Lattices and Emittance

- Introduction

  Design phases ◦ Interfaces ◦ Space

- Lattice building blocks

  local vs. global ◦ Approximations ◦ Fields and Magnets

- Beam dynamics pocket tools

  Transfer matrices and betafunctions ◦ Design code

- Emittance and lattices

  Emittance ◦ Lattice cells ◦ Minimum emittance ◦ Vertical emittance

- Other lattice parameters

  Circumference & periodicity ◦ Tune and chromaticity

- Acceptance

  physical ◦ momentum ◦ dynamic ◦ Optimization

Lattice Design Phases

1. **Preparation**

   Performance • Boundary conditions • Building blocks (magnets)

2. **Linear lattice**

   Global quantities • Cells, matching sections, insertions, etc.

3. **Nonlinear lattice**

   Sextupoles • Dynamic acceptance

4. **Real lattice**

   Closed orbit • Alignment errors • Multipolar errors

Lattice Design Interfaces

Magnet Design: Technological limits, coil space, multipolar errors

Vacuum: Impedance, pressure, physical apertures, space

Radiofrequency: Momentum acceptance, bunchlength, space

Diagnostics: Beam position monitors, space

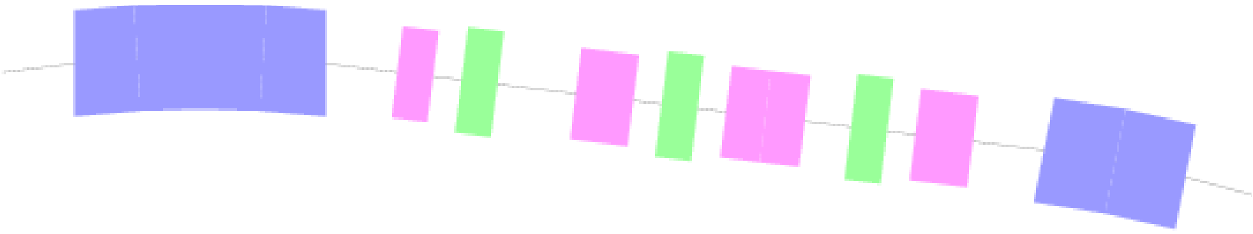
Alignment: Orbit distortions and correction

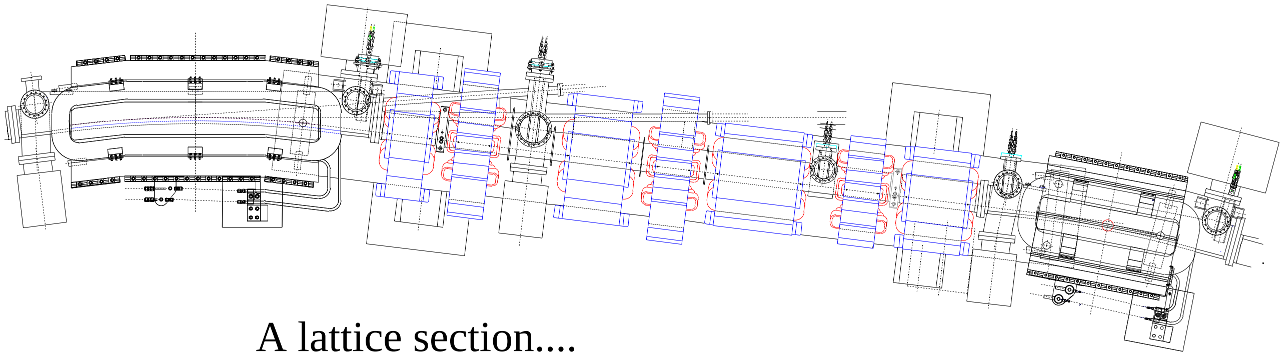
Mechanical engineering: Girders, vibrations

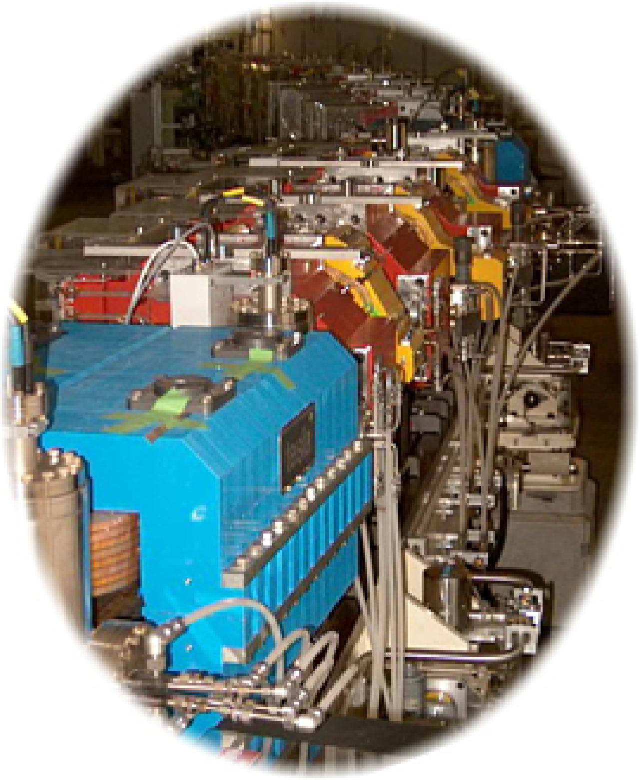
Design engineering: Assembly, feasibility

$\Longrightarrow$ Beam current $\longrightarrow$ Vacuum, Radiofrequency
$\Longrightarrow$ Space requirements $\longrightarrow$ Magnet, Vacuum, RF, Diagnostics, Engineering

Space requirements







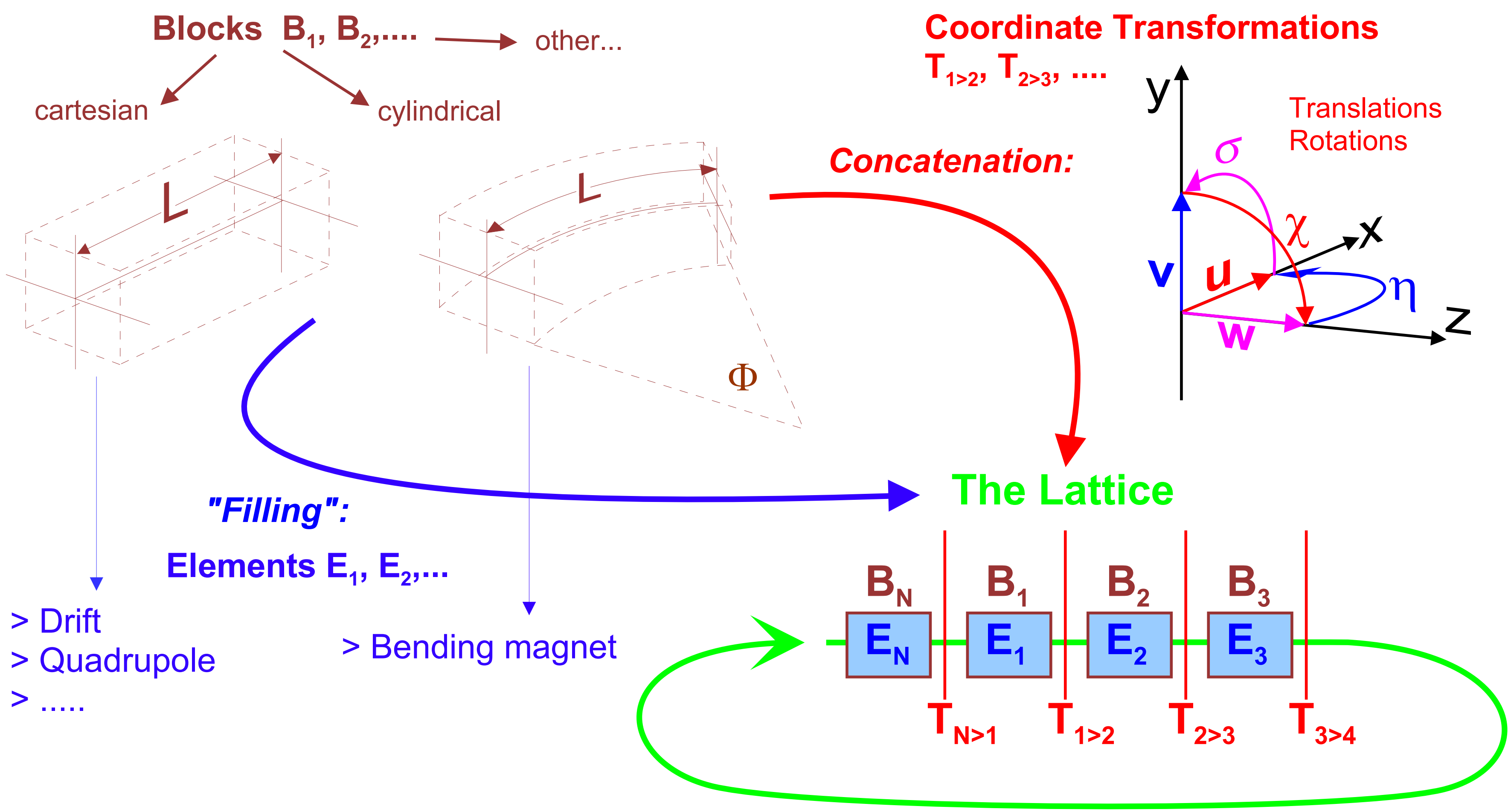
A lattice section....

(top) .....as seen by the lattice designer

(bottom) ..... as seen by the design engineer

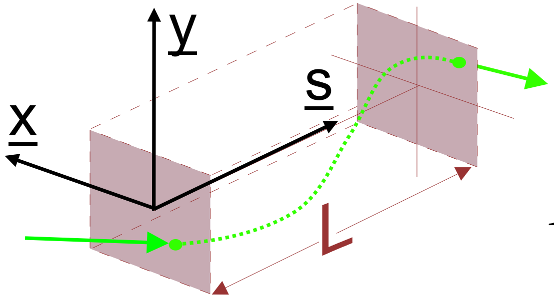
(right) ..... and how it looks in reality

Lattice composition



Ref.: E.Forrest & K.Hirata, A contemporary guide to beam dynamics, KEK 92-12

local $\Longleftrightarrow$ global



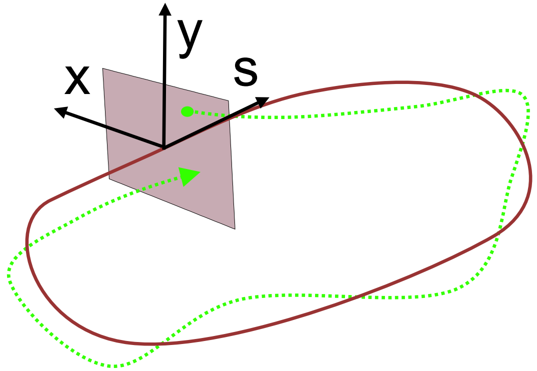
Block

$\vec{X}_{\text{in}} \longrightarrow \vec{X}_{\text{out}}$

$$\vec{X} = (x, p_x, y, p_y, \delta, \Delta s)$$

$$\delta := \frac{\Delta p}{p_o} \quad \Delta s = s - s_o$$

$\Longrightarrow$ Concatenate $\Longrightarrow$



Lattice

One turn map:

$\vec{X}_n \longrightarrow \vec{X}_{n+1}$

$$\left.\begin{array}{lcl} \textbf{Closed Orbit} & = & \text{Fixpoint} \\ \textbf{Transfermatrix} & = & \text{Linearization} \end{array}\right\} \text{of one turn map}$$

Approximations

Ideal lattice: no translations: closed orbit $\hat{=}$ block symmetry axis

Decoupling: Betatron frequencies $\gg$ Synchrotron frequency:

$$\{\underbrace{x, p_x, y, p_y,}_{\text{fast (MHz)}}\ \underbrace{\delta, \Delta s}_{\text{slow (kHz)}}\} \longrightarrow \delta, \Delta s \approx \text{constant.}$$

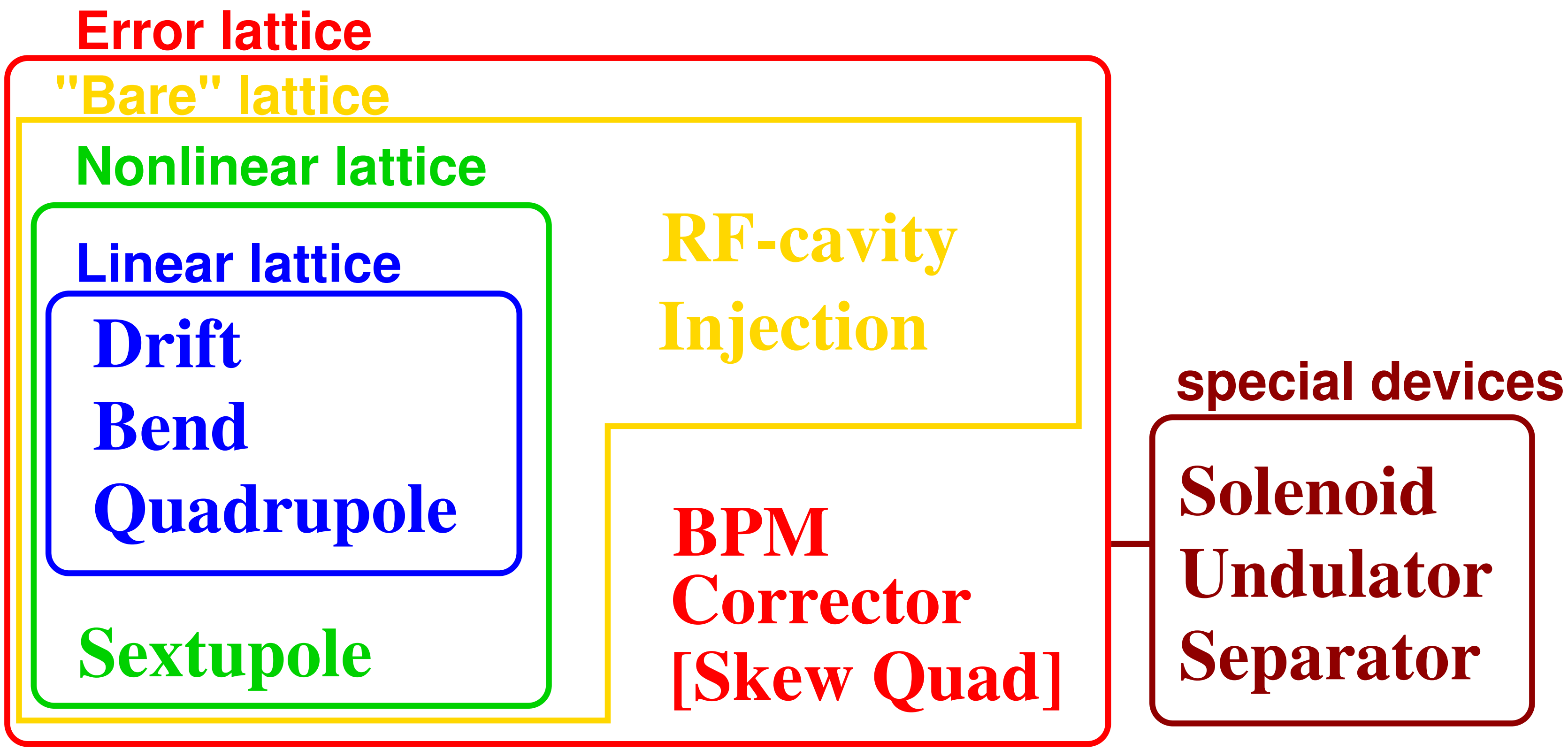
Horizontal-Vertical coupling $\kappa \ll 1$:

$$\{\ \underbrace{\{x, p_x\}}_{\text{horizontal}} \mid \underbrace{\{y, p_y\}}_{\text{vertical}} \mid \underbrace{\{\delta, \Delta s\}}_{\text{longitudinal}}\ \} \longrightarrow \text{Tracking}$$

Linear Lattice: betafunctions, betatron phases, etc.

Nonlinear latice: perturbative treatment of nonlinearities

Hierarchy of building blocks



Field and multipole definition

$$B_y(x,y) + iB_x(x,y) = (B\rho)\sum_n (ia_n + b_n)(x+iy)^{n-1}$$

$2n$-pole magnet: $n = 1, 2, 3 \ldots$ = dipole, quadrupole, sextupole

b_n regular, a_n skew (rotated by $90°/n$)

Magnetic rigidity: $B\rho = -\frac{p}{q} = -\frac{\beta E/e}{n_e c} = 3.3356\, E$[GeV] for electrons

Regular multipole: $b_n = \frac{1}{B\rho}\frac{1}{(n-1)!}\frac{\partial^{(n-1)} B_y(x,y)}{\partial x^{n-1}}\Big|_{y=0}$

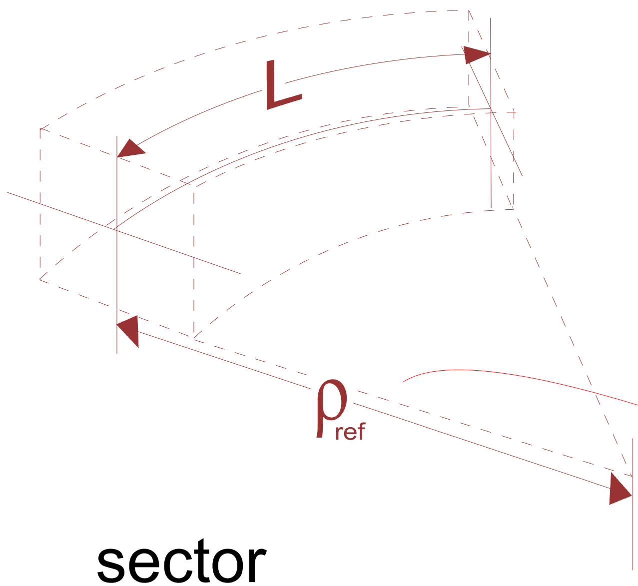
Poletip field: $B_{\rm pt} = (B\rho) b_n R^{n-1} = \frac{R^{n-1}}{(n-1)!}\frac{\partial^{(n-1)} B_y(x,y)}{\partial x^{n-1}}\Big|_{y=0}$

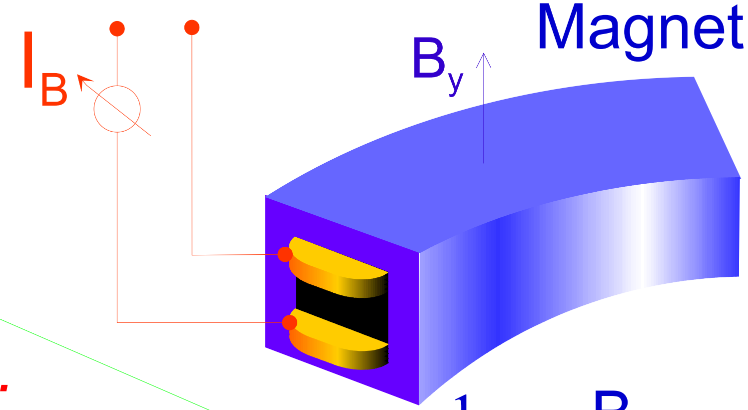
R pole inscribed radius

Conventions: $h = 1/\rho = b_1$ (Dip.), $k = -b_2$ (Quad.), $m = -b_3$ (Sext.)

Bending magnet

Cylindrical Block





Reference Particle:

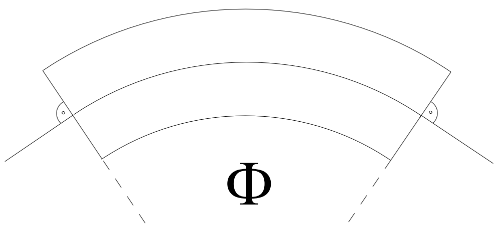
$(B\rho) = p/e$

adjust by:

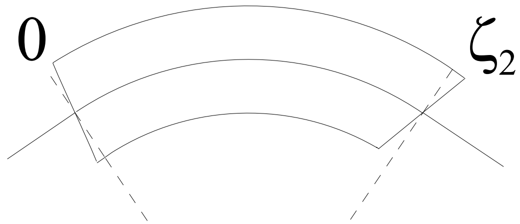
$\rho_{ref} \overset{!}{=} \rho$

$$b_1 = \frac{1}{\rho} = \frac{B_y}{(B\rho)}$$

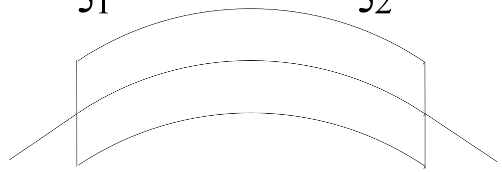
sector



$\zeta_1 > 0$ $\zeta_2 < 0$



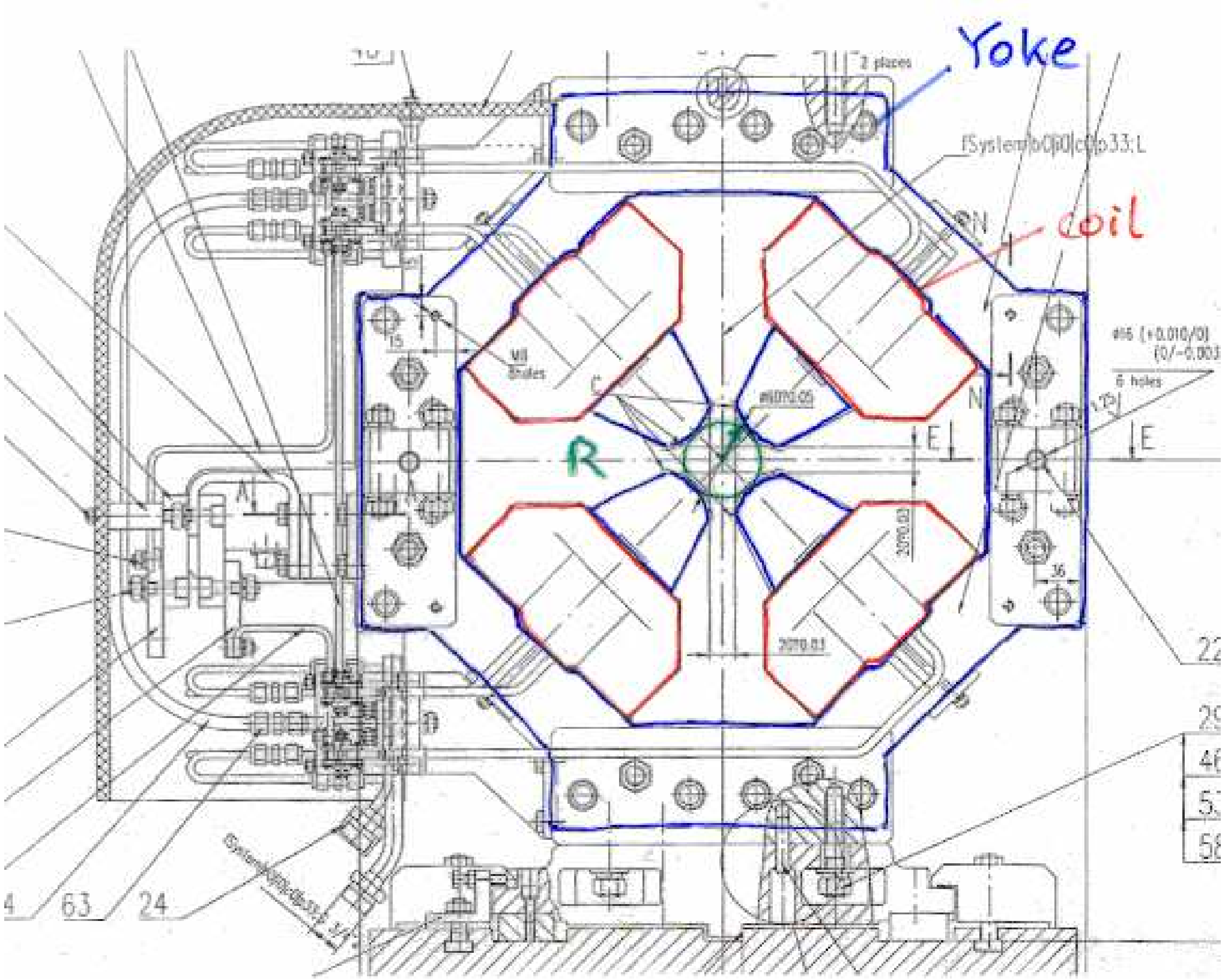
$\zeta_1 = \Phi/2 = \zeta_2$



rectangular bend

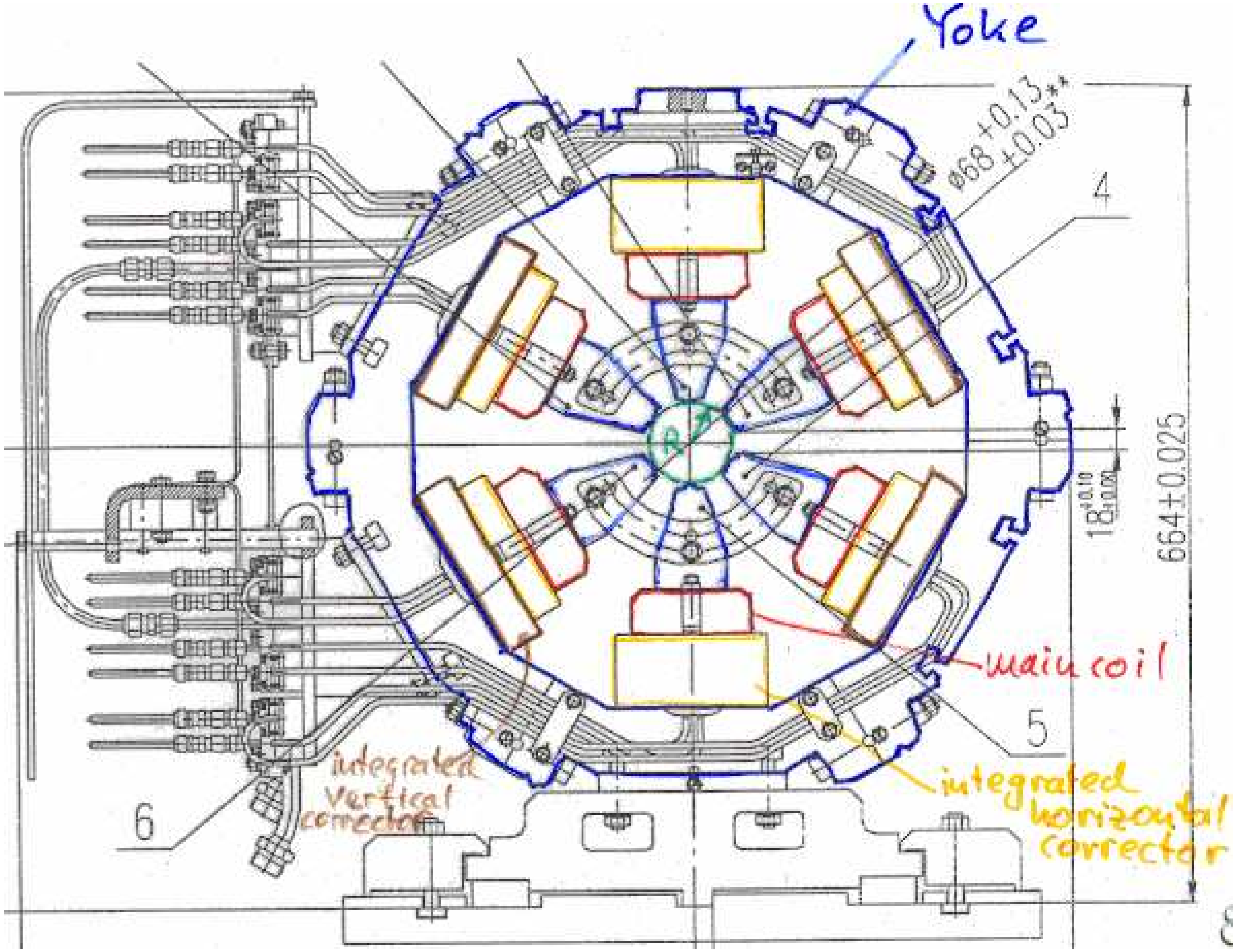
Parameters: $L,\ \frac{1}{\rho_{\rm ref}} \overset{!}{=} b_1, b_2, \zeta_1, \zeta_2, [g, k_1, k_2]\ ,\ [b_n, n \geq 3]$

Quadrupole



Parameters: L, b_2, R

Sextupole



Parameters: L, b_3, R

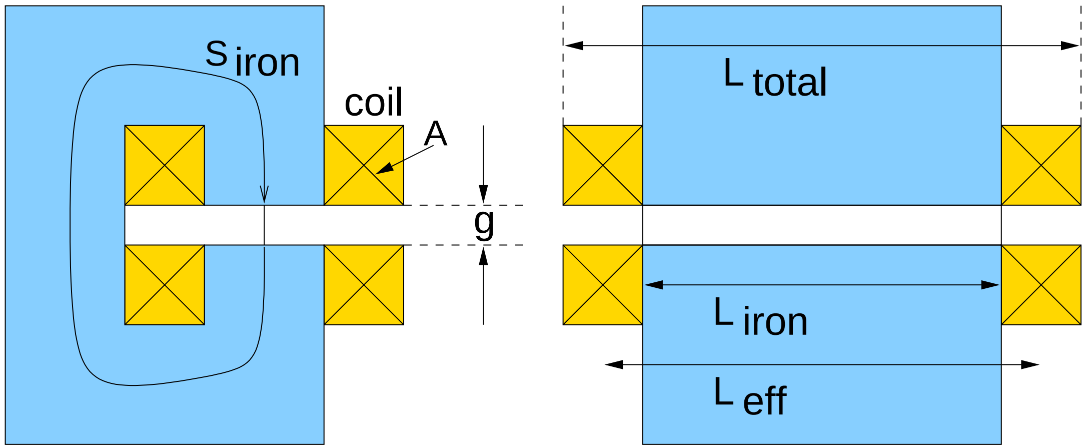
Other devices

	Parameters	Purpose
RF cavity	$\lambda_{\rm rf}$, $V_{\rm rf}$	acceleration, long. focussing
Septum magnet	position & width	injection
Kicker magnet	$\int b_1(t)dl$	injection
Correctors	$\int b_1 dl$, $\int a_1 dl$	orbit correction
BPM	passiv	orbit measurement
Skew Quadrupole	$\int a_2 dl$	coupling correction
Undulator	λ_u, N_u, B_u, g	$\rightarrow$ synchrotron light

Iron dominated dipole magnet

$$\oint H ds = \int\int j da \quad \Longrightarrow \quad \text{Coil cross section } A$$

$$A = \frac{B}{2j_c}\left(\frac{S_{\text{iron}}}{\mu_o \mu_r} + \frac{g}{\mu_o}\right) \xrightarrow{\mu_r \gg 1} A \approx \frac{Bg}{2j_c \mu_o}$$



Magnet poletip fields and apertures

	coil width $\frac{L_{\rm tot}-L_{\rm eff}}{2}$ [mm]	poletip field $B_{\rm pt}$ [T]	aperture R [mm]
Bending magnets:	$65 \ldots 150$	1.5	$20 \ldots 35$ (=$g/2$)
Quadrupoles:	$40 \ldots 70$	0.75	$30 \ldots 43$
Sextupoles:	$40 \ldots 80$	0.6	$30 \ldots 50$

(data from various light sources)

$\boxed{\text{Magnet current} \propto R^n}$

$\Longrightarrow$ **Apertures:** As small as possible

As large as necessary $\longrightarrow$ acceptance

Local: Transfer matrix of a gradient bend

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \delta \end{pmatrix}_{\text{out}} = \begin{pmatrix} c_x & \frac{1}{\sqrt{K}}\,s_x & 0 & 0 & \frac{b_1}{K}(1-c_x) \\ -\sqrt{K}\,s_x & c_x & 0 & 0 & \frac{b_1}{\sqrt{K}}\,s_x \\ 0 & 0 & c_y & \frac{1}{\sqrt{-b_2}}\,s_y & 0 \\ 0 & 0 & -\sqrt{-b_2}\,s_y & c_y & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ y \\ y' \\ \delta \end{pmatrix}_{\text{in}}$$

with: $c_x[s_x] = \cos\,[\sin](\sqrt{K}\,L)$, $c_y[s_y] = \cos\,[\sin](\sqrt{-b_2}\,L)$, $K = b_1^2 + b_2$

$\cos ix = \cosh x$, $\sin ix = i \sinh x \rightarrow$ can be focussing or defocussing

Special cases:

$b_2 = 0 \rightarrow$ Dipole $b_1 = 0 \rightarrow$ Quadrupol $b_1 = 0$ and $b_2 = 0 \rightarrow$ Drift

Transfer matrix of a gradient bend, alternative convention

$$\begin{pmatrix} x \\ x' \\ y \\ y' \\ \delta \end{pmatrix}_{\text{out}} = \begin{pmatrix} c_x & \frac{1}{\sqrt{K}}\, s_x & 0 & 0 & \frac{h}{K}\,(1-c_x) \\ -\sqrt{K}\, s_x & c_x & 0 & 0 & \frac{h}{\sqrt{K}}\, s_x \\ 0 & 0 & c_y & \frac{1}{\sqrt{k}}\, s_y & 0 \\ 0 & 0 & -\sqrt{k}\, s_y & c_y & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ y \\ y' \\ \delta \end{pmatrix}_{\text{in}}$$

with: $c_x[s_x] = \cos\,[\sin](\sqrt{K}\,L)$, $\quad c_y[s_y] = \cos\,[\sin](\sqrt{k}\,L)$, $\quad K = h^2 - k$

$\cos ix = \cosh x$, $\sin ix = i \sinh x \rightarrow$ can be focussing or defocussing

Special cases:

$k = 0 \rightarrow$ Dipole $\quad h = 0 \rightarrow$ Quadrupol $\quad h = 0$ and $k = 0 \rightarrow$ Drift

Global: Betafunction and Emittance

Practical view: $\sigma_x = \underbrace{\sqrt{\text{Betafunction } \beta}}_{\text{magnet structure}} \times \underbrace{\sqrt{\text{Emittance } \epsilon}}_{\text{particle ensemble}}$

Theoretical view: $H = \frac{p_x^2}{2} + \frac{k(s)x^2}{2} \xrightarrow{\text{c.t.}} \tilde{H} = \frac{J}{\beta(s)}, \quad \epsilon = \langle J \rangle$

$\longrightarrow$ Betatron oscillation: $x(s) = \sqrt{2J_x \cdot \beta_x(s)} \cos\phi(s) + D(s) \cdot \delta$

$\longrightarrow$ Twiss parameters: $\phi = \int \frac{1}{\beta}\, ds \quad \alpha = -\frac{\beta'}{2} \quad \gamma = \frac{1+\alpha^2}{\beta}$

Transformation of twiss parameters:

$$\begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_b = \begin{pmatrix} C^2 & -2SC & S^2 \\ -CC' & S'C + SC' & -SS' \\ C'^2 & -2S'C' & S'^2 \end{pmatrix} \cdot \begin{pmatrix} \beta \\ \alpha \\ \gamma \end{pmatrix}_a \quad \text{with} \quad M_{a\to b} = \begin{pmatrix} C & S \\ S' & C' \end{pmatrix}$$

Ref.: R. D .Ruth, Single particle dynamics in circular accelerators, AIP Conf.Proc. 153 (1987) 150

Global: Transfermatrix of a lattice section

Transfermatrix $a \to b$: Transformation to normalized phase space at a
$\to$ Rotation by $\Delta\phi = \phi_b - \phi_a$ in normalized phase space
$\to$ Backtransformation to real phase space at b

$$M_{a\to b} = T_b^{-1} \begin{pmatrix} \cos\Delta\phi & \sin\Delta\phi \\ -\sin\Delta\phi & \cos\Delta\phi \end{pmatrix} T_a \quad \text{with} \quad T = \begin{pmatrix} \frac{1}{\sqrt{\beta}} & 0 \\ \frac{\alpha}{\sqrt{\beta}} & \sqrt{\beta} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{\frac{\beta_b}{\beta_a}}(\cos\Delta\phi + \alpha_a \sin\Delta\phi) & \sqrt{\beta_a\beta_b}\sin\Delta\phi \\ \frac{(\alpha_a-\alpha_b)\cos\Delta\phi-(1+\alpha_a\alpha_b)\sin\Delta\phi}{\sqrt{\beta_a\beta_b}} & \sqrt{\frac{\beta_a}{\beta_b}}(\cos\Delta\phi - \alpha_b \sin\Delta\phi) \end{pmatrix}$$

Periodic structure ($a = b$) $\to$ One turn matrix ($\mu = 2\pi Q$, Q betatron tune):

$$M_a = \begin{pmatrix} \cos\mu + \alpha_a \sin\mu & \beta_a \sin\mu \\ -\gamma_a \sin\mu & \cos\mu - \alpha_a \sin\mu \end{pmatrix} \xrightarrow{\text{symmetry point}} \begin{pmatrix} \cos\mu & \beta_a \sin\mu \\ -\frac{1}{\beta_a}\sin\mu & \cos\mu \end{pmatrix}$$

Lattice Design Code

Model: complete set of elements, correct methods for tracking and concatenation, well documented approximations

Elementary functions: beta functions and dispersions, periodic solutions, closed orbit finder, energy variations, tracking, matching

Toolbox: Fourier transforms of particle data ($\longrightarrow$ resonance analysis), minimizaton routines ($\longrightarrow$ dynamic aperture optimization, coupling suppression) , linear algebra package ($\longrightarrow$ orbit correction)

User convenience: editor functions, graphical user interface, editable text files

Extended functions: RF dimensioning, geometry plots, lifetime calculations, injection design, alignment errors, multipolar errors, ground vibrations

Connectivity: database access, control system access ($\longrightarrow$ real machine)

Natural horizontal emittance

Flat lattice:

$$\epsilon_{xo}[\text{nm}\cdot\text{rad}] = \underbrace{\frac{55\,\hbar c}{32\sqrt{3}\,m_e c^2}}_{3.83\cdot 10^{-13}\,\text{m}} \gamma^2 \frac{I_5}{J_x I_2} = 1470\,(E[\text{GeV}])^2 \frac{\langle \mathcal{H}/\rho^3 \rangle}{J_x \langle 1/\rho^2 \rangle}$$

Lattice invariant (or "dispersion's emittance"):

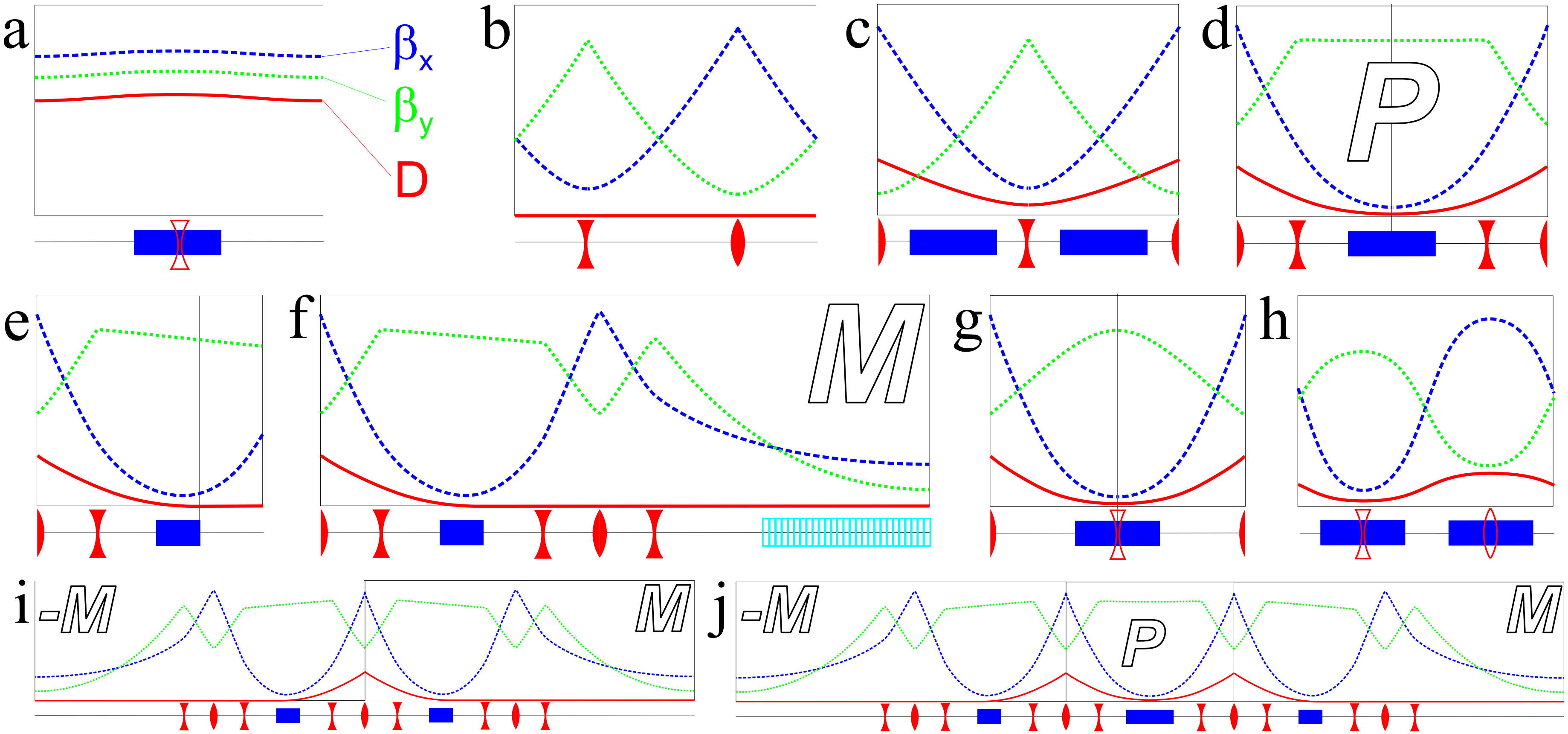
$$\mathcal{H}(s) = \gamma_x(s) D(s)^2 + 2\alpha_x(s) D(s) D'(s) + \beta_x(s) D'(s)^2$$

Horizontal damping partition $J_x \approx 1$ $\langle \ldots \rangle$ lattice average $\langle \ldots \rangle_{\text{mag}}$ magnets average

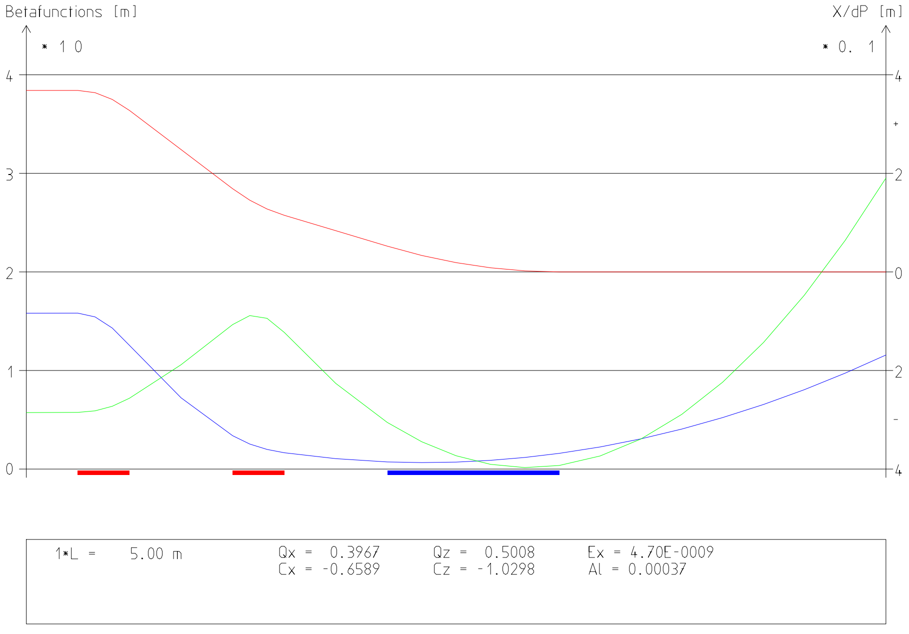
Simplification for isomagnetic lattice:

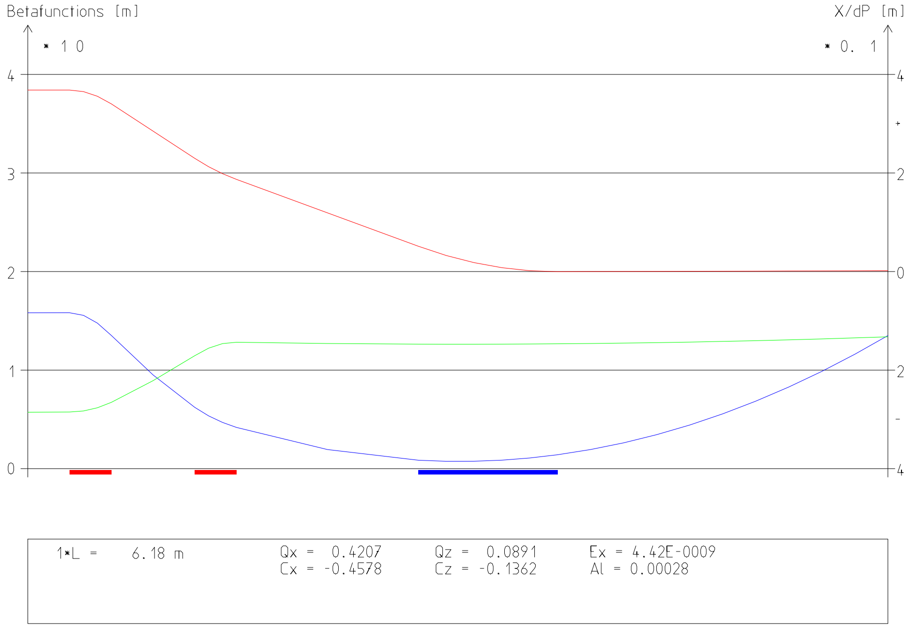
$$\epsilon_{xo}[\text{nm}\cdot\text{rad}] = 1470\,(E[\text{GeV}])^2 \frac{\langle \mathcal{H} \rangle_{\text{mag}}}{\rho J_x}$$

Building low emittance lattices . . .



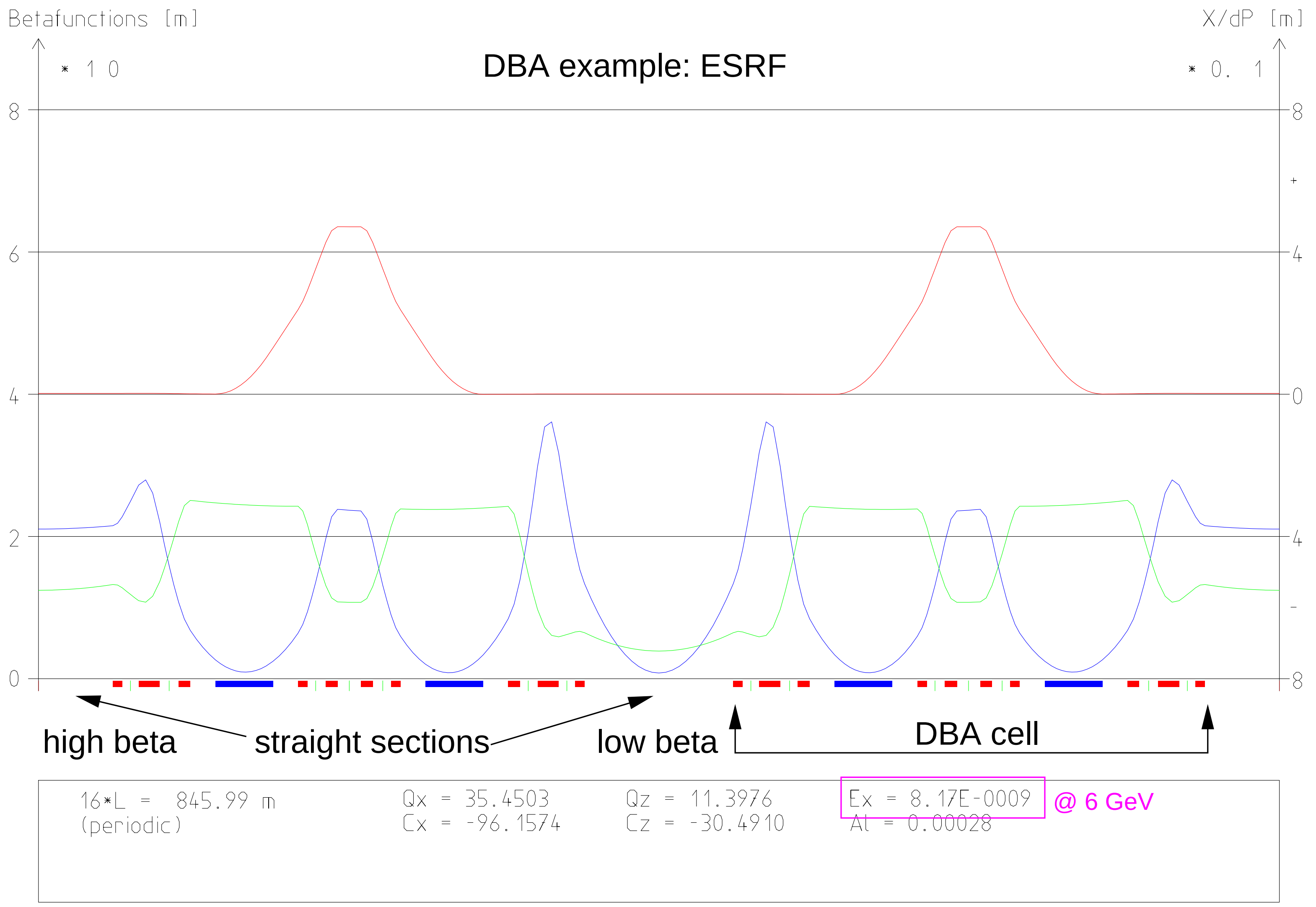
Dispersion suppression by using a half bending magnet:



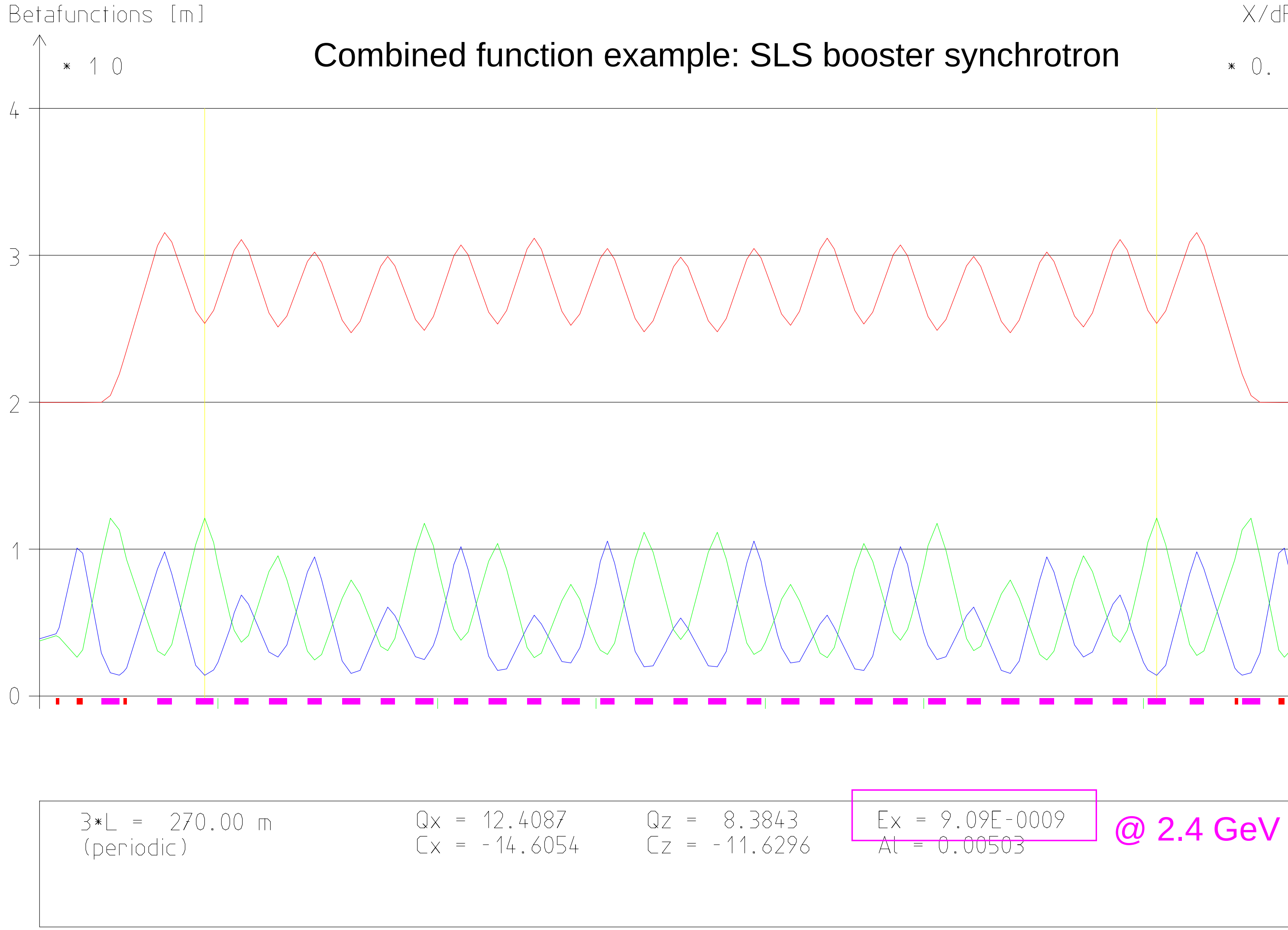


Increased quadrupole strength
(lengthes constant)

Incrased length before half bend
(quadrupoles constant)


Betafunctions [m]
* 1 0
DBA example: ESRF
X/dP [m]
* 0. 1
8
6
4
2
0
8
4
0
4
8
high beta
straight sections
low beta
DBA cell
16*L = 845.99 m
(periodic)
Qx = 35.4503
Cx = -96.1574
Qz = 11.3976
Cz = -30.4910
Ex = 8.17E-0009
Al = 0.00028
@ 6 GeV

Lattice Cells

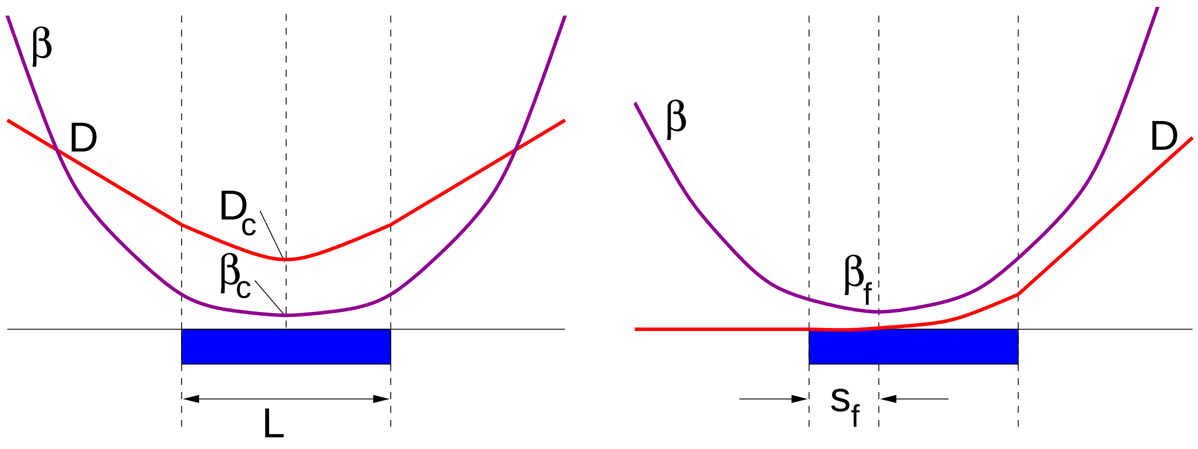

Betafunctions [m]
X/dP [m]
Combined function example: SLS booster synchrotron
* 1 0
* 0. 1
4
3
2
1
0
8
4
0
4
8
3*L = 270.00 m
(periodic)
Qx = 12.4087
Cx = -14.6054
Qz = 8.3843
Cz = -11.6296
Ex = 9.09E-0009
Al = 0.00503
@ 2.4 GeV

Minimum Emittance

$d\langle\mathcal{H}(\alpha_{xc}, \beta_{xc}, D_c, D_c')\rangle_{\mathrm{mag}} = 0 \implies$ Minimum emittance:

$$\epsilon_{xo}[\mathrm{nm{\cdot}rad}] = 1470\,\frac{(E[\mathrm{GeV}])^2}{J_x}\,\frac{\Phi^3 F}{12\sqrt{15}}$$

Φ [rad] magnet deflection angle ($\Phi/2 \ll 1$)



$F = 1$

$$\beta_{xc} = \frac{1}{2\sqrt{15}}L \quad D_c = \frac{1}{24\rho}L^2$$

$F = 3$

$$s_f = \frac{3}{8}L \quad \beta_{xf} = \sqrt{\frac{3}{320}}L$$

Minimum emittance 2

Deviations from minimum:

$$b = \frac{\beta_{xc}}{\beta_{xc,\min}} \qquad F = \frac{\epsilon_{xo}}{\epsilon_{xo,min}}$$
$$d = \frac{D_c}{D_{c,\min}}$$

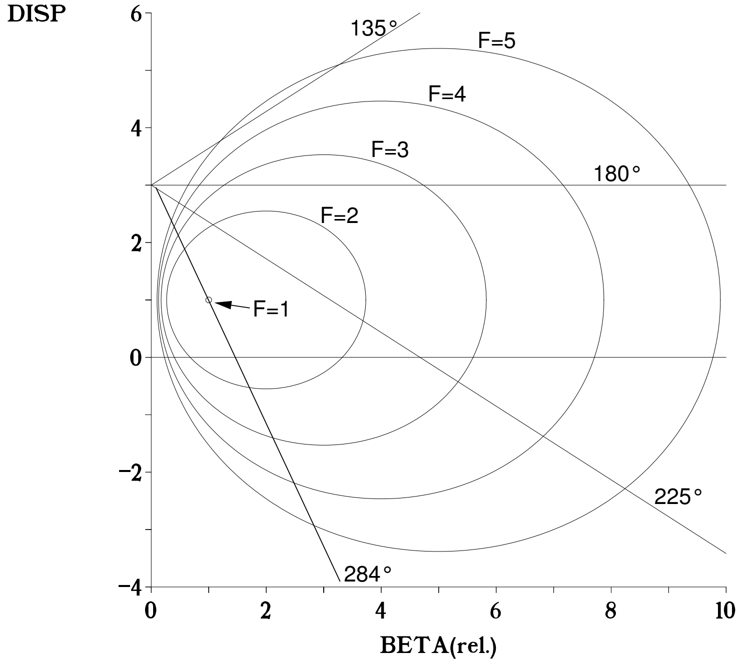
Relative emittance F:

$$\frac{5}{4}(d-1)^2 + (b-F)^2 = F^2$$

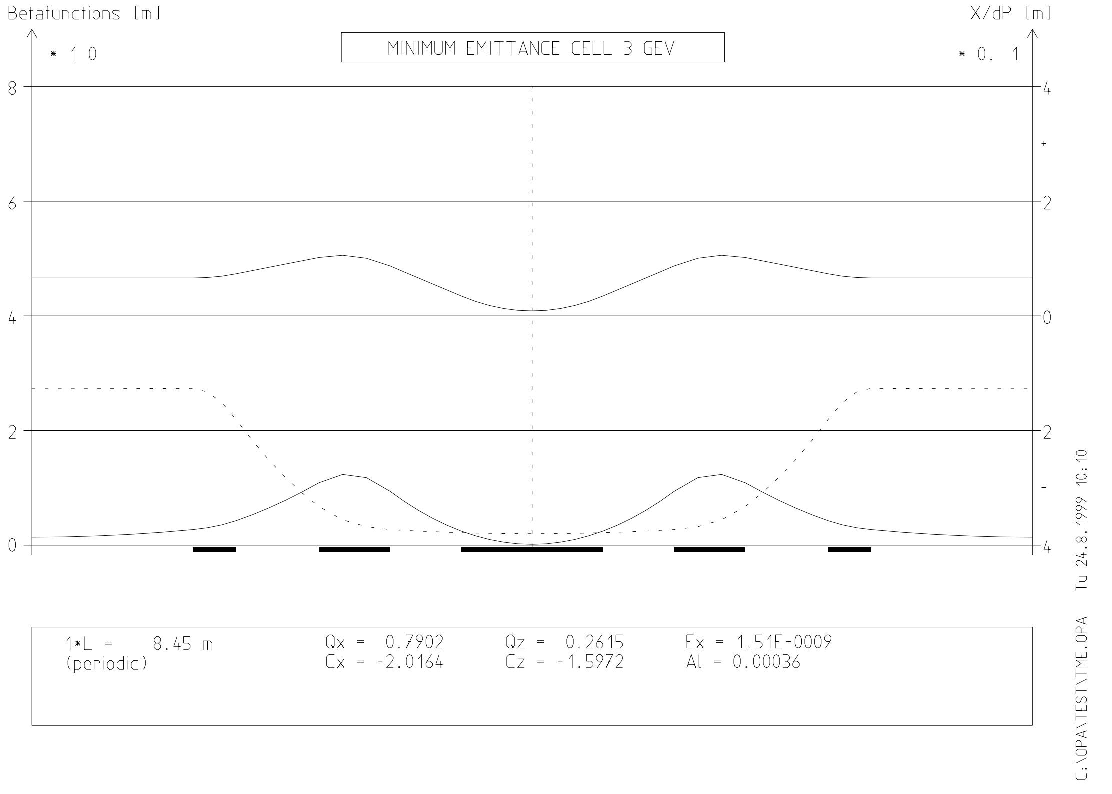
Phase advance in cell:

$$\Psi = 2\arctan\left(\frac{6}{\sqrt{15}}\frac{b}{(d-3)}\right)$$

$$F = 1 \implies \Psi = 284.5°$$



Minimum emittance cell



10° gradient free sector bend, b=d=1, $E = 3$ GeV

$\Longrightarrow F = 1$: Theoretical minimum emittance (`Ex`)= 1.5 nm·rad

Tune advance (`Qx`)=0.7902 $\Longleftrightarrow$ Ideal phase advance $\Psi = 284.5°$.

Damping times

$$\tau_i = 6.67\ \mathrm{ms}\ \frac{C\ [\mathrm{m}]\ E\ [\mathrm{GeV}]}{J_i\ U\ [\mathrm{keV}]} \qquad J_x = 1 - \mathcal{D} \quad J_y = 1 \quad J_s = 2 + \mathcal{D}$$

$$\mathcal{D} = \frac{1}{2\pi} \int_{\mathrm{mag}} D(s)\left[b_1(s)^2 + 2b_2(s)\right]\,ds$$

Energy loss per turn: $U\ [\mathrm{keV}] = 26.5\,\left(E[\mathrm{GeV}]\right)^{\mathbf{3}}\ B[\mathrm{T}]$

Stability requirement: $-2 < \mathcal{D} < 1$

Separate function bends: $\mathcal{D} \ll 1$ in light sources.

Combined function bending magnets: Adjust gradients!
Option: Vertical focusing in bending magnet: $b_2 < 0 \rightarrow J_x \rightarrow 2$: half emittance!

Energy spread and Beam size

r.m.s. natural energy spread:

$$\sigma_e = 6.64 \cdot 10^{-4} \cdot \sqrt{\frac{B[T]\,E[GeV]}{J_s}} \qquad J_s \approx 2$$

Beam size and effective emittance:

$$\sigma_x(s) = \sqrt{\epsilon_x\,\beta_x(s) + (\sigma_e\,D(s))^2} \qquad \sigma_y(s) = \sqrt{\epsilon_y\,\beta_y(s)}$$

$$\epsilon_{x,\mathrm{eff}}(s) = \sqrt{\epsilon_{xo}^2 + \epsilon_{xo}\mathcal{H}(s)\sigma_e^2}$$

Vertical emittance

Ideal flat Lattice: $\mathcal{H}_y \equiv 0 \quad \longrightarrow \quad \epsilon_y = 0$

Real Lattice: Errors as sources of vertical emittance ϵ_y

Vertical dipoles (a_1):	Skew quadrupoles (a_2):	
Dipole rolls	Quadrupole rolls	roll $= s$-rotation
Quadrupole heaves	Sextupole heaves	heave $= \Delta y$ displacement
Vertical dispersion (D_y)	**Linear coupling (κ)**	
$\rightarrow$ orbit correction	$\rightarrow$ skew quadrupoles	for suppression

Emittance ratio $g = \frac{\epsilon_y}{\epsilon_x} \quad \rightarrow \quad \epsilon_x = \frac{1}{1+g}\epsilon_{xo} \quad \epsilon_y = \frac{g}{1+g}\epsilon_{xo}$

Coupling corrected lattices: $g \approx 10^{-3}$

BUT: Diffraction limitation $\rightarrow$ Brightness $\sim 1/g$ only for hard X-rays
Touschek lifetime $\sim$ (bunch volume) $\sim \sqrt{g}$

Circumference and periodicity

Circumference C

- Area $\rightarrow$ minimize
- Optics $\rightarrow$ relax
- Spaces $\rightarrow$ reserve
- RF harmonic number
  $\rightarrow C = h\lambda_{rf}$
  $\rightarrow h = h_1 \cdot h_2 \cdot h_3 \ldots$

Ritsumeikan PSR	$C = 98$ cm
LEP	$C = 27$ km

Periodicity N_{per}

Advantages of large periodicity:

- simplicity: design & operation
- stability: resonances
- cost efficiency: few types

DORIS:	$N_{\mathrm{per}} = 1$
APS:	$N_{\mathrm{per}} = 40$

Working point

Betatron resonances:

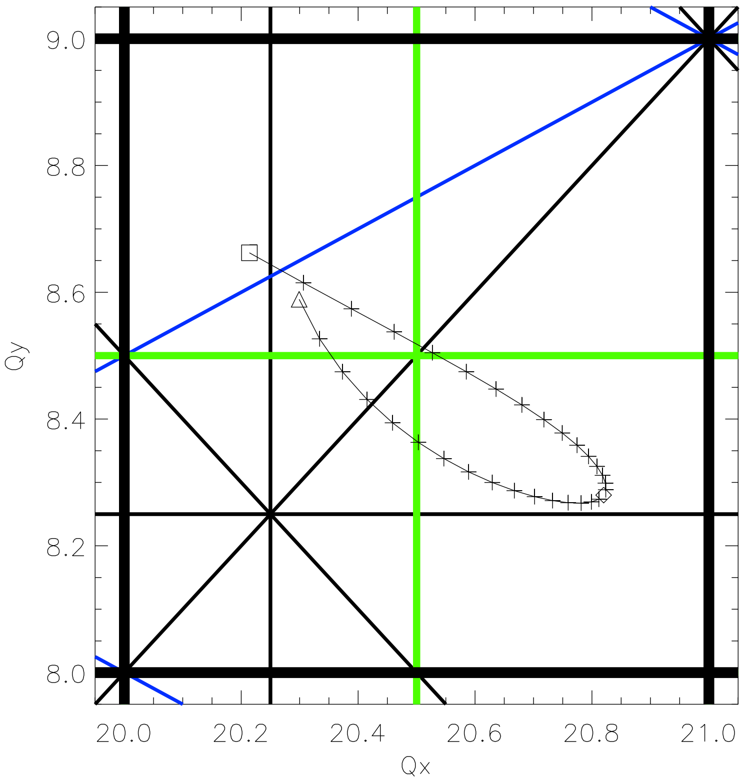
$$\boxed{aQ_x + bQ_y = p}$$

order: $n = |a| + |b|$
systematic: N_{per}/p = integer
regular: b even, skew: b odd
(a, b, k, n, N_{per}, p integers)

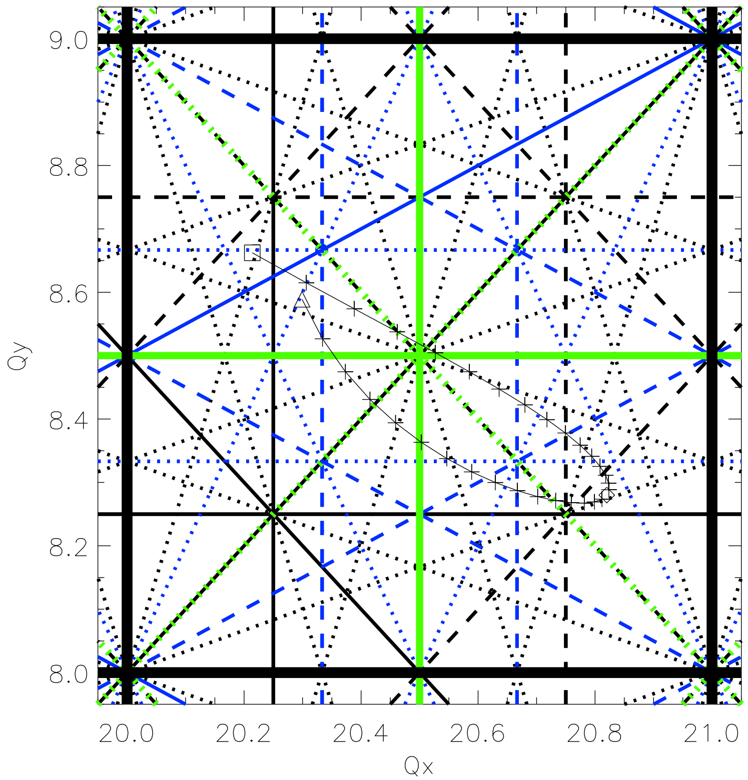
Tune constraints:

- NO integer — $Q_{x;y} = k \longrightarrow$ dipolar errors
- NO half integer — $Q_{x;y} = (2k + 1)/2 \longrightarrow$ gradient errors
- NO sum resonance — $Q_x + Q_y = p \longrightarrow$ coupling
- NO sextupole resonances — $Q_x = p$, $3Q_x = p$, $Q_x \pm 2Q_y = p$ $\longrightarrow$ dynamic acceptance
- Multiturn injection: — $|\text{frac}(Q)| \geq 0.2 \longrightarrow$ septum
- and more…

Working point: Example



Ideal lattice



Real Lattice

Chromaticity

Chromatic aberrations: $b_2(\delta) = b_2/(1+\delta) \implies Q = Q_o + \xi\delta$

Chromaticity $\xi = dQ/d\delta < 0$:

- Tune spread, resonance crossings
- Head-tail instability

Correction by sextupoles in dispersive regions:

$$\begin{aligned} \xi_x &= \frac{1}{4\pi}\oint_C \left[2b_3(s)D(s) - b_2(s)\right]\beta_x(s)\,ds \\ \xi_y &= \frac{1}{4\pi}\oint_C \left[-2b_3(s)D(s) + b_2(s)\right]\beta_y(s)\,ds \end{aligned}$$

Sextupole nonlinearity ($B_y(x) \sim x^2$) $\implies$ dynamic acceptance problems

Acceptance Definitions

Acceptance: 6D volume of stable particles $\longrightarrow$ decoupling:
horizontal, vertical and longitudinal 2D-acceptances

Physical acceptance Linear lattice $\longrightarrow$ vaccum chamber $\longrightarrow$ “known”

Dynamic acceptance Nonlinear lattice $\longrightarrow$ separatrix $\longrightarrow$ “unknown”

Longitudinal acceptance

- RF momentum acceptance (bucket height)
- Lattice momentum acceptance = δ-dependant horizontal acceptance

Dynamic aperture = *local* projection of dynamic acceptance
acceptance [mm·mrad] $\longleftrightarrow$ aperture [mm]

Design criterion: Dynamic acceptance > physical acceptance

Physical acceptance

Linear lattice (quads and bends only): “infinite” dynamic acceptance

Particle at acceptance limit A_x:

$$x(s) = \sqrt{A_x \cdot \beta_x(s)} \cos(\phi(s)) + D(s) \cdot \delta$$

Particle loss: $|x(s)| \geq a_x(s)$ somewhere.

Acceptance

$$A_x = \min \left(\frac{(a_x(s) - |D(s) \cdot \delta|)^2}{\beta_x(s)} \right)$$

A_x invariant of betatron motion. $\rightarrow$ Projection:

$$x_{\max}(s) = \pm \sqrt{A_x \cdot \beta_x(s)} + D(s) \cdot \delta$$

Momentum acceptance

Horizontal acceptance $A_x = 0$ for $|\delta| > \min(a_x(s)/|D(s)|)$

BUT:

Scattering processes $\rightarrow$ momentum change of core particles:
$\vec{X} = (\approx 0, \approx 0, \approx 0, \approx 0, \delta, 0)$

Betatron oscillation around dispersive orbit with amplitude A_x

$$A_x = \gamma_{xo}(D_o\delta)^2 + 2\alpha_{xo}(D_o\delta)(D'_o\delta) + \beta_{xo}(D'_o\delta)^2 = \mathcal{H}_o\delta^2$$

$\beta_{xo} := \beta_x(s_o)$ etc., s_o = location of scattering event!

Maximum value of betatron oscillation:

$$x(s) = \sqrt{A_x\beta_x(s)} + |D(s)\delta| = \left(\sqrt{\mathcal{H}_o\beta_x(s)} + |D(s)|\right) \cdot |\delta|$$

Local momentum acceptance:

$$\delta_{\mathrm{acc}}(s_o) = \pm \min\left(\frac{a_x(s)}{\sqrt{\mathcal{H}_o \beta_x(s)} + |D(s)|}\right)$$

Momentum acceptance for different lattice locations ($a_x(s) = a_x$):

In dispersionfree section:
$\mathcal{H}_o = 0 \quad \rightarrow \quad \delta_{\mathrm{acc}} = \pm a_x / D_{\max}$

At location of maximum dispersion:
$\mathcal{H}_o = \gamma_o D_{\max}^2 \quad \rightarrow \quad \delta_{\mathrm{acc}} = \pm a_x / (2 D_{\max})$

Dynamic acceptance

$$
\begin{array}{llll}
\text{Quadrupole:} & \Delta x' = -b_2 L x & & \Delta y' = b_2 L y \\
\text{Sextupole:} & \Delta x' = -b_3 L (x^2 - y^2) & & \Delta y' = 2 b_3 L x y
\end{array}
$$

Quadrupole: chromatic aberration $\quad b_2(\delta) = b_2/(1+\delta) \approx b_2(1-\delta)$

$\rightarrow$ compensation by sextupole in dispersive region ($x \rightarrow D\delta + x$, $y \rightarrow y$):

$$
\begin{array}{lll}
\text{Quadrupole:} & \Delta(\Delta x') = [b_2 L]\,\delta\, x & \Delta(\Delta y') = -[b_2 L]\,\delta\, y \\
\text{Sextupole:} & \Delta(\Delta x') = -[2 b_3 L D]\,\delta\, x - b_3 L D^2 \delta^2 - b_3 L (x^2 - y^2) & \\
 & \Delta(\Delta y') = [2 b_3 L D]\,\delta\, y + 2 b_3 L\, x y &
\end{array}
$$

$\downarrow$ **nonlinear kicks:** $\leftarrow \overset{\infty}{\frown} \Longleftrightarrow \overset{\bullet\,\bullet}{\smile} \rightarrow (b_2 L \overset{!}{=} 2 b_3 L D) \rightarrow$ Chromaticity correction

nonlinear resonance driving horizontal/vertical coupling amplitude dependant tune shift	$\Longrightarrow$	**CHAOS !** Restriction of dynamic acceptance

Sextupole effects

9 first order terms:

- 2 chromaticities ξ_x, ξ_y
- 2 off-momentum resonances $2Q_x, 2Q_y \rightarrow d\beta/d\delta \rightarrow \xi^{(2)} = \partial^2 Q/\partial\delta^2$
- 2 terms $\rightarrow$ integer resonances Q_x
- 1 term $\rightarrow$ 3^{rd} integer resonances $3Q_x$
- 2 terms $\rightarrow$ coupling resonances $Q_x \pm 2Q_y$

13 second order terms:

- 3 tune shifts with amplitude: $\partial Q_x/\partial J_x,\;\; \partial Q_x/\partial J_y = \partial Q_y/\partial J_x,\;\; \partial Q_y/\partial J_y$
- 8 terms $\rightarrow$ octupole like resonances: $4Q_x, 2Q_x \pm 2Q_y, 4Q_y, 2Q_x, 2Q_y$
- 2 second order chromaticities: $\partial^2 Q_x/\partial\delta^2$ and $\partial^2 Q_y/\partial\delta^2$

Ref.: J.Bengtsson, The Sextupole Scheme for the Swiss Light Source: An Analytic Approach, SLS-Note 9/97, PSI 1997

Nonlinear Lattice Design

Optimization of sextupole patterns:

Chromaticity correction: Decoupling: $\rightarrow$ keep strength low

Sextupoles in quadrupoles: $b_3 = b_2/D \rightarrow$ inflexible!

Non interleaved sextupoles: "$-I$ transformer" (KEK-B scheme)

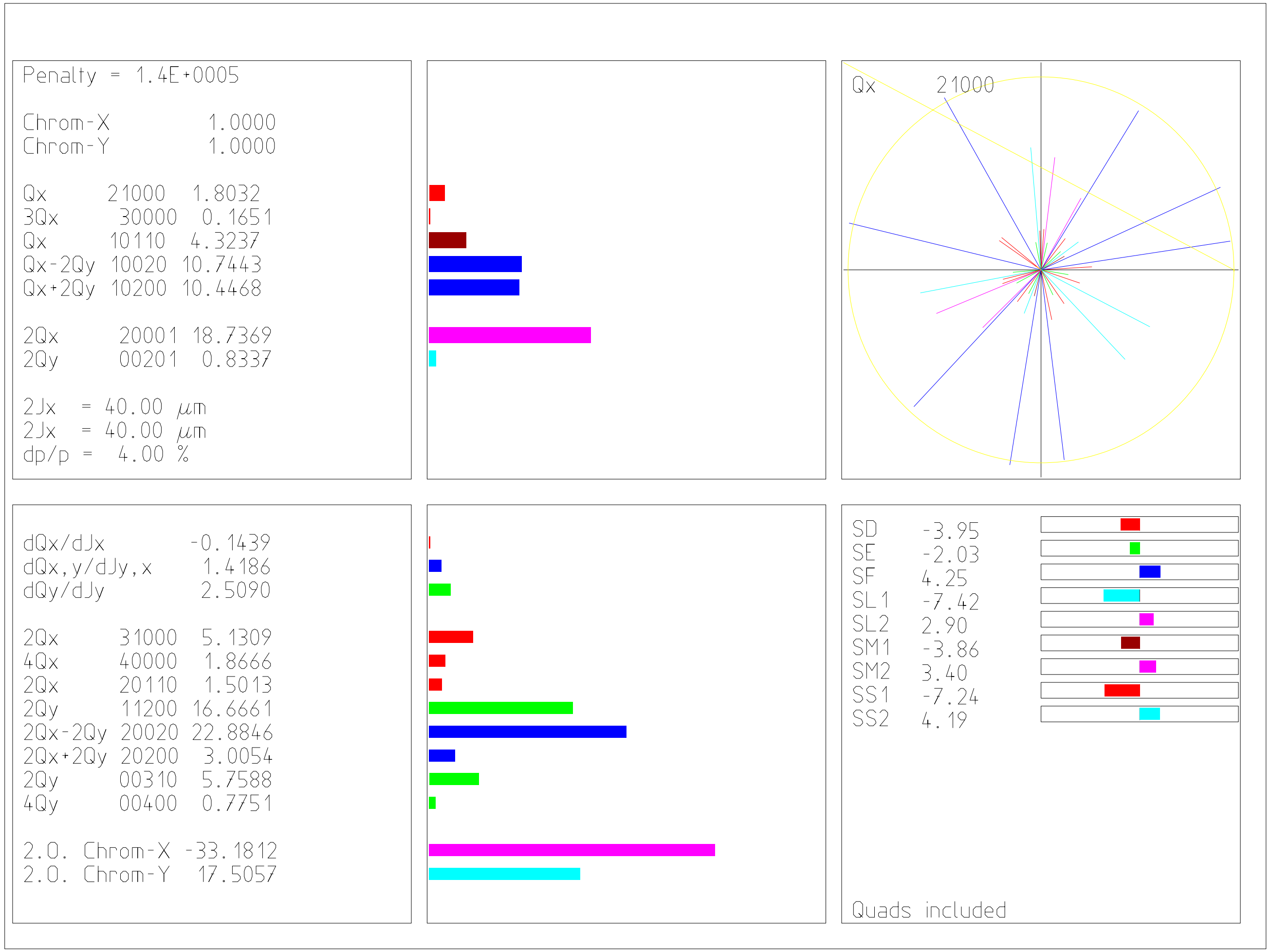
Multicell cancellation: N cells:

$$N\Delta Q_x, \quad 3N\Delta Q_x, \quad 2N\Delta Q_x, \quad 2N\Delta Q_y \quad \longrightarrow \text{integer!}$$

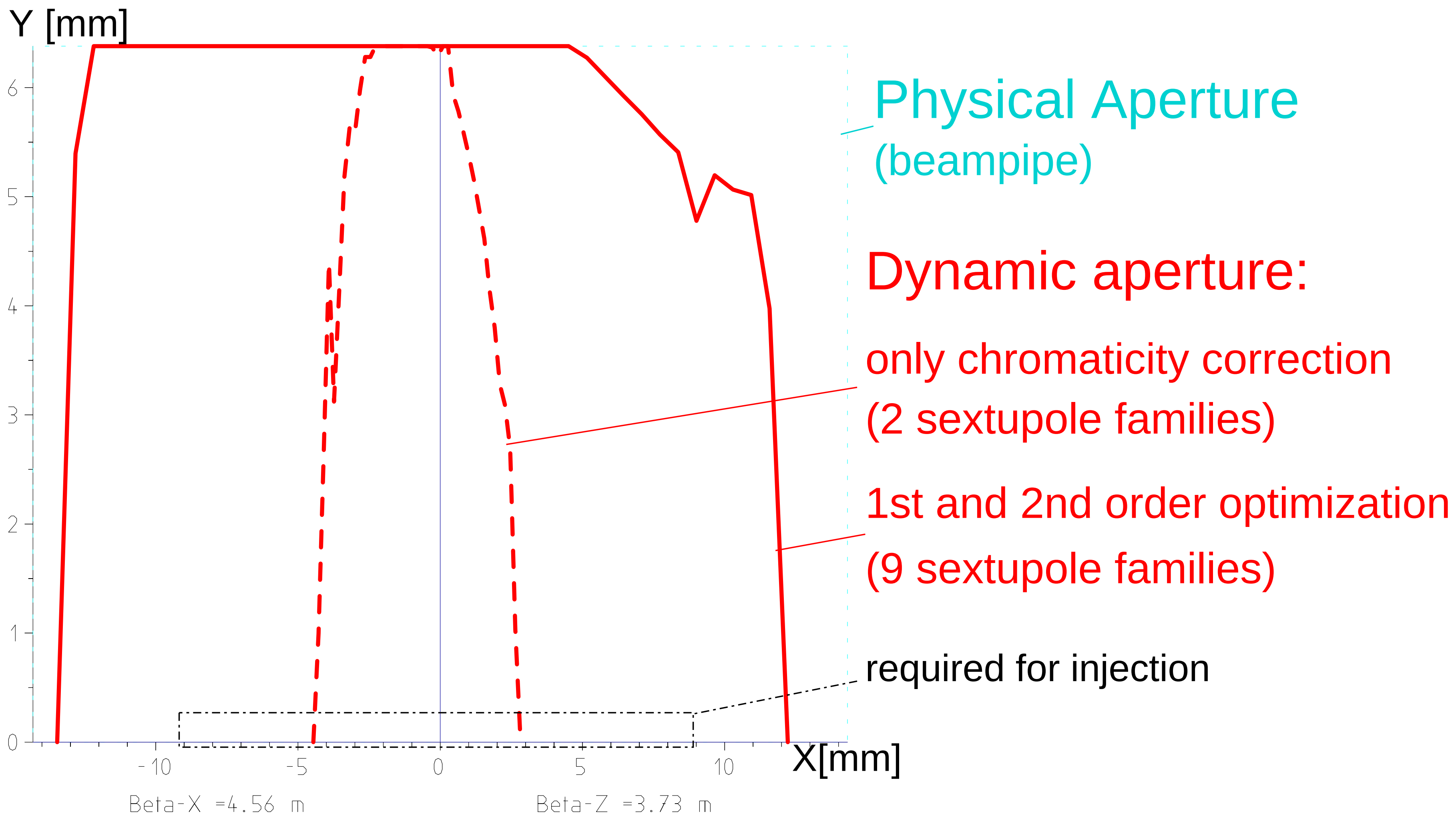
Cancellation between sections: lattice section vs. mirror image

$\Longrightarrow$ Iterate: linear $\Longleftrightarrow$ nonlinear lattice design

Sextupole pattern optimization



Dynamic aperture optimization



else . . .

Impact on lattice design:

The Injection Process: multi turn accumulation.

Lattice Errors:

- Magnet misalignments
  - Closed Orbit distortion and correction $\longrightarrow$ BPMs and correctors
  - Correlated misalignments: magnet girders and dynamic alignment concepts
  - Ground waves and vibrations: orbit feedback
  - Beam rotation and coupling control
- Multipolar errors (Magnets and Undulators): Dynamic acceptance