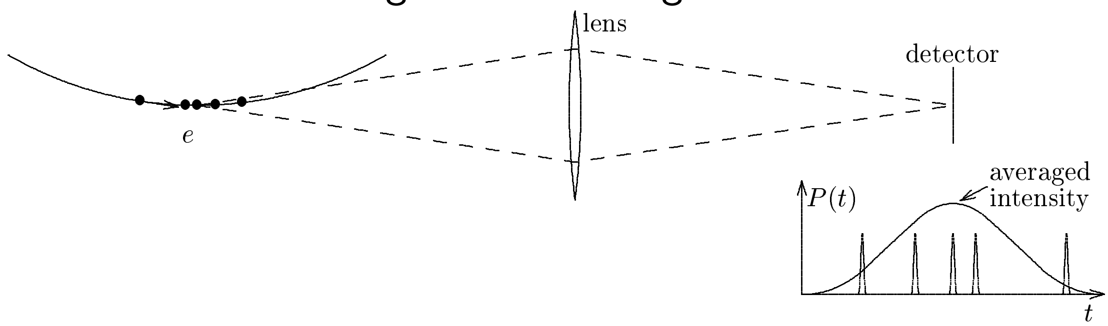
Imaging with SR gives beam cross section



Direct observation gives angular spread (vertical for SR)



Time structure of SR gives bunch length

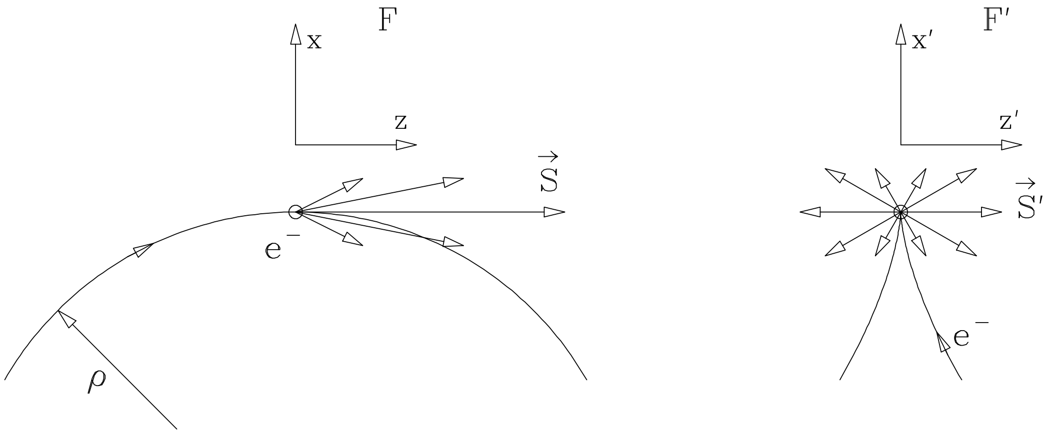


SYNCHROTRON RADIATION

QUALITATIVE PROPERTIES OF SR

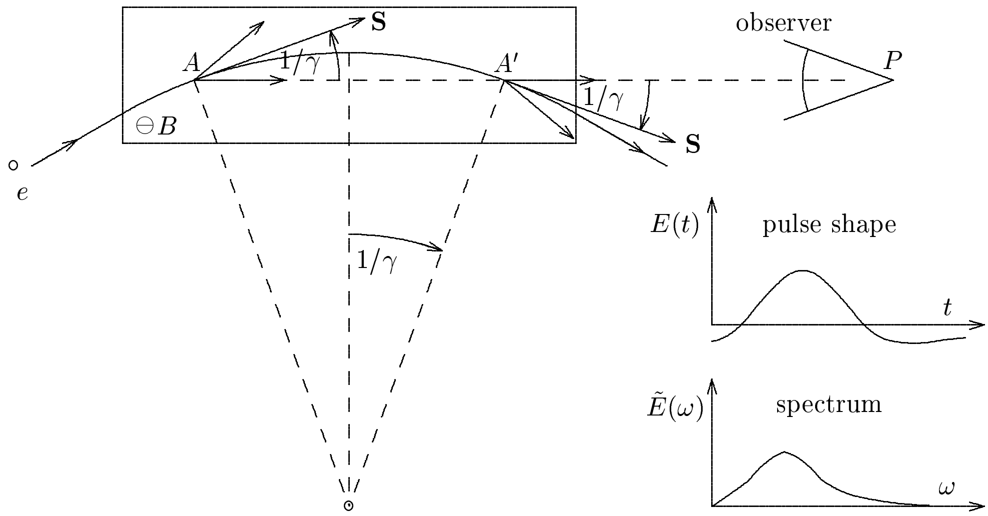
Opening angle

An electron moving in the laboratory frame F on a circular orbit and emitting synchrotron radiation. In a frame F' which moves at one instant with the trajectory is a cycloid with a cusp where the electron is accelerated in the $-x'$ direction. It radiates in this frame with about uniform distribution. Lorentz transforming back into F this radiation is peaked forward. A photon emitted along the x'-axis in F' appears at an angle $1/\gamma$ in F. The typical opening angle of SR is therefore of order $1/\gamma$, usually $1/\gamma \ll 1$.



Opening angle of SR due to moving source

Synchrotron radiation spectrum



How long the radiation pulse seen by the observer? Since opening angle $\approx 1/\gamma$, the radiation first and last seen is emitted at A and A' where the trajectory has an angle of $\pm 1/\gamma$. The pulse length is the travel time difference for particle and photon going from A to A'

$$\Delta t = \frac{2\rho}{\beta\gamma c} - \frac{2\rho \sin(1/\gamma)}{c} \approx \frac{2\rho}{\beta\gamma c}\left(1 - \beta + \frac{\beta}{6\gamma^2}\right)$$

We use $(1-\beta) \approx 1/2\gamma^2$ for $\gamma \gg 1$.

$$\Delta t \approx \frac{2\rho}{\beta\gamma c}\left(\frac{1}{2\gamma^2} + \frac{1}{6\gamma^2}\right) \approx \frac{4\rho}{3c\gamma^3}$$

$$\omega_{\text{typ}} \approx \frac{2\pi}{\Delta t} \approx \frac{3\pi c\gamma^3}{2\rho} \quad \left(\text{critical freq. } \omega_c = \frac{3\pi c\gamma^3}{2\rho}\right)$$

The frequency $\omega \propto \gamma^3$, a factor γ^2 is due to velocity difference for particle and photon, a factor γ is due to path difference.

QUANTITATIVE PROPERTIES of SR

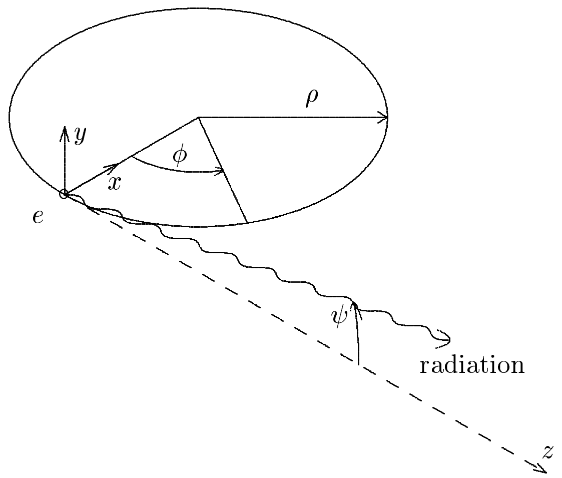
Radiated power

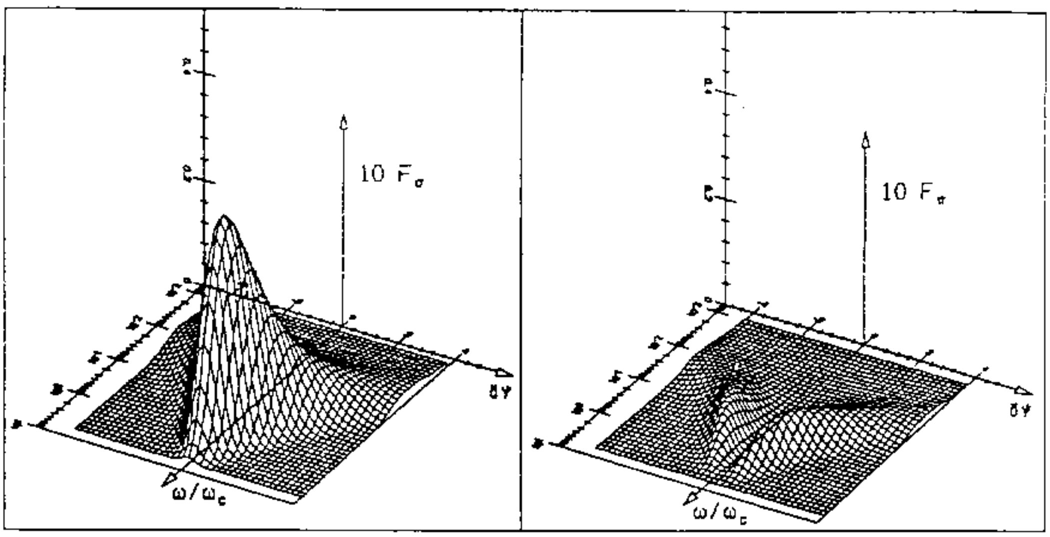
Charge e of energy E in a magnetic field with curvature $1/\rho$

$$P_0 = \frac{2r_0 c m_0 c^2 \gamma^4}{3\rho^2} = \frac{2r_0 e^2 c^3 E^2 B^2}{3\left(m_0 c^2\right)^3},$$

Angular spectral distribution

Vertical opening angle $\psi_{\text{RMS}} \approx 1/\gamma$ being smaller at high and larger at low frequencies. The spectrum is characterized by the the critical frequency $\omega_c = 3\pi c\gamma^3/2\rho$. Horizontal polarization (σ-mode) is dominant (7/8). The vertical polarization (π-mode) vanishes in the median plane $\psi = 0$.





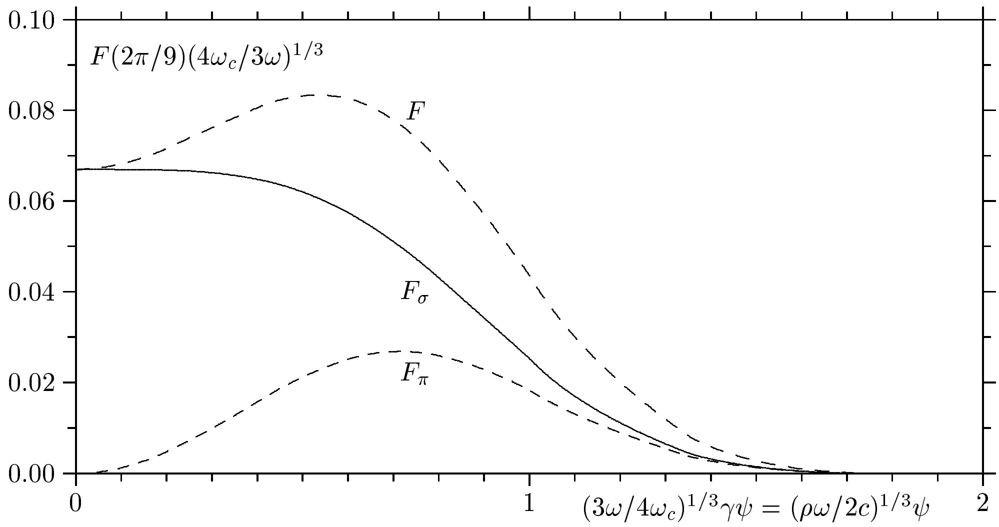
Properties at low frequencies

For diagnostics we use visible light with $\omega \ll \omega_c$ allowing some approximations. The angular spectral distribution is *independent* of γ and given by ρ and λ. The vertical opening angle of the σ-mode is

$$\psi_{\sigma\mathsf{RMS}} \approx 0.41\,(\lambda/\rho)^{1/3}\,.$$

The polarization distribution of the power is

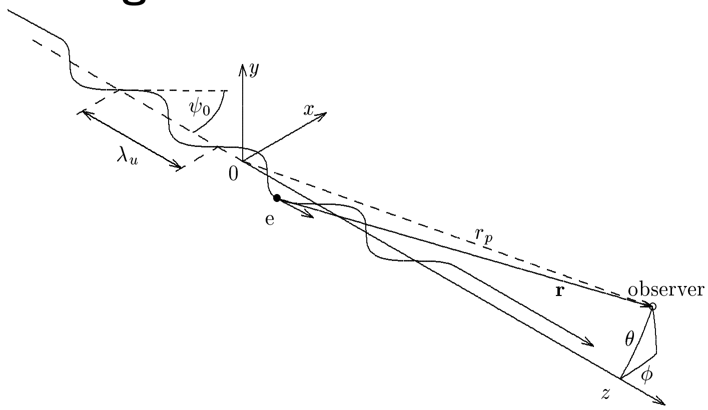
$$P_\sigma = \frac{3}{4}P_0\;,\;P_\pi = \frac{1}{4}P_0$$



Vertical distribution of SR for $\omega \ll \omega_c$

UNDULATORS

Weak and strong undulator radiation

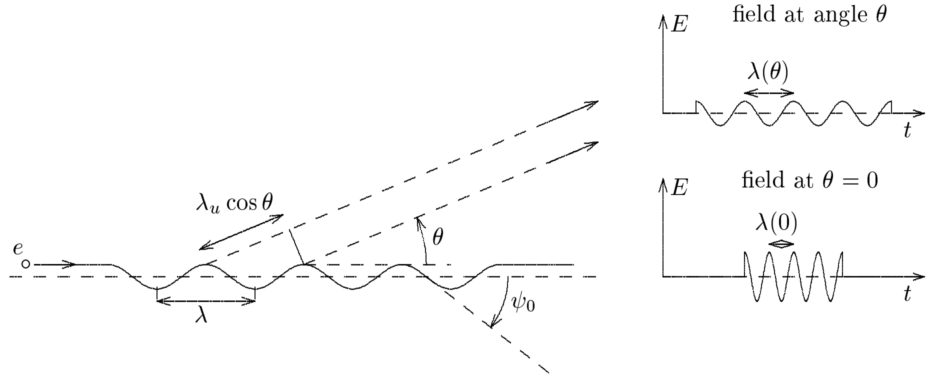


A plane undulator with harmonic field $B(z) = B_y(z) = B_0 \cos(k_u z)$, $k_u = 2\pi/\lambda_u$ gives a harmonic particle trajectory with amplitude a and maximum angle ψ_0

$$x(z) = a\cos(k_u z) \ \ , \ a = \frac{eB_0}{m_0 c\gamma k_u^2} \ \ , \ \ \psi_0 = \frac{eB_0}{m_0 c\gamma k_u} = \frac{K}{\gamma}$$

$K = \gamma\psi_0$, ratio between deflection angle ψ_0 and natural radiation opening angle $\approx 1/\gamma$. For $K < 1$ emitted light is modulated weakly. For $K > 1$ strong modulation gives harmonics.

Spectrum



Radiation contributions from adjacent periods have a time difference $\Delta\theta$ at angle θ being for $\gamma \gg 1 \rightarrow \theta \ll 1$

$$\Delta t = \frac{\lambda_u}{\beta c} - \frac{\lambda_u \cos\theta}{c} = \frac{\lambda_u(1-\beta\cos\theta)}{\beta c} \approx \frac{\lambda_u}{2c\gamma^2}(1+\gamma^2\theta^2).$$

Constructive interference at $\lambda = \Delta t c$ gives main frequency at θ

$$\omega_1 = \frac{\omega_{10}}{(1+\gamma^2\theta^2)} \ \ , \ \omega_{10} = \frac{4\pi c\gamma^2}{\lambda_u} \text{ and harmonics } \omega_m = m\omega_1.$$

Angular spectral distribution

Weak undulator radiation of $K < 1$ and many periods $N_u \gg 1$ has an angular spectral distribution of σ and π-modes

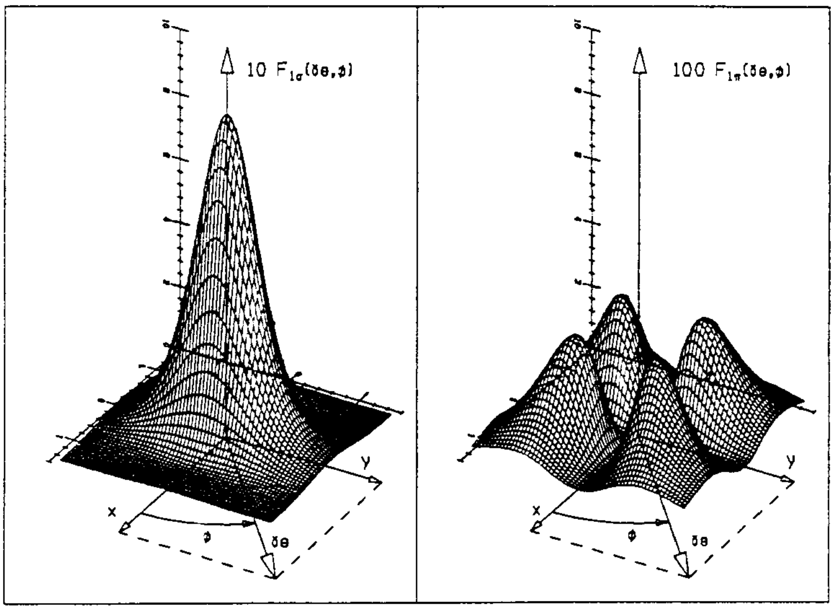
$$\frac{d^2 P_u}{d\Omega d\omega} = P_u \gamma^2 \left(F_{u\sigma}(\theta,\phi) + F_{u\pi}(\theta,\phi) \right) f_N(\Delta\omega).$$

$$[F_{u\sigma}, F_{u\pi}] = \frac{3}{\pi} \frac{\left[\left(1 - \gamma^2\theta^2 \cos(2\phi)\right)^2, \, \left(\gamma^2\theta^2 \sin(2\phi)\right)^2 \right]}{\left(1+\gamma^2\theta^2\right)^5}.$$

and the spectral distribution at a given θ

$$f_N(\Delta\omega) = \frac{N_u}{\omega_1} \left(\frac{\sin\left(\frac{(\omega-\omega_1)\pi N_u}{\omega_1}\right)}{\frac{(\omega-\omega_1)\pi N_u}{\omega_1}} \right)^2 \quad \text{with} \quad \omega_1 = \frac{2\gamma^2 c k_u}{1+\gamma^2\theta^2}$$

$f_N(\Delta\omega) \rightarrow \delta(\omega - \omega_1)$ for $N_u \rightarrow \infty$ giving monochromatic radiation $\omega = \omega_1$ at a given θ.

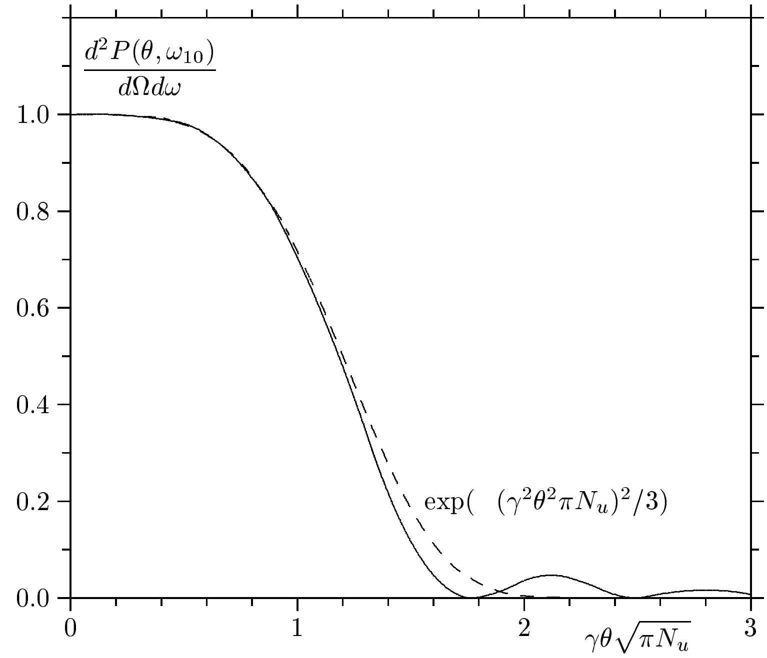


$K > 1$ gives more complicated radiation with higher harmonics

$$\omega_m = m \frac{2\gamma^2 c k_u}{1+\gamma^2\theta^2 + K^2/2}.$$

For diagnostics we filter $\omega = \omega_{10} = 2\gamma^2 c k_u$ appearing around $\theta = 0$ giving the angular distribution for finite but large N_u

$$\frac{d^2P(\omega_{10})}{d\Omega d\omega} \approx P_u \gamma^2 \frac{3}{\pi}\frac{N_u}{\omega_{10}} \left(\frac{\sin(\gamma^2\theta^2\pi N_u)}{\gamma^2\theta^2\pi N_u}\right)^2 .$$



Angular distribution of UR for $\omega = \omega_{10}$

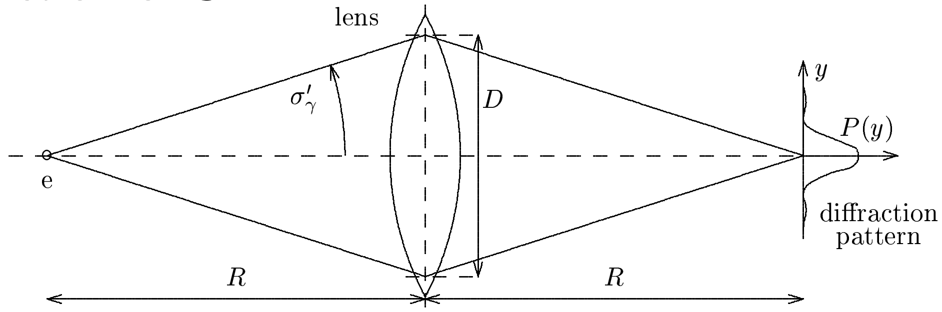
It has a divergent RMS value due to abrupt termination of undulator field. An exponential fit gives equivalent value

$$\theta_{\text{RMS-eq}} \approx \frac{1}{\gamma}\sqrt{\frac{\sqrt{3\pi}}{\pi^2 N_u}} = \frac{0.558}{\gamma}\frac{1}{\sqrt{N_u}}$$

IMAGING WITH SR AND UR

QUALITATIVE

Diffraction of SR



We use synchrotron radiation to form a 1:1 image the beam cross section with a single lens. Due to the small vertical opening angle of $\approx 1/\gamma$ only the central part of the lens is illuminated. Similar to optical imaging with limited lens aperture D this leads to **diffraction** limiting the resolution d (half image size)

$$d \approx \frac{\lambda}{2D/R},$$

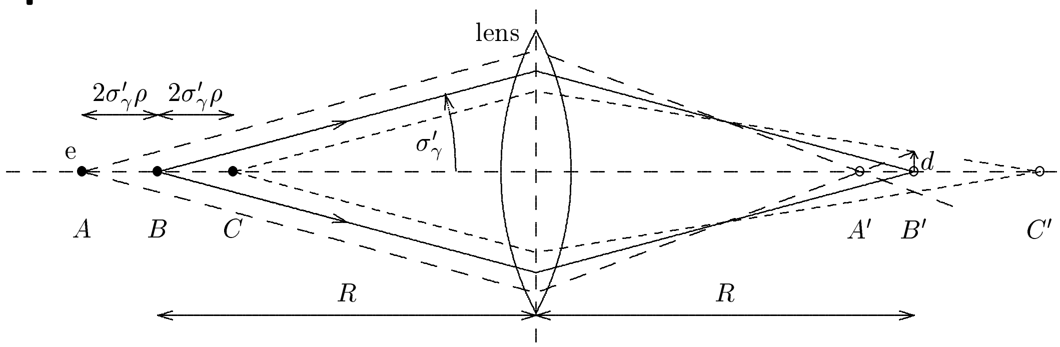
For SR from long magnets we found before an opening angle $\sigma'_\gamma = \psi_{\sigma\mathsf{RMS}} \approx 0.41(\lambda/\rho)^{1/3}$ and take $D \approx 4\sigma_\gamma R$ we find

$$d \approx 0.3(\lambda^2\rho)^{1/3}.$$

The resolution improves with small λ and ρ. It can be quite limited in large machines

machine	ρ	λ	σ'_γ	d
	m	nm	m rad	mm
EPA (CERN)	1.43	400	2.7	0.018
LEP (CERN)	3096	400	0.21	0.24

Depth of field of SR



The length of orbit from which radiation is received has a finite length of

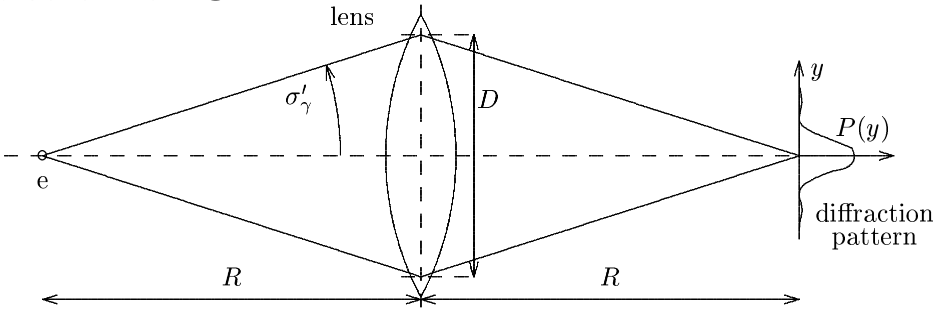
$$\Delta L \approx \pm 2\sigma'_\gamma \rho$$

The A, B and C at the center and ends of this source are imaged to A' and C' having about spacing. At the central point B' the radiation from A and B has an extension

$$d \approx 2{\sigma'_\gamma}^2 \rho = 0.34(\lambda^2 \rho)^{1/3}.$$

Note that resolution limitations due to the depth of field and diffraction have the same parameter dependence and are of the same magnitude. The reason for this is the relation between the length of the source and the opening angle of the radiation.

Diffraction of UR



For undulator radiation we had a full opening angle at $\omega_{10} = 2\gamma^2 c k_u$ or $\lambda_{10} = \lambda_u / 2\gamma^2$

$$\theta_{\text{RMS-eq}} = \frac{0.558}{\gamma\sqrt{N_u}}$$

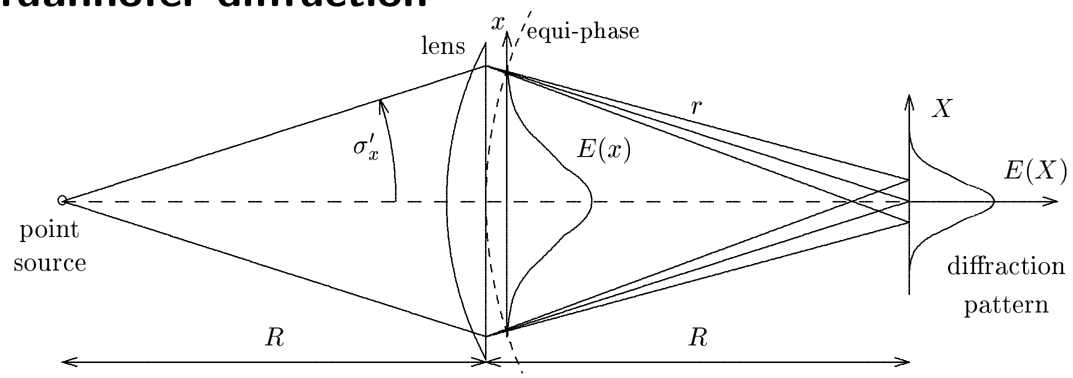
Taking of the illuminated aperture $D = 4R\theta_{\text{RMS-eq}}$ we get for the resolution

$$d \approx \frac{\lambda_{10}}{2D/R} \approx 0.224\gamma\lambda\sqrt{\lambda_{10}}$$

Again a short λ gives better resolution. Also a short undulator length $L_u = \lambda_u N_u$ looks better but we assumed $N_u \gg 1$.

QUANTITATIVE

Fraunhofer diffraction



Forming a 1:1 image with SR with a lens at distance R transforms the angular radiation distribution $\tilde{E}(x', y')$ in a spacial field distribution at the lens

$$\tilde{E}(x, y) = \tilde{E}(Rx'_\gamma, Ry'_\gamma).$$

All rays between source and image have the same optical length and a sphere of radius R around the image point is an equi-phase surface on which each point is a radiation source (Huygens) of strength $\propto \tilde{E}(x, y)$ which propagates to the image giving

$$\delta\tilde{E}(X, Y) \propto \tilde{E}(x, y)\mathrm{e}^{i(kr-\omega t)} \quad , \quad k = 2\pi/\lambda$$

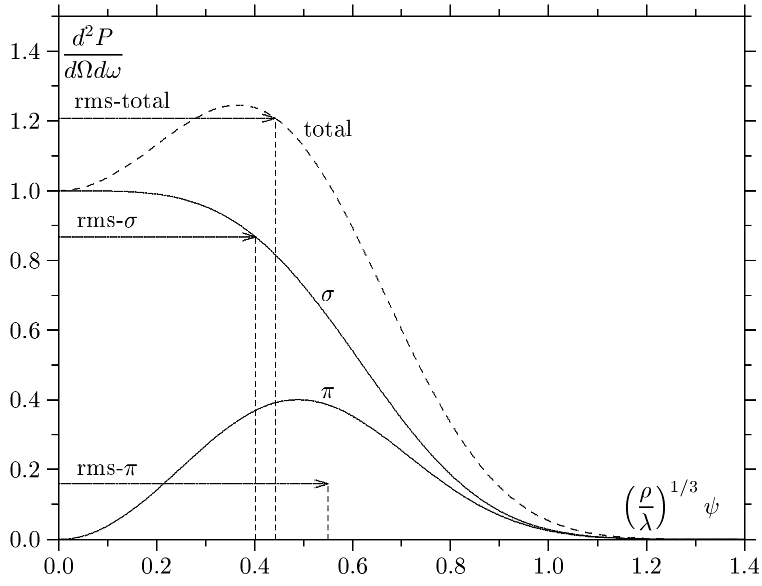
$$r = \sqrt{R^2 - 2xX - 2yY + X^2 + Y^2} \approx R\left(1 + -\frac{xX}{R^2} - \frac{yY}{R^2} + ...\right).$$

with $x' = x/R$ we get the image field

$$\tilde{E}(X, Y) \propto \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{E}(Rx', Ry')\mathrm{e}^{-i\left(x'kX + y'kY\right)}dx'dy'.$$

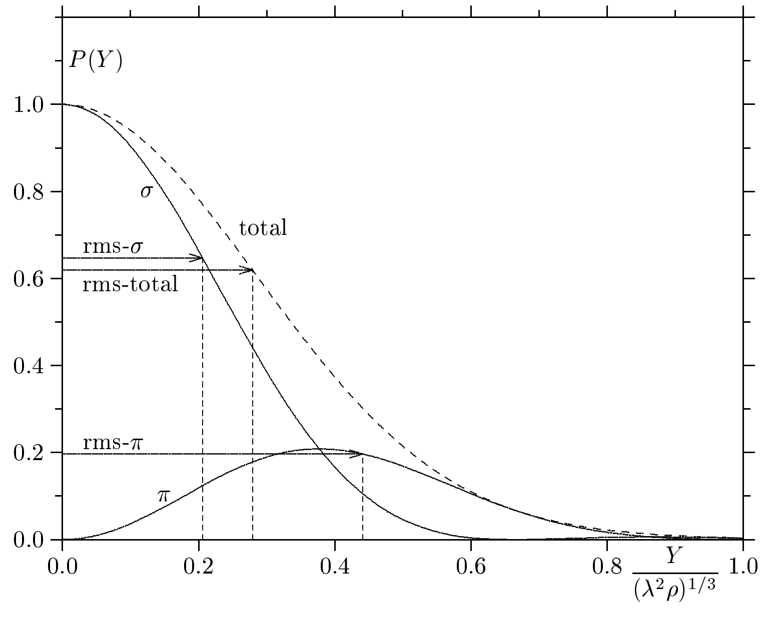
The field distribution at the image is the Fourier transform of the angular distribution.

Fraunhofer diffraction of SR



Vertical angular distribution of SR

$$\sigma'_{y,\sigma} = 0.4097\left(\frac{\lambda}{\rho}\right)^{\frac{1}{3}} \;,\; \sigma'_{y,\pi} = 0.5497\left(\frac{\lambda}{\rho}\right)^{\frac{1}{3}} \;,\; \sigma'_{y,tot} = 0.4488\left(\frac{\lambda}{\rho}\right)^{\frac{1}{3}} .$$

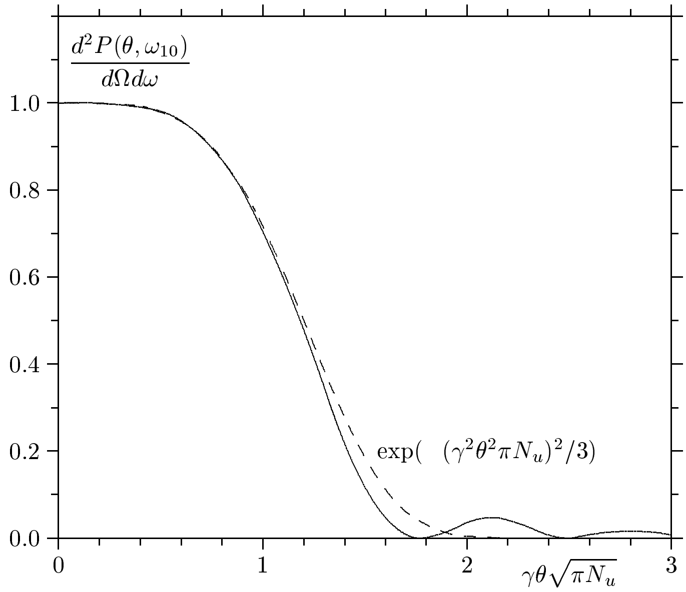


Vertical diffraction pattern of SR

$$\sigma_{Y,\sigma} = 0.206(\lambda^2\rho)^{\frac{1}{3}} \;,\; \sigma_{Y,\pi} = 0.429(\lambda^2\rho)^{\frac{1}{3}} \;,\; \sigma_{Y,tot.} = 0.279(\lambda^2\rho)^{\frac{1}{3}}.$$

This confirms the simple calculation done before. Selecting the σ-mode with a filter improves the resolution.

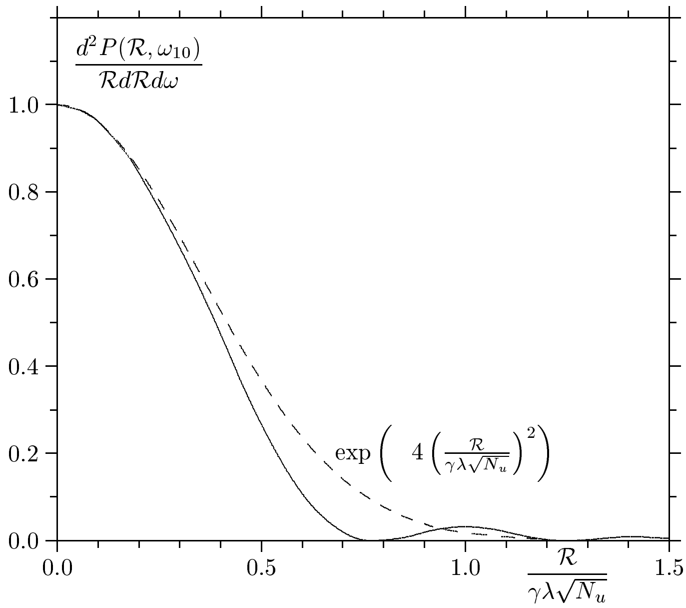
Fraunhofer diffraction of UR



Angular distribution of UR at ω_{10}

θ_{rms} diverges but the exponential fit has

$$\theta_{\text{RMS-eq}} \approx \frac{1}{\pi\gamma}\sqrt{\frac{\sqrt{3\pi}}{N_u}} = \frac{0.558}{\gamma}\frac{1}{\sqrt{N_u}} \quad , \quad \sigma'_x = \sigma'_y = \frac{\theta_{\text{RMS-eq}}}{\sqrt{2}}$$



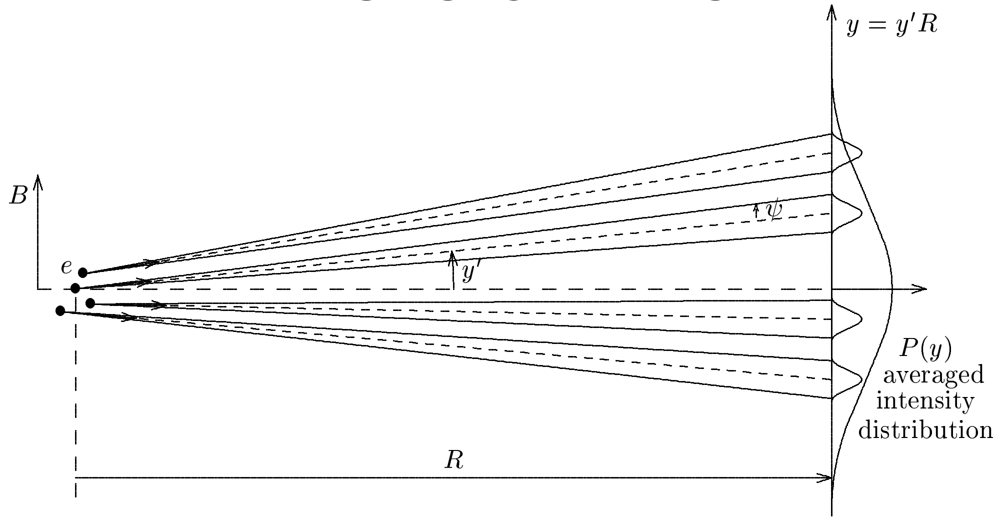
Diffraction for UR at ω_{10}

$\mathcal{R}_{\text{RMS}}$ diverges but the exponential fit has

$$\mathcal{R}_{\text{RMS-eq}} \approx \frac{\gamma\lambda_{10}\sqrt{N_u}}{2} \quad , \quad \sigma_X = \sigma_Y = \frac{\mathcal{R}_{\text{RMS-eq}}}{\sqrt{2}} = 0.354\gamma\lambda_{10}\sqrt{N_u}$$

which agrees approximately with the simple calculation.

DIRECT OBSERVATION



Observing SR and its opening angle directly gives the angular spread of the particles. Knowing $\beta_w, \alpha_w, \gamma_w$ at the source the beam emittance ϵ_{ew} ($\epsilon_{ew} = \sigma_w \sigma'_w$ at a waist) is obtained from this where w stands for x or y. Neglecting the opening angle of SR, the emitted photons have the same emittance as the electrons and continue the particle trajectory. We apply the lattice functions to them and get $\beta_{\gamma w}$ at the observation screen at distance R by using the propagation of $\beta_w(s)$ in a straight section (M. Placidi)

$$\beta_{\gamma w}(R) = \beta_w(0) - 2\alpha_w(0)R + \gamma_w(0)R^2.$$

The measured spot size $\sigma_{\gamma w}$ is related to ϵ

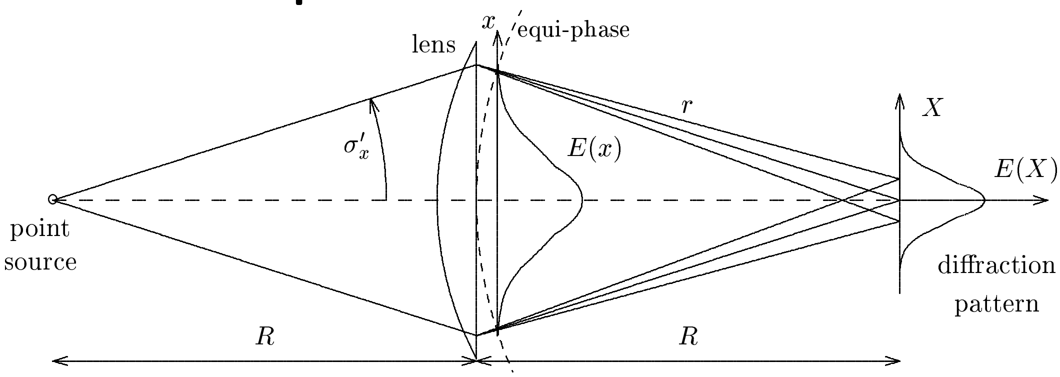
$$\epsilon_{ew} \approx \epsilon_{\gamma w} = \frac{\sigma_{\gamma w}^2}{\beta_{\gamma w}(R)}\ ,\ \sigma_{ew} = \sqrt{\epsilon_{ew}\beta_w(0)}$$

Correcting for the opening angle σ'_w of SR which gives the actual measured photon beam size σ_{mw} which can be used to correct and the corrected electron emittance

$$\sigma_{mw}^2 = R^2 \sigma_{\gamma' w}^2 + \sigma_{\gamma w}^2\ ,\ \epsilon_{ew} = \frac{\sigma_{mw}^2 - R^2 \sigma_{\gamma' w}}{\beta_{\gamma w}(R)}$$

EMITTANCE MEASUREMENT

Emittance of a photon beam



Imaging photons of Gaussian angular distr.

$$\tilde{E}(x') = E_0 \mathrm{e}^{-\frac{x'^2}{2\sigma'^2_E}} \rightarrow \tilde{E}_{lens}(x) = E_1 \mathrm{e}^{-\frac{x^2}{2(R\sigma'_E)^2}}$$

$$P(x') \propto |\tilde{E}(x')^2 = E_0^2 \mathrm{e}^{-\frac{x'^2}{\sigma^2_E}} = E_0^2 \mathrm{e}^{-\frac{x'^2}{2\sigma'^2_\gamma}}$$

With $P \propto |E|^2$ RMS angles of field and power (photons) are related $\sigma'_E = \sqrt{2}\sigma'_\gamma$. Image field:

$$\tilde{E}(X) \propto \int \mathrm{e}^{-\frac{x'^2}{2(\sigma'_E)^2}} \cos(kXx')dx' = \sqrt{2\pi}\sigma'_E \mathrm{e}^{-\frac{(kX\sigma'_E)^2}{2}}$$

Gaussian image with RMS values of field, power $\sigma_E = \sqrt{2}\sigma_\gamma$, $\sigma_P = \sigma_\gamma = 1/(2k\sigma'_\gamma)$. The product $\sigma_\gamma\sigma'_\gamma$ is the photon emittance

$$\epsilon_\gamma = \sigma_\gamma\sigma'_\gamma = \frac{1}{2k} = \frac{\lambda}{4\pi}$$

For other distributions the emittance is larger

source	σ_γ	σ'_γ	ϵ_γ
long magnet (y)	$0.21\sqrt[3]{\lambda^2\rho}$	$0.41\sqrt[3]{\frac{\lambda}{\rho}}$	$1.06\frac{\lambda}{4\pi}$
undulator (x,y)	$\sim \frac{\lambda\gamma\sqrt{2N_u}}{4}$	$\sim \frac{1}{\pi\gamma}\sqrt{\frac{\sqrt{3\pi}}{2N_u}}$	$\sim 1.75\frac{\lambda}{4\pi}$

Measuring σ_w and σ'_w

We measure the emittance $\epsilon_w = \sigma_w \sigma'_w / \sqrt{1+\alpha_w^2} \approx \sigma_w \sigma'_w$ where w stans for x or y. Measure σ_w and σ'_w has the limitation $\epsilon_{\gamma w} \geq \lambda/4\pi$ but some corrections can be made by a deconvolution. The direct observation can easily be made with x-rays which improves the resolution.

Measuring σ_w or σ'_w

If only one of the two is measured the emittance is obtained with the lattice functions

$$\epsilon_w = \frac{\sigma_w^2}{\beta_w} = \frac{\sigma'^2_w}{\gamma_w} \ , \quad \frac{\sigma_w}{\sigma'_w} = \frac{\beta_w}{\sqrt{1+\alpha_w^2}}.$$

$$\text{where } \alpha_w = -\frac{1}{2}\beta'_w \ , \ \gamma_w = \frac{1+\alpha_w}{\beta_w}.$$

Diffraction limits the accuracy of the σ_w measurement and the finite opening angle the one of the σ'_w measurements. Since for a fixed ϵ_w

$$\sigma_w = \sqrt{\epsilon_w \beta_w} \ , \quad \sigma'_w = \sqrt{\epsilon_{\gamma w}} \approx \sqrt{\epsilon_w / \beta_w}$$

a high β_w is desired in the first and a low β_w in the second measurement. In practice the limited knowledge of the lattice function is often the largest error. It is helpful to measure beta at the source, e.g. by varying a quadrupole and observing the tune change.