Accelerator Technology Department

Foundations of Analytical and Numerical Field Computation

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Overview

Rubbish in

Preprocessor

Solver

Postprocessor

Rubbish out

Optimization

Parametric geometric mod. (coil)
Parametric geometric mod. (iron)
Excitation field (coil)

Mesh-generator

Material data
Boundary conditions

M(B) iteration

FEM-BEM Models

Linear equation solver

Plotting

Field analysis (harmonics, inductance, forces)
(load-line, magnetization, eddy-currents)

Affine spaces, orientation

Vector-fields

Maxwell's equations

Potential formulations

Boundary and interface cond.

Harmonic fields, multipoles

BH curves

Biot-Savart's law

Magneto-motive force

Week formulation (1D, 2D, 3D)

Element shape functions

Negative Jacobians

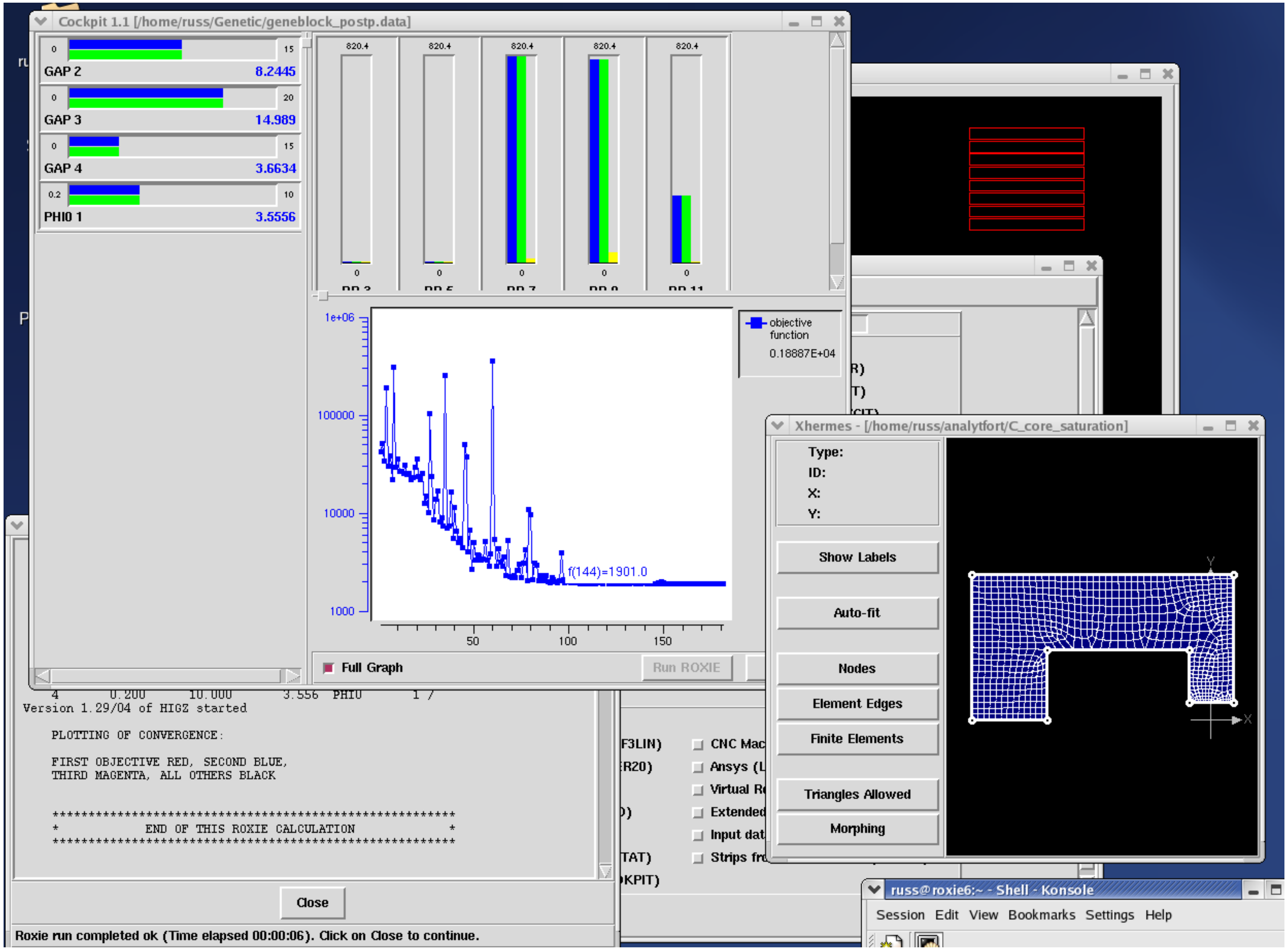
Domain decomposition (higher order elements)

Differential inductance

Energy and Forces

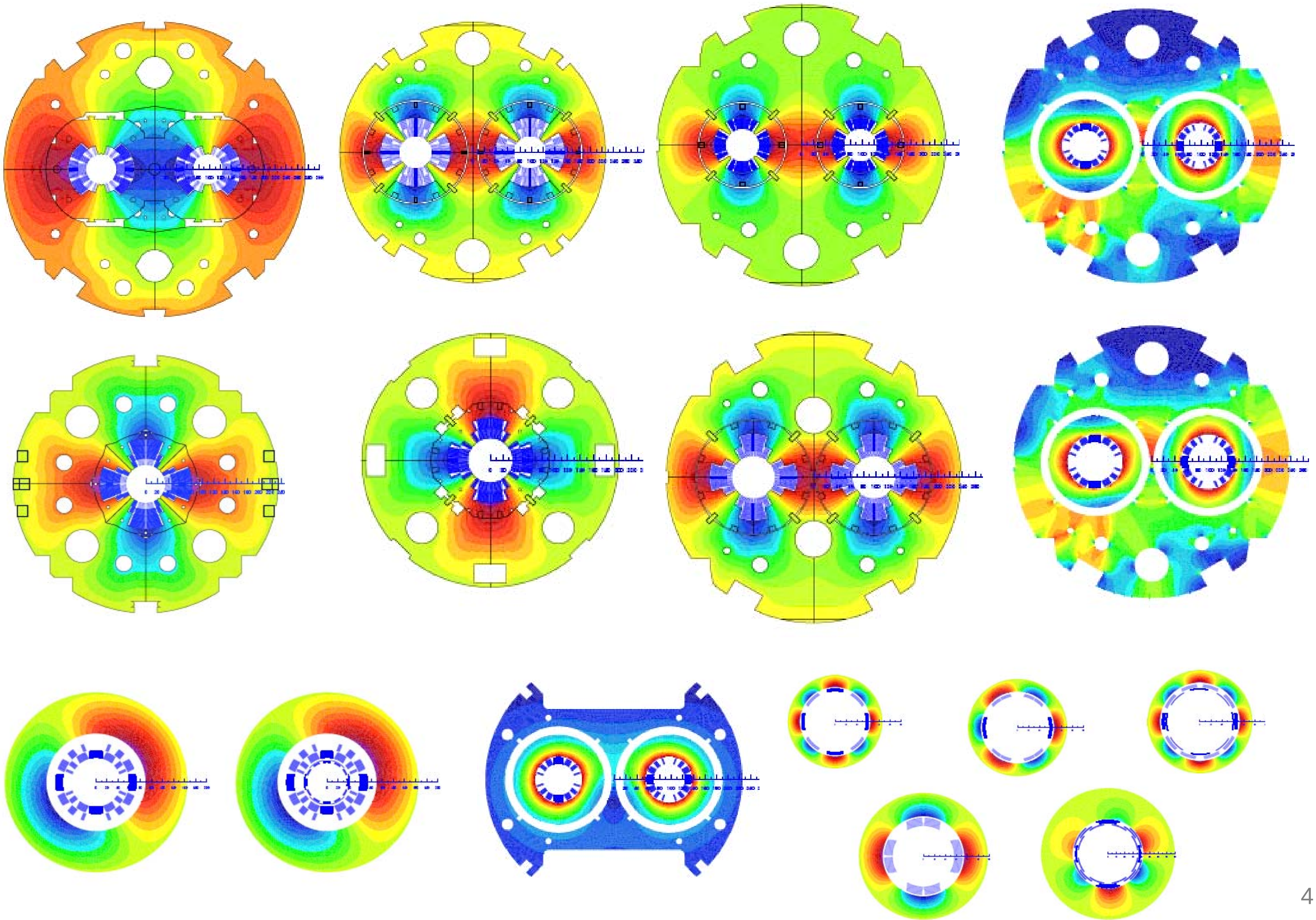
Objective conflict

CERN Field Computation Program ROXIE

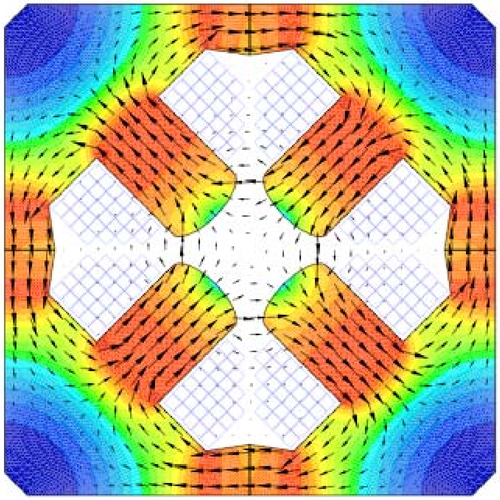


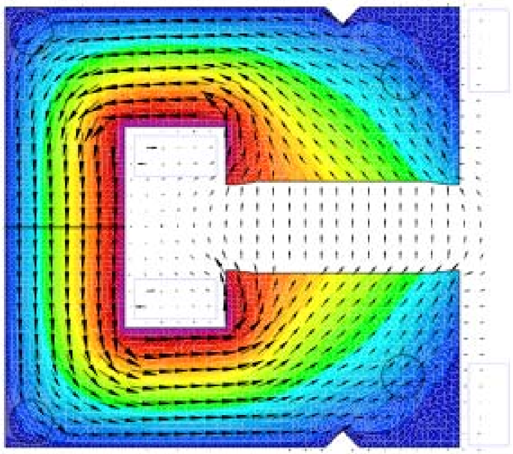
http://espace.cern.ch/roxie

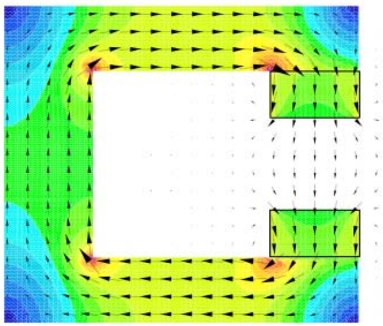
The LHC Magnet Zoo

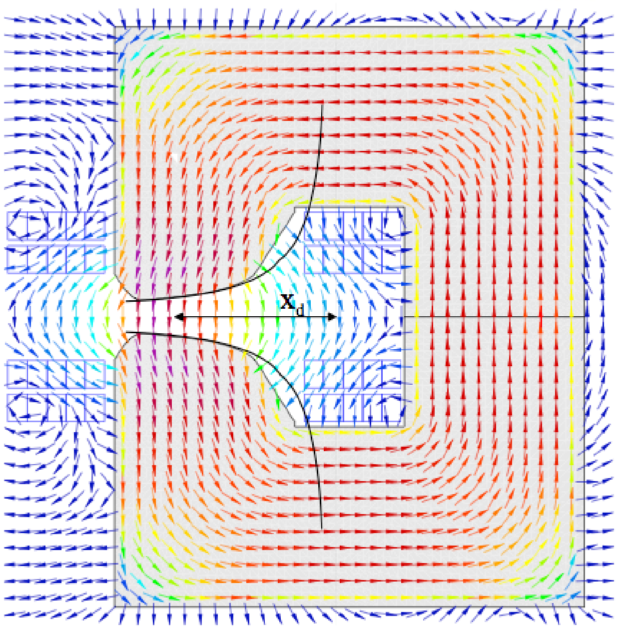


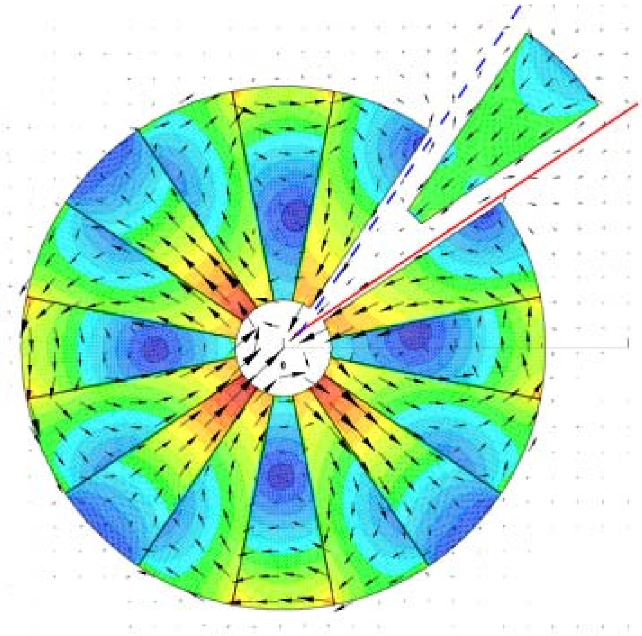
Normal Conducting Magnets



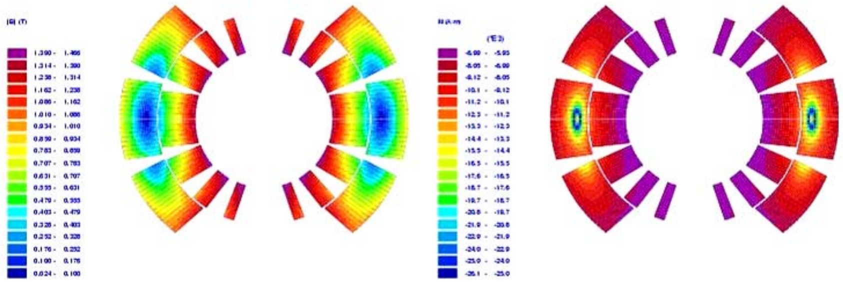




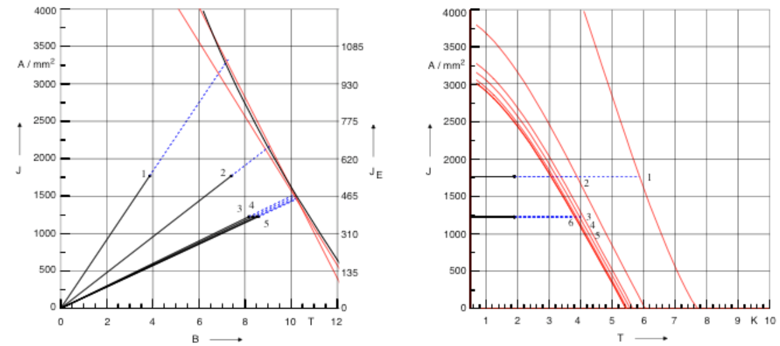




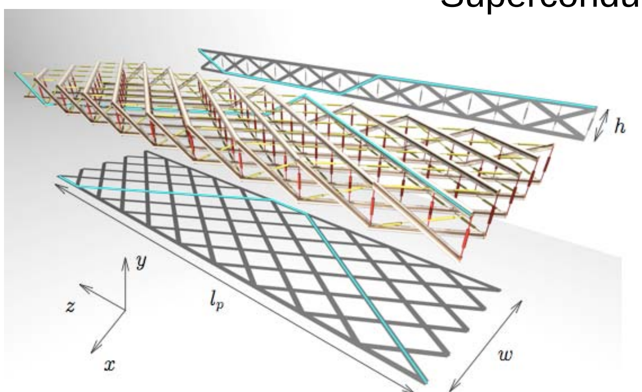
Postprocessing for Superconducting Magnets



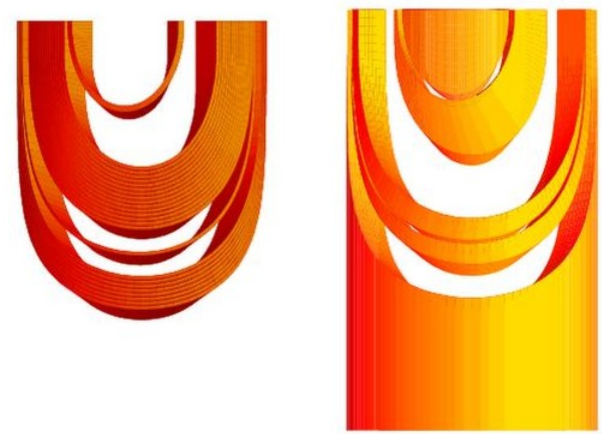
Superconductor magnetization



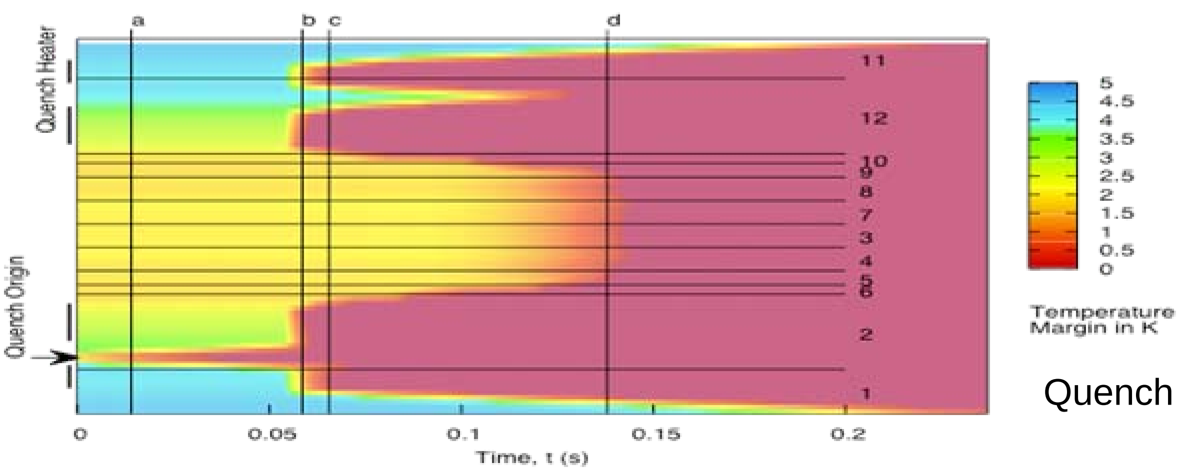
Load lines



Cable eddy currents



End-spacer design

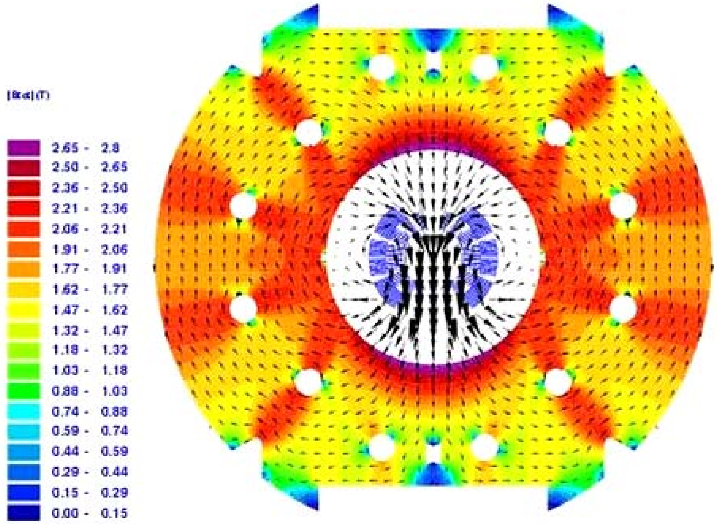


Quench simulation

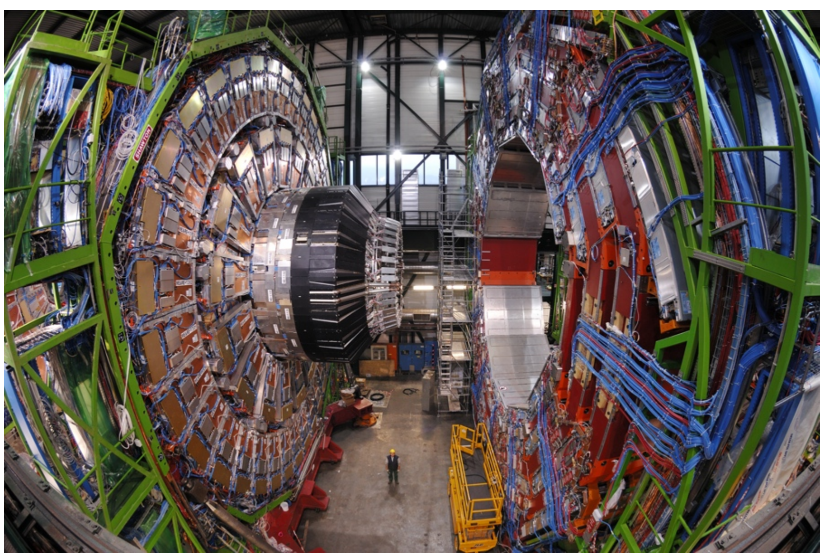
Learning Objectives

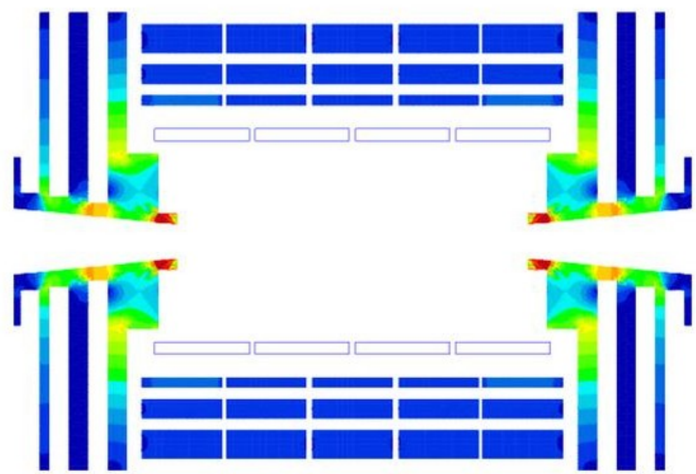
- This is not a ROXIE user's course, nor does it present syntax of any commercial software
- We glance at the foundations of FEM and BEM (basically for a 1-D example)
- But we want to give a deeper understanding of what the potential problems and limitations of both of semi-analytical and numerical field computation are
  - We provide questions to be asked at your next software user's meeting
- We discuss pre- and post-processing that can, if necessary, be appended to commercial solvers using modern script languages
  - To this end, we show examples from the ROXIE code

Coil Dominated Magnets

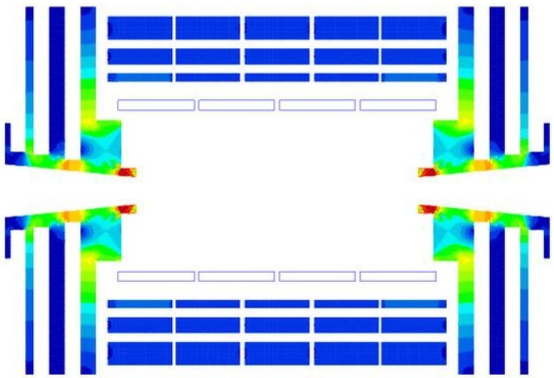


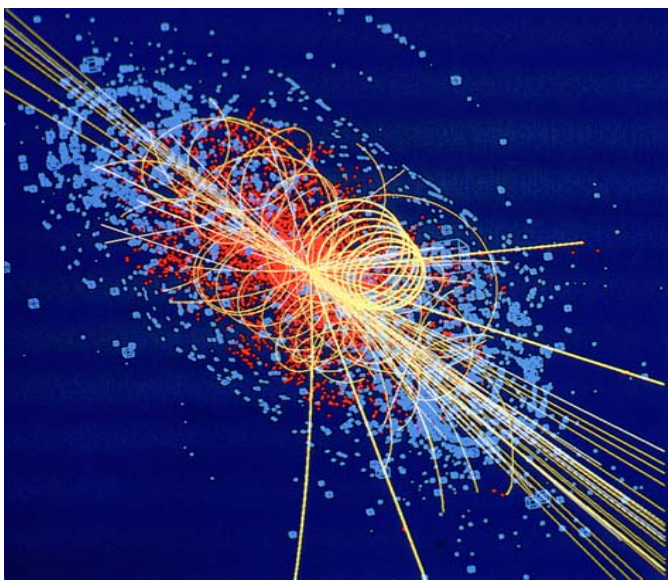
B = 8.33 T B_S = 7.77 T

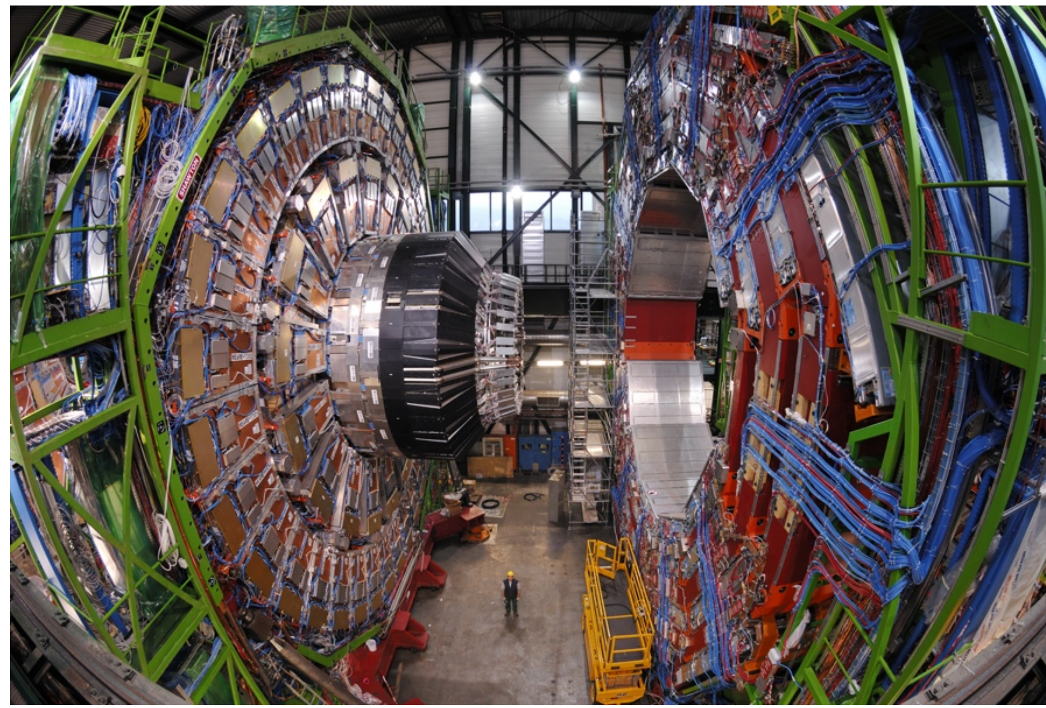




B = 4 T B_S = 3.69 T

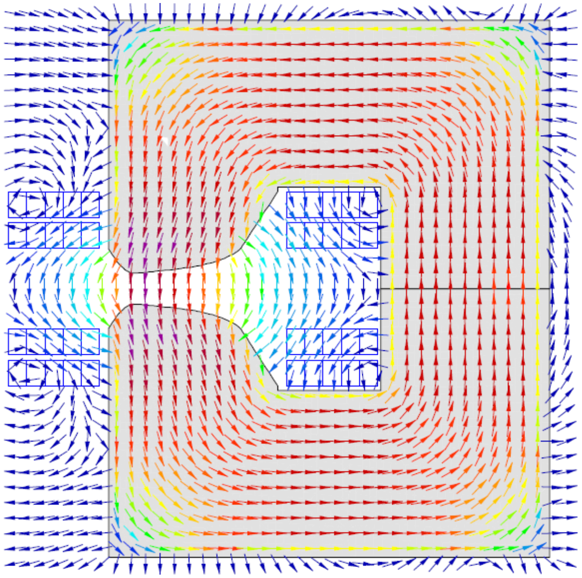


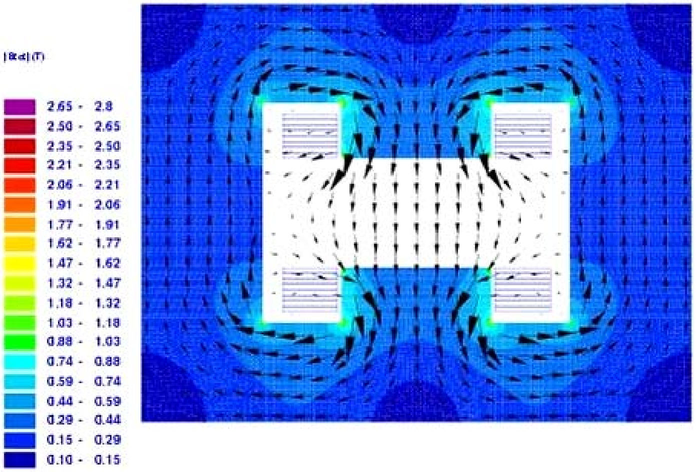




$$S = R(1 - \cos\frac{\alpha}{2}) \approx \frac{R\alpha^2}{8} = \frac{QBL^2}{8p}$$

Iron Dominated Magnets





$N \cdot I = 24000$ A $\quad B_1 = 0.3$ T $\quad B_s = 0.065$ T $\quad$ Fill.fac. 0.98

War es ein Gott der diese Zeichen schrieb,
die mit geheimnisvoll verborg'nem Trieb
die Kraefte der Natur um mich enthuellen
und mir das Herz mit stiller Freude fuellen.
Ludwig Bolzmann

Was it a god whose inspirations
led him to write these fine equations,
nature's field to me he shows
and so my heart with pleasure glows.
Translation by J.P. Blewett

Different Incarnations of Maxwell’s Equations

$$\int_{\partial a} \vec{H} \cdot \mathrm{d}\vec{s} = \int_a \vec{J} \cdot \mathrm{d}\vec{a} + \frac{\mathrm{d}}{\mathrm{d}t} \int_a \vec{D} \cdot \mathrm{d}\vec{a}$$
$$\int_{\partial a} \vec{E} \cdot \mathrm{d}\vec{s} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_a \vec{B} \cdot \mathrm{d}\vec{a}$$
$$\int_{\partial V} \vec{B} \cdot \mathrm{d}\vec{a} = 0$$
$$\int_{\partial V} \vec{D} \cdot \mathrm{d}\vec{a} = \int_V \rho \, \mathrm{d}V$$

$$\mathrm{d}H = J + \partial_t D$$
$$\mathrm{d}E = -\partial_t B$$
$$\mathrm{d}B = 0$$
$$\mathrm{d}D = \rho$$

$$V_{\mathrm{m}}(\partial a) = I(a) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(a)$$
$$U(\partial a) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(a)$$
$$\Phi(\partial V) = 0$$
$$\Psi(\partial V) = Q(V)$$

$$\mathrm{curl}\,\vec{H} = \vec{J} + \partial_t \vec{D}$$
$$\mathrm{curl}\,\vec{E} = -\partial_t \vec{B}$$
$$\mathrm{div}\,\vec{B} = 0$$
$$\mathrm{div}\,\vec{D} = \rho$$

$$a = \frac{\partial H}{\partial y} - \frac{\partial G}{\partial z}$$
$$b = \frac{\partial F}{\partial z} - \frac{\partial H}{\partial x}$$
$$c = \frac{\partial G}{\partial x} - \frac{\partial F}{\partial y}$$

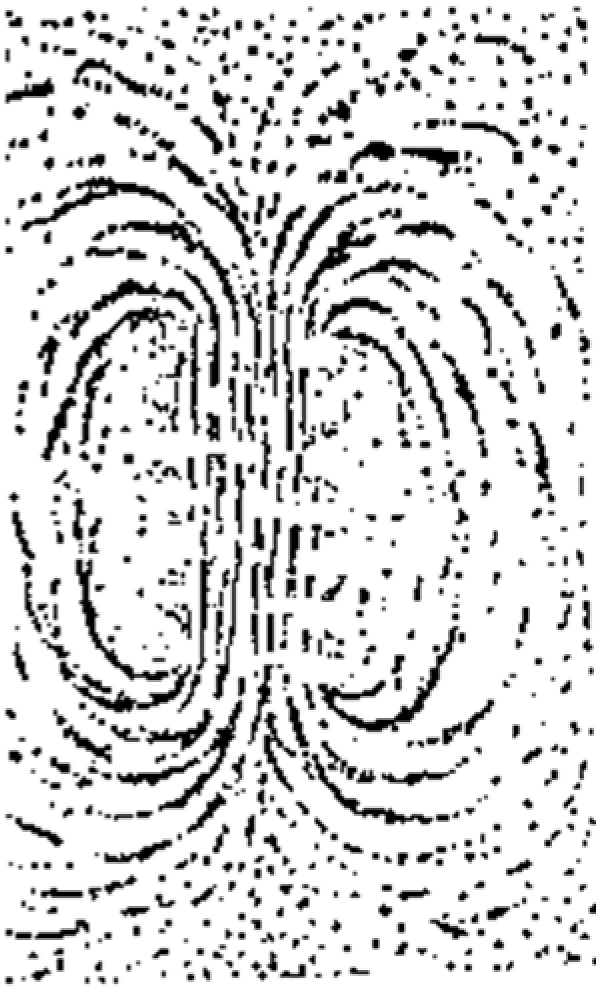
$$p + \frac{\partial f}{\partial t} = \frac{\partial \gamma}{\partial y} - \frac{\partial \beta}{\partial z}$$
$$q + \frac{\partial g}{\partial t} = \frac{\partial \alpha}{\partial z} - \frac{\partial \gamma}{\partial x}$$
$$r + \frac{\partial h}{\partial t} = \frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}$$

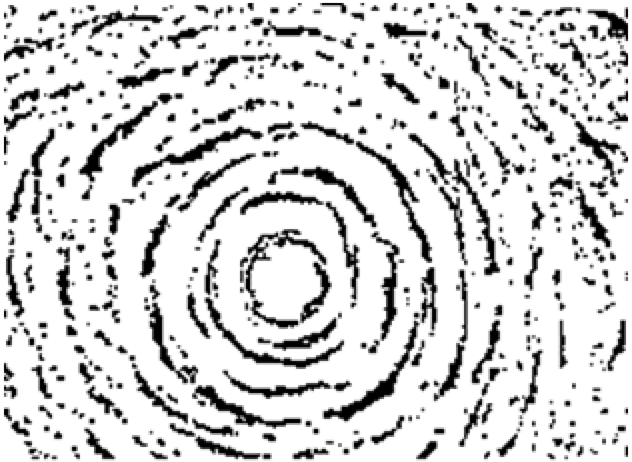
$$P = -\frac{\partial F}{\partial t} - \frac{\partial \varphi}{\partial x}$$
$$Q = -\frac{\partial G}{\partial t} - \frac{\partial \varphi}{\partial y}$$
$$R = -\frac{\partial H}{\partial t} - \frac{\partial \varphi}{\partial z}$$

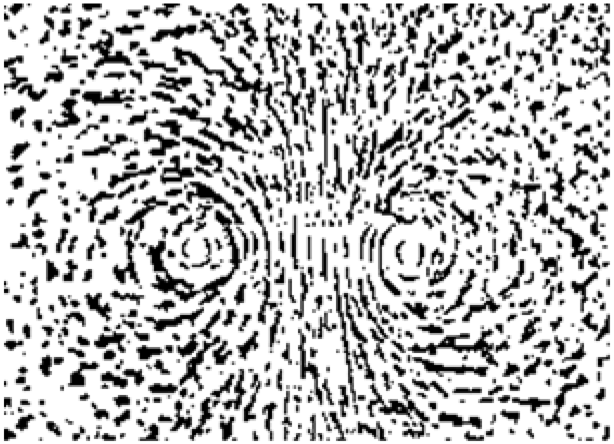
$$\rho = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

$$\mathcal{B} = V.\nabla\mathcal{U}$$
$$\mathcal{E} = V.\dot{\rho}\mathcal{B} - \mathcal{U} - \nabla\psi$$
$$\mathcal{C} = c\mathcal{E} + \mathcal{D}$$
$$\mathcal{B} = \mathcal{H} + 4\pi\mathcal{J}$$
$$4\pi c = V.\nabla\mathcal{H}$$
$$\mathcal{D} = \frac{1}{4\pi}\varkappa\mathcal{E}$$

Magnetic Flux Conservation Law (Inner Oriented Volume)

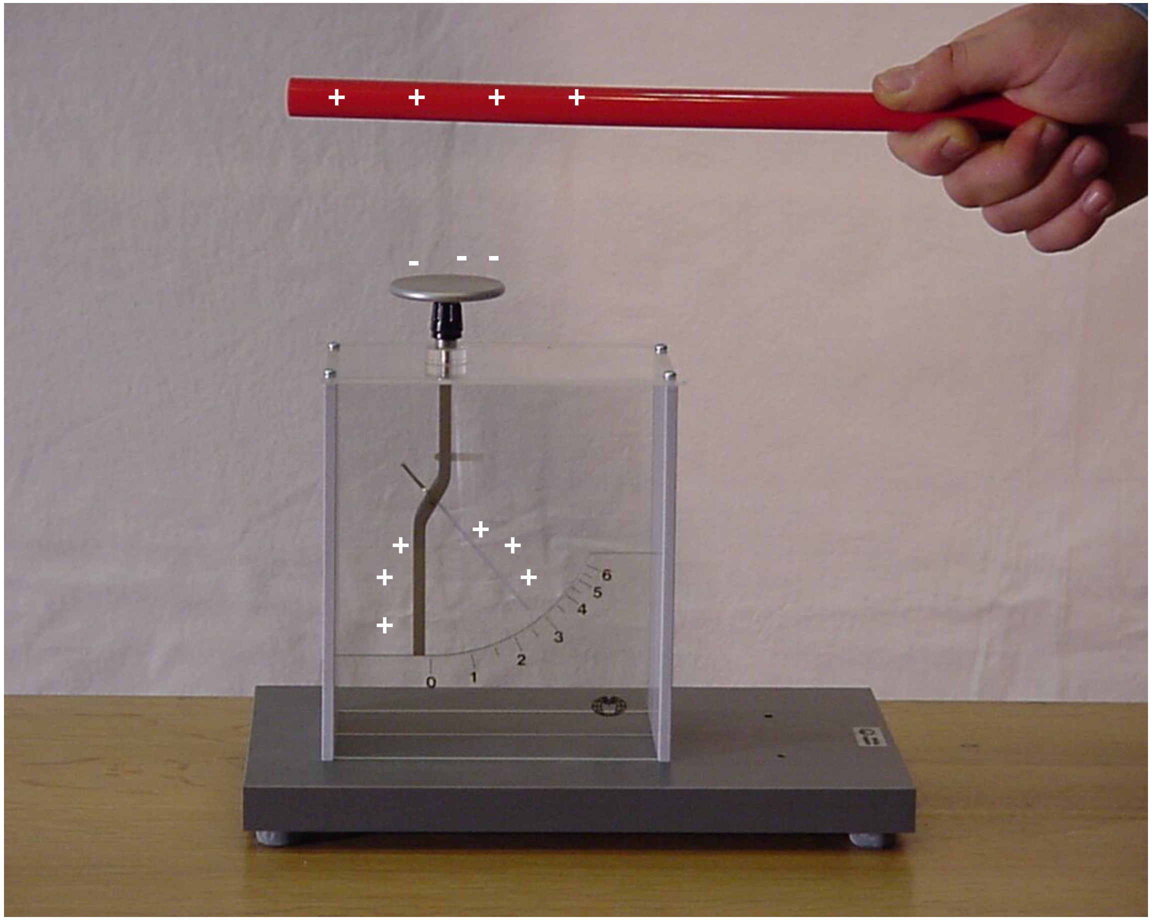






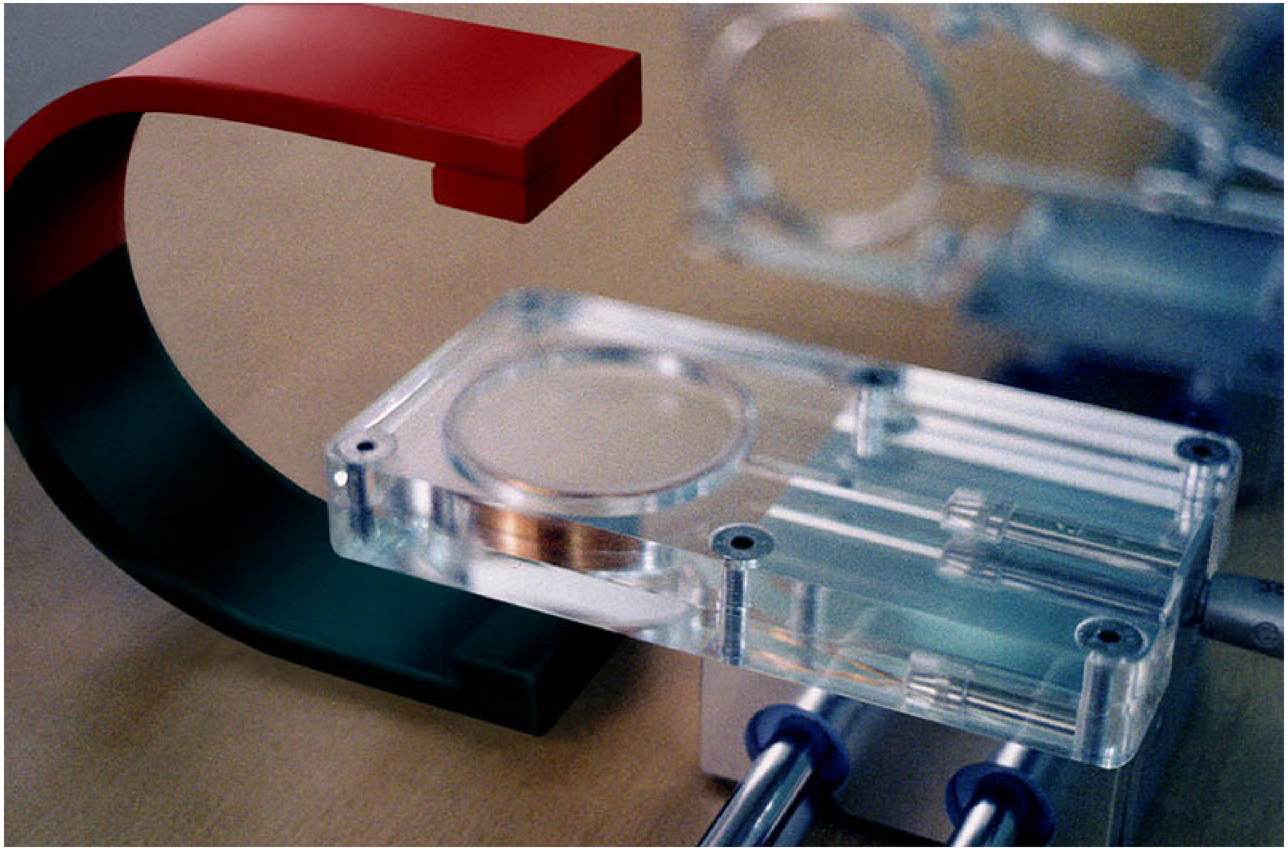
$$\Phi(\partial V) = 0$$

Gauss Law (Outer Oriented Volume)

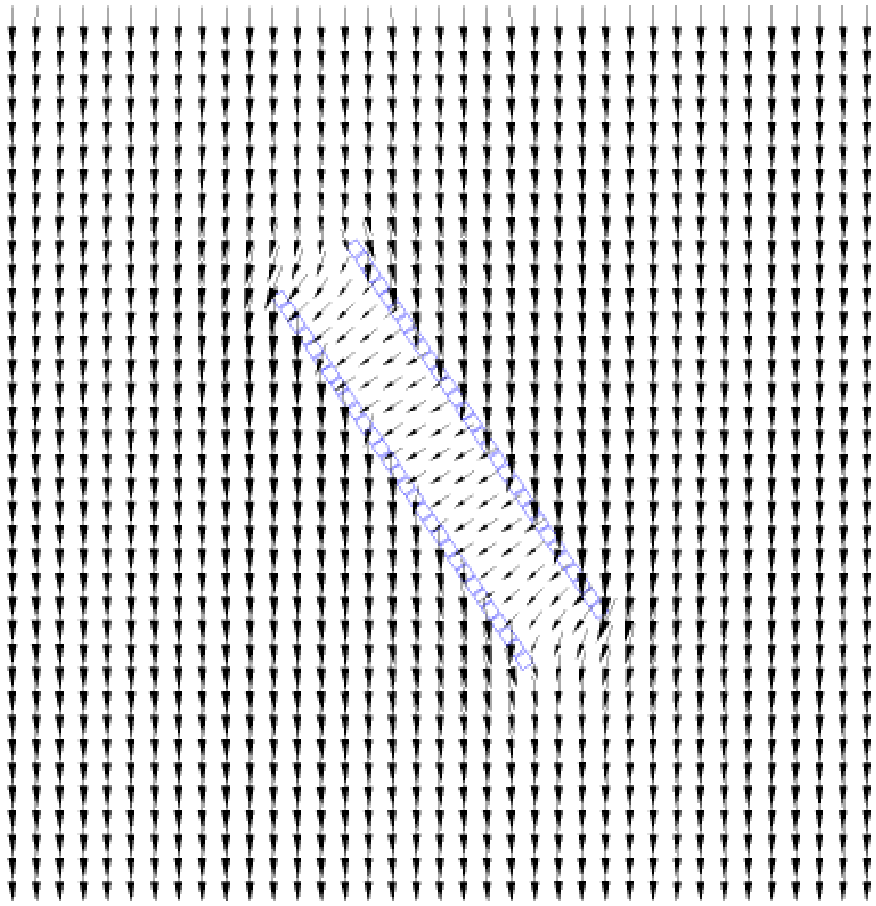


$$\Psi(\partial V) = Q(V)$$

Faraday's Law (Inner Oriented Surface)



$$U(\partial a) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(a)$$



$$V_{\mathrm{m}}(\partial a) = I(a)$$

Maxwell's Equations in Global Form

Ampere $$V_{\mathrm{m}}(\partial a) = I(a) + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(a)$$

Faraday $$U(\partial a) = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(a)$$

Flux conservation $$\Phi(\partial V) = 0$$

Gauss $$\Psi(\partial V) = Q(V)$$

Conservation of charge / Kirchhoff law

$$V_{\mathrm{m}}(\partial(\partial V)) = 0 = I(\partial V) + \tfrac{\mathrm{d}}{\mathrm{d}t}Q(V)$$

For its unworthy of excellent men

to loose hours like slaves in the labour of calculation

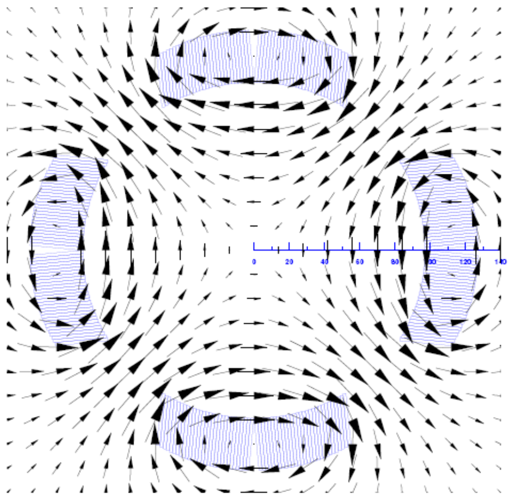
which could safely be regulated to anyone else if

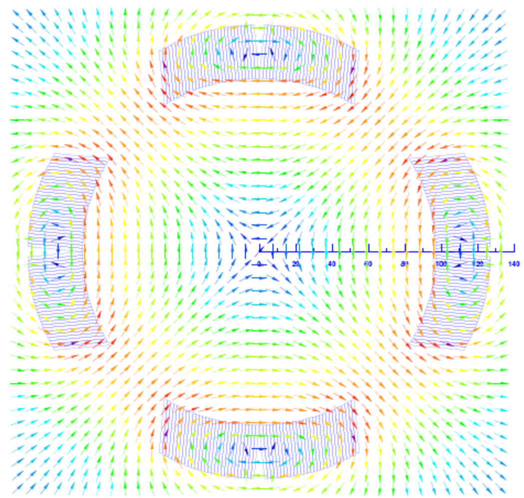
machines were used

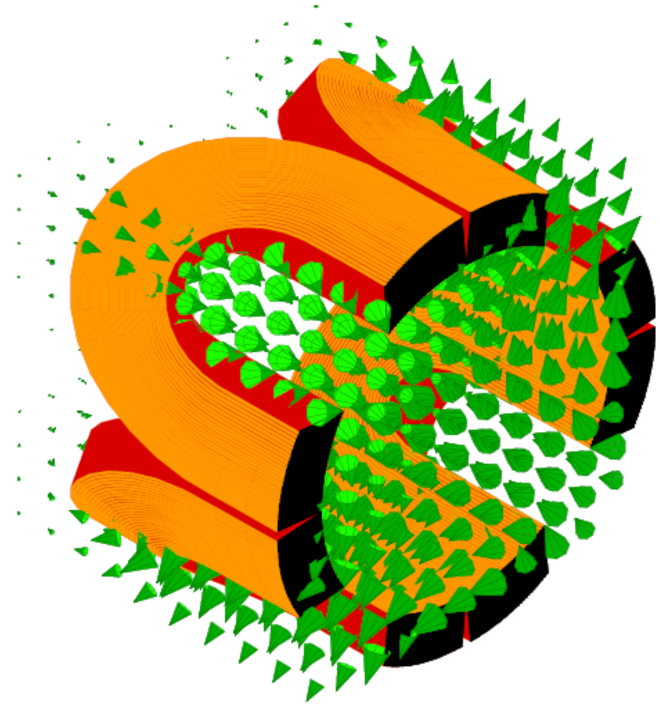
Gottfried Wilhelm Leibnitz (1646-1716)

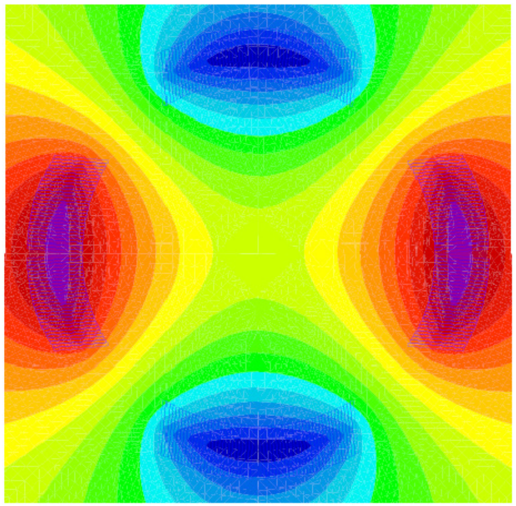
SI-unit	Relation	SI-unit
$1\mathrm{A}$	$V_m(s) = \int_s \mathbf{H} \cdot \mathrm{ds}$	$1\mathrm{A} \cdot \mathrm{m}^{-1}$
$1\mathrm{V}$	$U(s) = \int_s \mathbf{E} \cdot \mathrm{ds}$	$1\mathrm{V} \cdot \mathrm{m}^{-1}$
$1\mathrm{V} \cdot \mathrm{s}$	$\Phi(a) = \int_a \mathbf{B} \cdot \mathrm{d}\mathbf{a}$	$1\mathrm{V} \cdot \mathrm{s} \cdot \mathrm{m}^{-2}$
$1\mathrm{A} \cdot \mathrm{s}$	$\Psi(a) = \int_a \mathbf{D} \cdot \mathrm{d}\mathbf{a}$	$1\mathrm{A} \cdot \mathrm{s} \cdot \mathrm{m}^{-2}$
$1\mathrm{A}$	$I(a) = \int_a \mathbf{J} \cdot \mathrm{d}\mathbf{a}$	$1\mathrm{A} \cdot \mathrm{m}^{-2}$
$1\mathrm{A} \cdot \mathrm{s}$	$Q(V) = \int_V \rho \cdot \mathrm{d}V$	$1\mathrm{A} \cdot \mathrm{s} \cdot \mathrm{m}^{-3}$

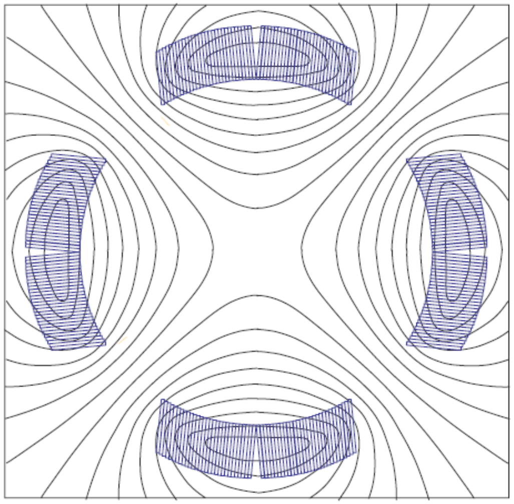
Vector Field









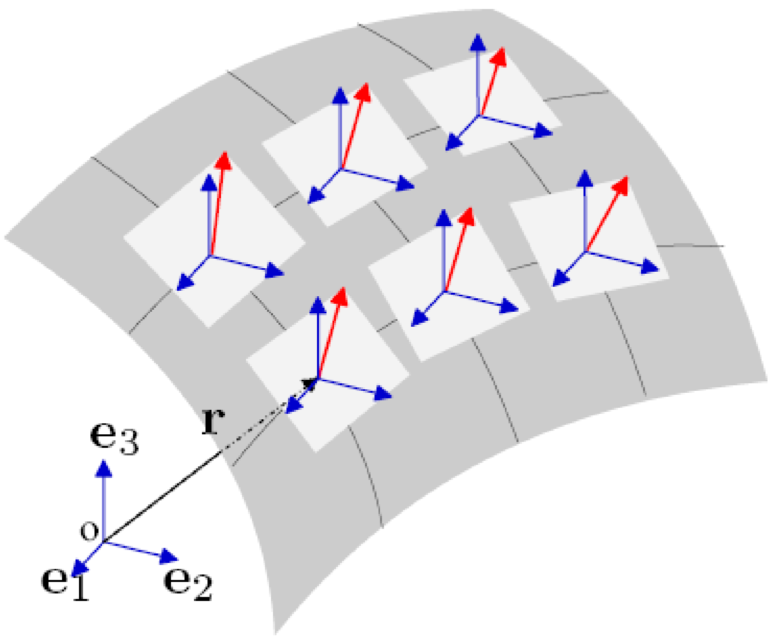


Vector and Scalar Fields

$$\mathbf{a} : \Omega \to \mathbb{R}^3 : \mathbf{r} \mapsto \mathbf{a}(\mathbf{r}) : \mathbf{a}(\mathbf{r}) = (a^1(\mathbf{r}), a^2(\mathbf{r}), a^3(\mathbf{r}))$$

$$\mathcal{V}(\Omega) := C^\infty(\Omega, \mathbb{R}^3)$$

$$\Omega \subset \mathbb{R}^3$$



$$\phi : \Omega \to \mathbb{R} : \phi \mapsto \phi(\mathbf{r})$$

$$\mathcal{S}(\Omega) := C^\infty(\Omega, \mathbb{R})$$

Maxwell's Equations in Integral Form

$$\int_{\partial a} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int_a \mathbf{J} \cdot \mathrm{d}\mathbf{a} + \frac{\mathrm{d}}{\mathrm{d}t} \int_a \mathbf{D} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial a} \mathbf{E} \cdot \mathrm{d}\mathbf{s} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_a \mathbf{B} \cdot \mathrm{d}\mathbf{a},$$

$$\int_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{a} = 0,$$

$$\int_{\partial V} \mathbf{D} \cdot \mathrm{d}\mathbf{a} = \int_V \rho \, \mathrm{d}V.$$

$$V_{\mathrm{m}}(\partial a) = I(a) + \frac{\mathrm{d}}{\mathrm{d}t} \Psi(a),$$

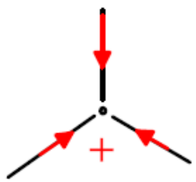
$$U(\partial a) = -\frac{\mathrm{d}}{\mathrm{d}t} \Phi(a),$$

$$\Phi(\partial V) = 0,$$

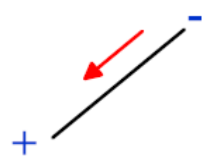
$$\Psi(\partial V) = Q(V).$$

Orientation of Space Elements

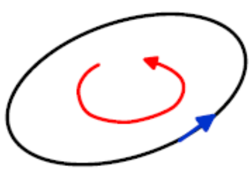
Inner orientation



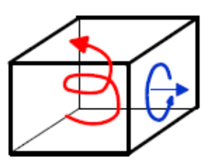
Point:
A positive point is oriented as a sink



Line/Edge:
Selecting a vector pointing in forward direction

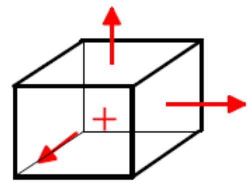


Surface:
Sense of rotation

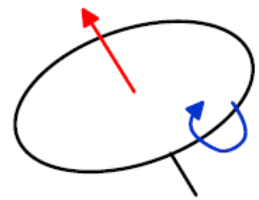


Volume:
Sense of a screw

Outer orientation

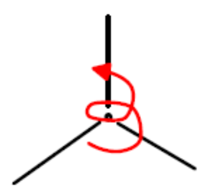


Volume:
Choice of outward normals



Surface:
Crossing direction of a line

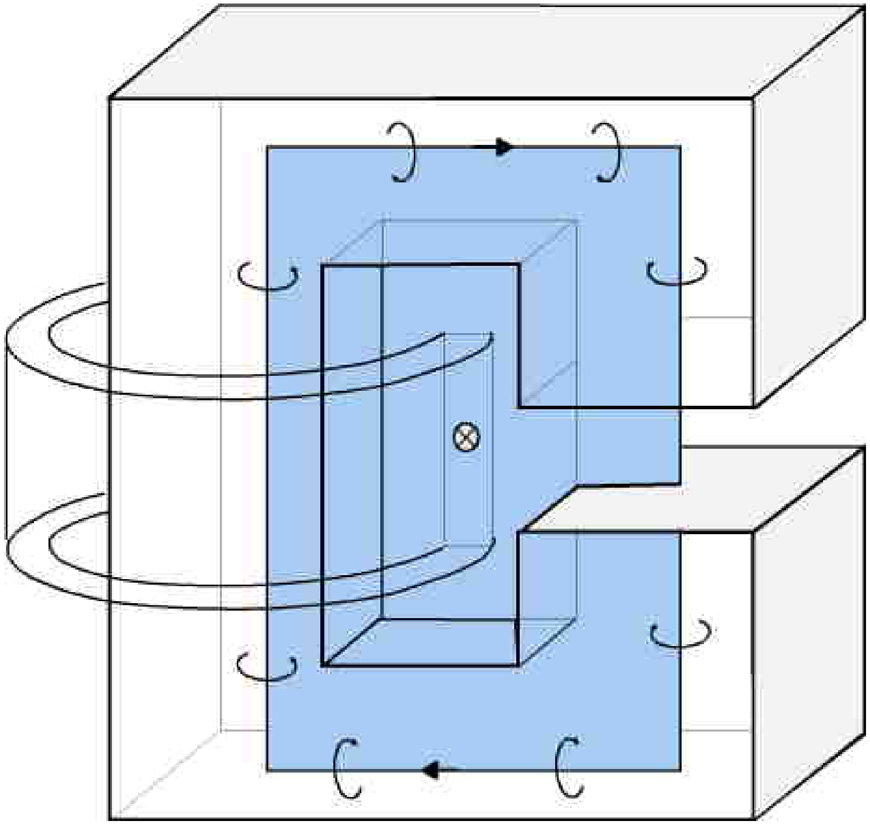
Line/Edge:
Direction of circulation of a surface around this line



Point:
The inner orientation of the volume containing the point

Inner and Outer Oriented Surfaces

$$\int_{\partial a} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int_a \mathbf{J} \cdot \mathrm{d}\mathbf{a}$$



$$\Phi(a) = \int_a \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

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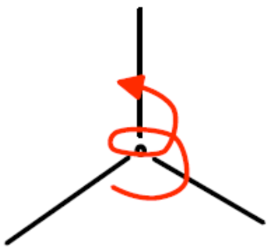
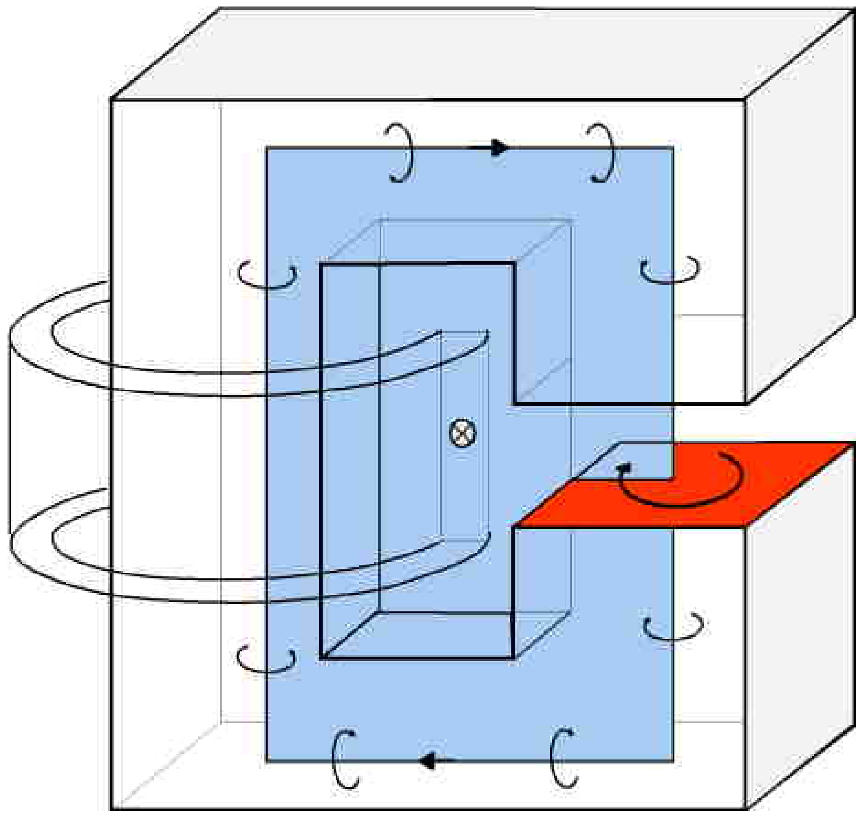
Inner and Outer Oriented Surfaces

$$\int_{\partial a} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int_{a} \mathbf{J} \cdot \mathrm{d}\mathbf{a}$$





Embedding into oriented ambient space (Origin, coordinates)

$$\Phi(a) = \int_a \mathbf{B} \cdot \mathrm{d}\mathbf{a}$$

Magnetic Discussion (Bruno Touschek 1921-1978)

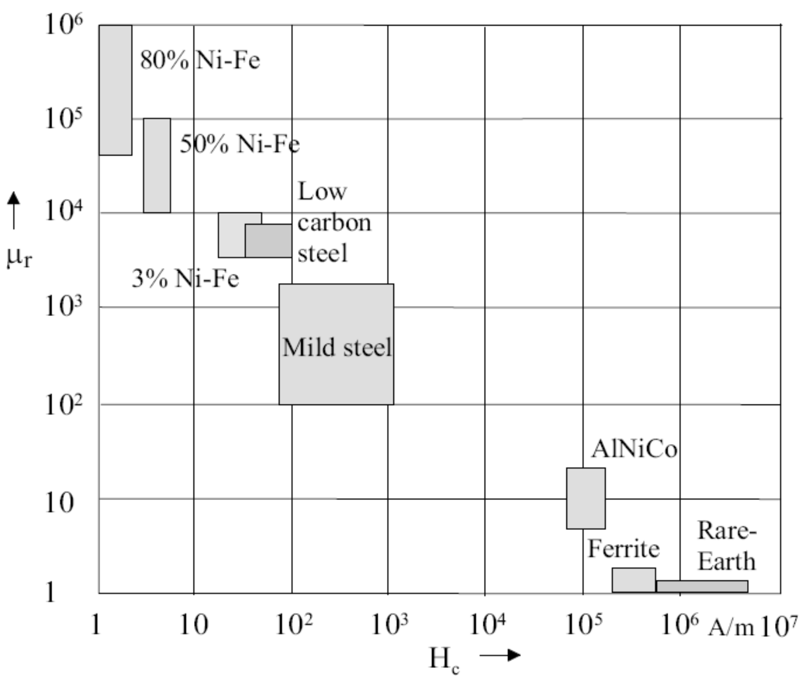
$$\mathbf{B} = \mu \mathbf{H} \qquad \mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{J} = \kappa \mathbf{E}$$

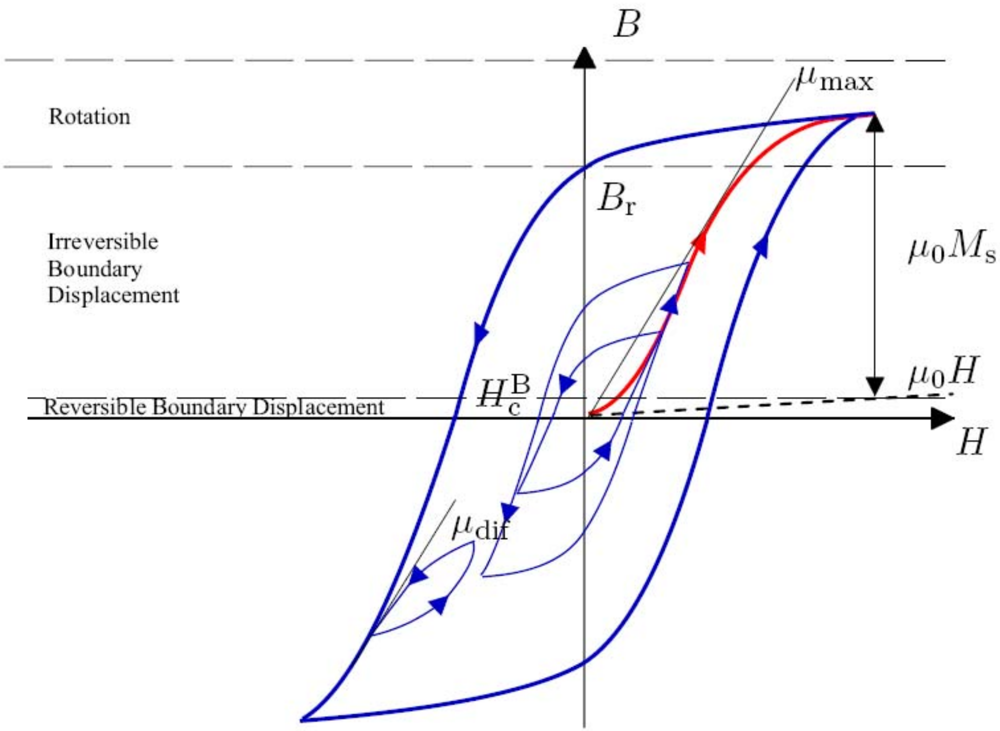
Permeability: $[\mu] = 1\,\mathrm{V} \cdot \mathrm{s} \cdot \mathrm{A}^{-1} \cdot \mathrm{m}^{-1} = 1\,\mathrm{H} \cdot \mathrm{m}^{-1}$

Permittivity: $[\epsilon] = 1\,\mathrm{A} \cdot \mathrm{s} \cdot \mathrm{V}^{-1} \cdot \mathrm{m}^{-1}$

Conductivity: $[\kappa] = 1\mathrm{A} \cdot \mathrm{V}^{-1} \cdot \mathrm{m}^{-1}$

Linear (field independent), homogeneous (position independent), isotropic (direction independent) and stationary media. The material parameters may depend on the spatial position.





$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{P}_{\mathsf{mag}}(\mathbf{H}) = \mu_0(\mathbf{H} + \mathbf{M}(\mathbf{H}))$$

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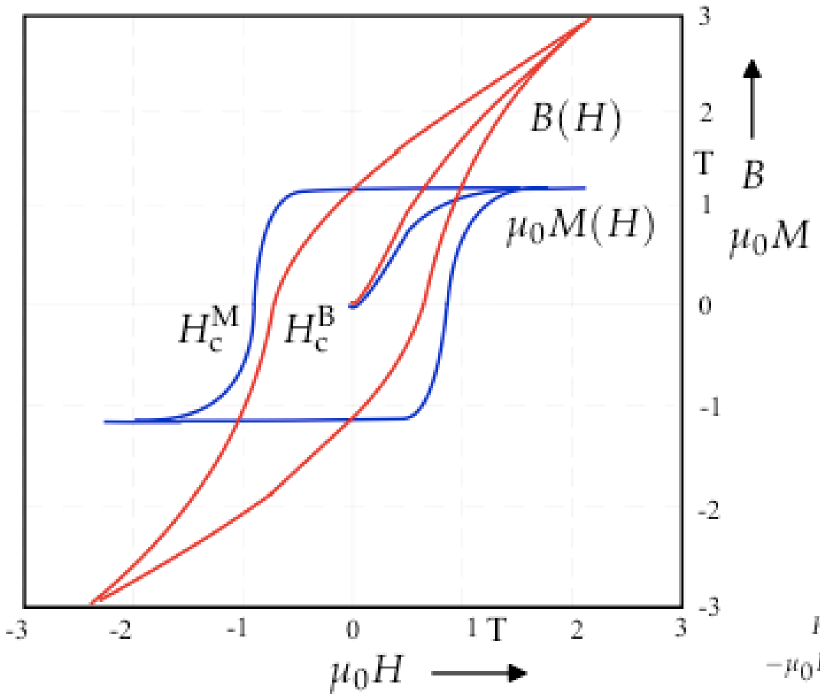
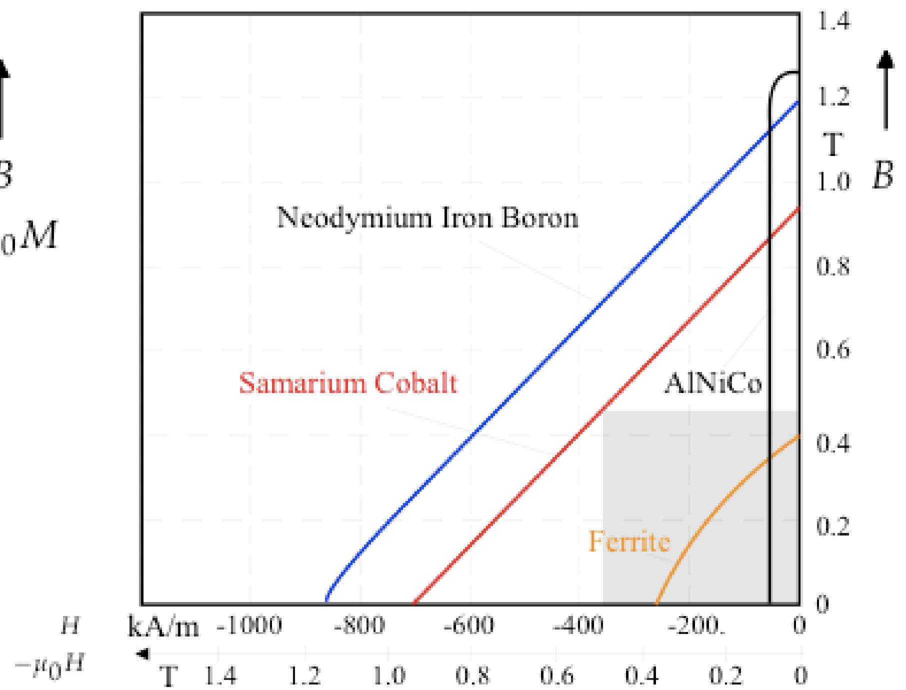
M(H) and B(H) for Permanent Magnets





$$H = \frac{NI}{2\pi r}$$

$$\overline{\mu} = \frac{\overline{B}}{\overline{H}}$$

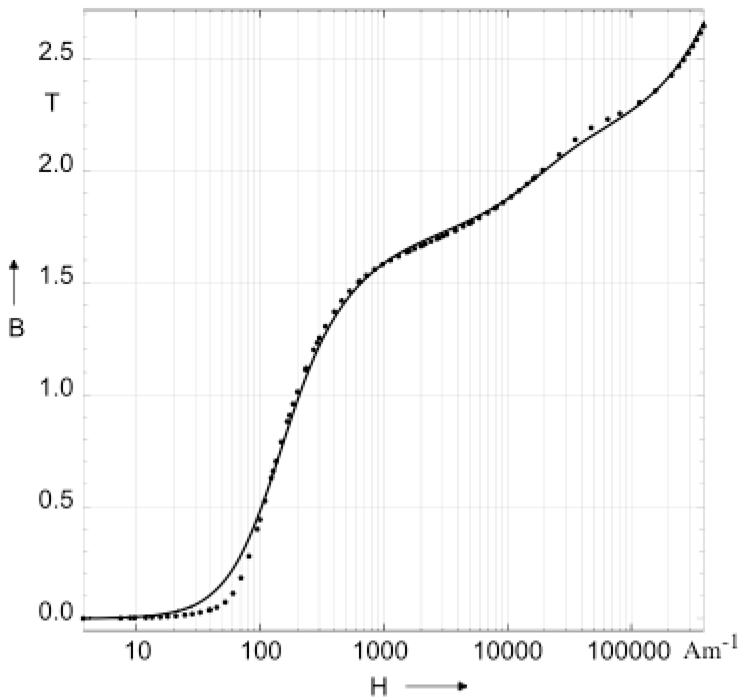
$$\overline{H} = \frac{NI}{2\pi(r_2 - r_1)} \ln \frac{r_2}{r_1}$$

$$U = \frac{\mathrm{d}}{\mathrm{d}t}\Phi = \frac{\mathrm{d}}{\mathrm{d}t}\overline{B}a$$

$$\overline{B} = \overline{B}_{\mathrm{meas.}} + B_{\mathrm{corr.}}$$

Temperature T K	Stress MPa	Coercive field H_c^B $\mathrm{A \cdot m^{-1}}$	Remanence B_r T	max μ_r
300	0	68.4	1.07	5900
77	0	79.6	1.12	5600
4.2	0	85.1	1.06	4800
4.2	20	110.	0.67	2460

Always check conditions of measurements



$$L\left(\frac{H}{a}\right) := \coth\left(\frac{H}{a}\right) - \left(\frac{a}{H}\right)$$

$$M(H) = M_{\mathrm{a}} L\left(\frac{H}{a}\right) + M_{\mathrm{b}} \tanh\left(\frac{|H|}{b}\right) L\left(\frac{H}{b}\right)$$

Always fulfilling the smoothness requirements for M(B) and Newton-Raphson iterative solvers

Closed domain (2-D or 3-D) will be denoted as Ω. Boundary $\Gamma = \partial\Omega$ with $\Gamma = \Gamma_H \cup \Gamma_B$

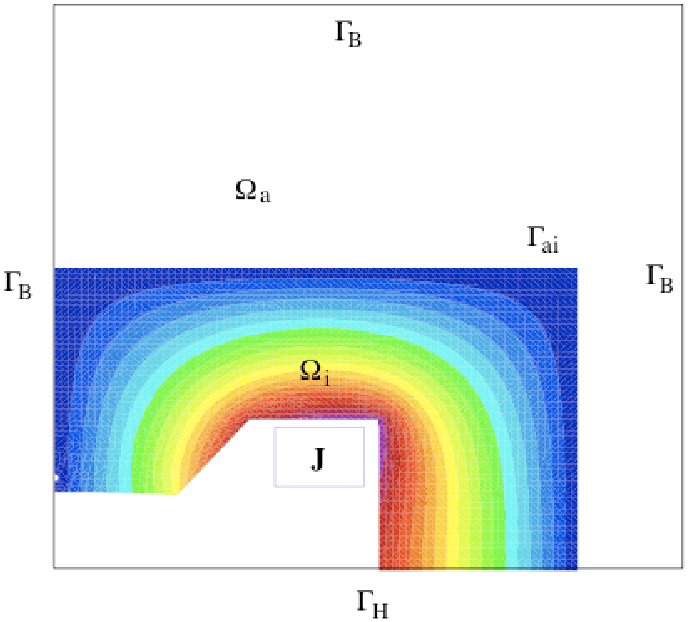
On Γ_B the normal component of the magnetic flux density is prescribed (symmetry planes parallel to the field, on far boundaries etc.)

$$B_n = \mathbf{B} \cdot \mathbf{n} = \sigma_{mag} \qquad \text{on } \Gamma_B$$

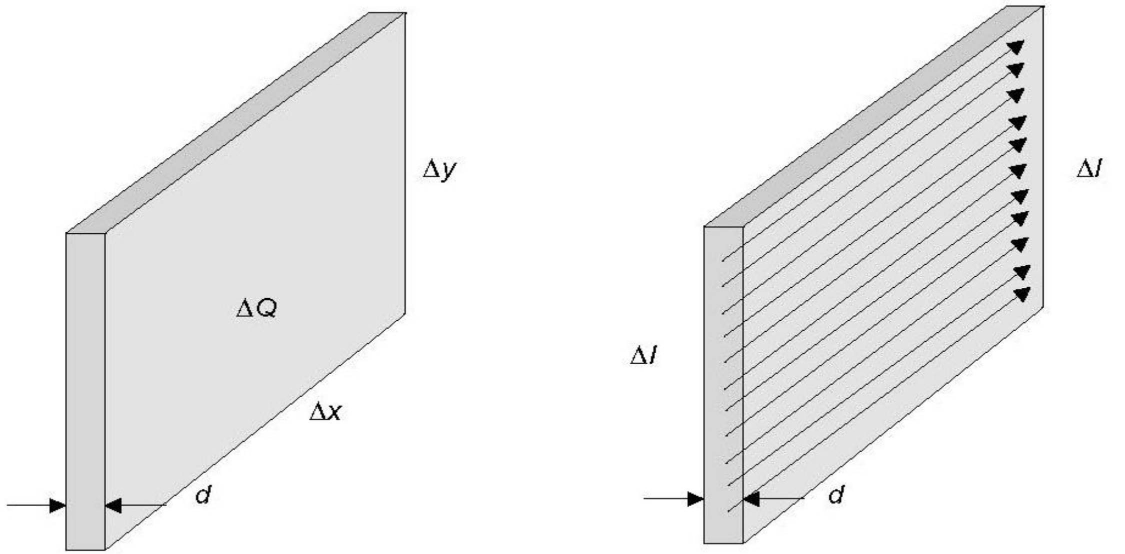
On Γ_H of the boundary the tangential components of the magnetic field are prescribed. In general

$$\mathbf{H} \times \mathbf{n} = \boldsymbol{\alpha} \qquad \text{on } \Gamma_H$$

$$\mathbf{H}_t = \mathbf{0} \quad \rightarrow \quad \mathbf{n} \times (\mathbf{H} \times \mathbf{n}) = \mathbf{0}$$



Surface Charge and Surface Current



Thin layer with ρ_{mag}

$\Delta Q = \Delta x \Delta y d \rho_{\text{mag}}$

$\rho_{\text{mag}} \to \infty$ and $d \to 0$

$\sigma_{\text{mag}} = d\rho_{\text{mag}}$

$[\sigma_{\text{mag}}] = 1 \ \text{V{\cdot}s/m}^2$

Thin layer with J

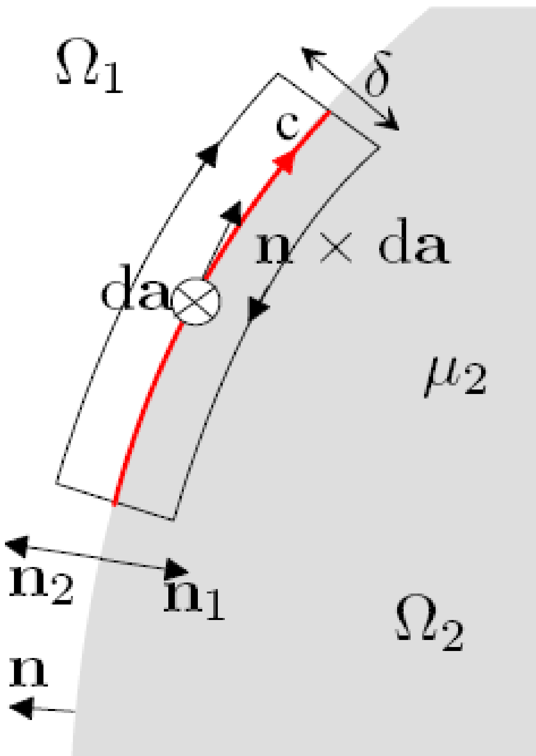
$\Delta I = J d \Delta l$

$J \to \infty$ and $d \to 0$

$\boldsymbol{\alpha} = \mathbf{J} d$

$[\alpha] = 1 \ \text{A} \cdot \text{m}^{-1}$

Fictitious quantities to define boundary values



$$\int_{\partial a} \mathbf{H} \cdot d\mathbf{s} = \int_a \mathbf{J} \cdot d\mathbf{a} \qquad \delta \to 0$$

$$\int_{s_2} \mathbf{H}_2 \cdot d\mathbf{s} + \int_{s_1} \mathbf{H}_1 \cdot d\mathbf{s} = -\int_c (\mathbf{n} \times \boldsymbol{\alpha}) \cdot d\mathbf{s}$$

$$\int_c (\mathbf{H}_1 - \mathbf{H}_2) \cdot d\mathbf{s} = -\int_c (\mathbf{n} \times \boldsymbol{\alpha}) \cdot d\mathbf{s}$$

Holds for any curve c if the integrands are equal, except for a possible normal component $\lambda \mathbf{n}$

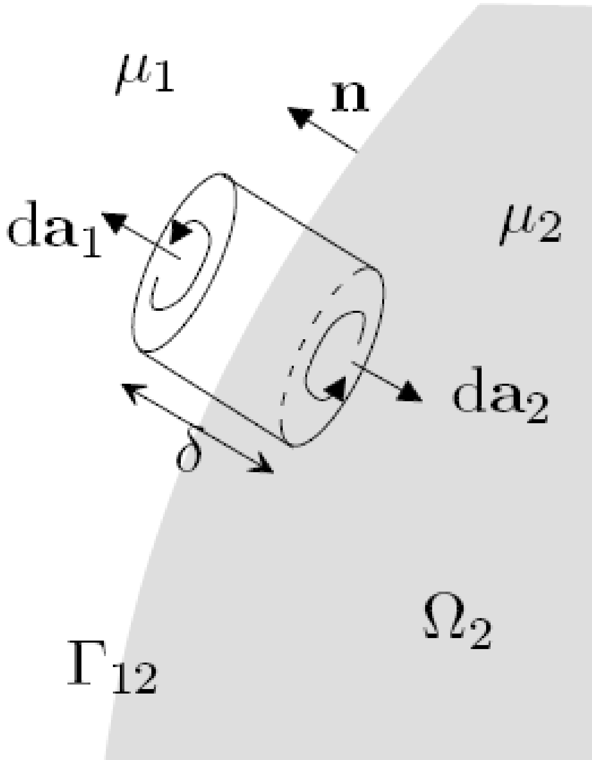
$$(\mathbf{H}_1 - \mathbf{H}_2) + \lambda \mathbf{n} = -\mathbf{n} \times \boldsymbol{\alpha}$$

With $\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\alpha}) = (\mathbf{n} \cdot \boldsymbol{\alpha})\mathbf{n} - (\mathbf{n} \cdot \mathbf{n})\boldsymbol{\alpha} = -\boldsymbol{\alpha}$

$$\boldsymbol{\alpha} = (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{n} = [\mathbf{H} \times \mathbf{n}]_{12}$$

$$\mathbf{H}_{t1} = \mathbf{H}_{t2} \equiv (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{n} = \mathbf{0} \equiv [\mathbf{H} \times \mathbf{n}]_{12} = \mathbf{0}$$

Continuity Conditions (2)



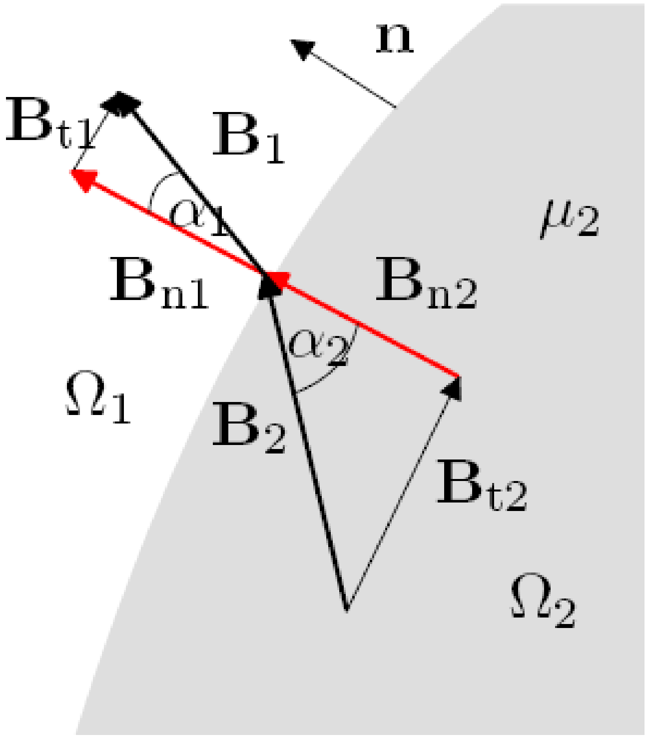
$$\int_{\partial V} \mathbf{B} \cdot \mathbf{da} = 0 \qquad \delta \to 0$$

$$\begin{aligned} \int_a \sigma_{\text{mag}} \mathrm{d}a &= \int_a \mathbf{B}_1 \cdot \mathbf{da}_1 + \mathbf{B}_2 \cdot \mathbf{da}_2 \\ &= \int_a (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n}_1 \mathrm{d}a \end{aligned}$$

Holds for any surface a if

$$\begin{aligned} \sigma_{\text{mag}} &= (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} \\ &= [\mathbf{B} \cdot \mathbf{n}]_{12} \end{aligned}$$

$$B_{\text{n}1} = B_{\text{n}2} \quad \equiv \quad (\mathbf{B}_1 - \mathbf{B}_2) \cdot \mathbf{n} = 0 \quad \equiv \quad [\mathbf{B} \cdot \mathbf{n}]_{12} = 0$$

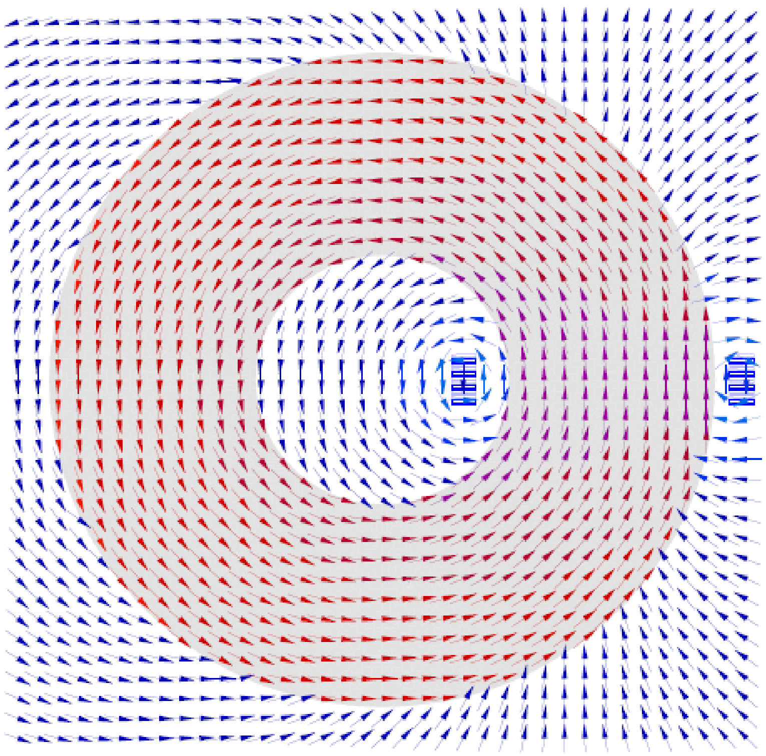


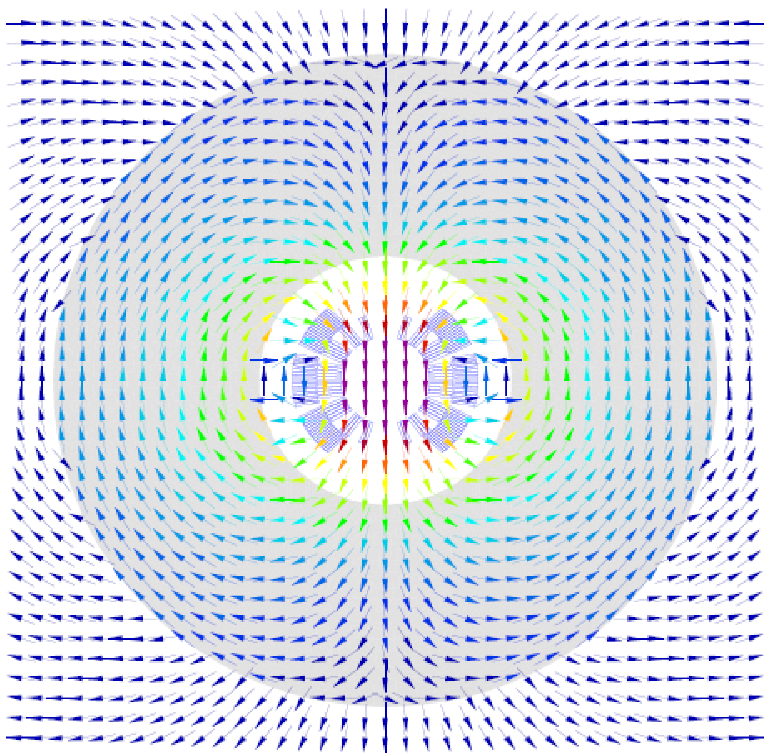
No surface currents:

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\frac{B_{t1}}{B_{n1}}}{\frac{B_{t2}}{B_{n2}}} = \frac{\mu_1 H_{t1}}{\mu_2 H_{t2}} = \frac{\mu_1}{\mu_2}$$

$$\mu_2 \gg \mu_1$$

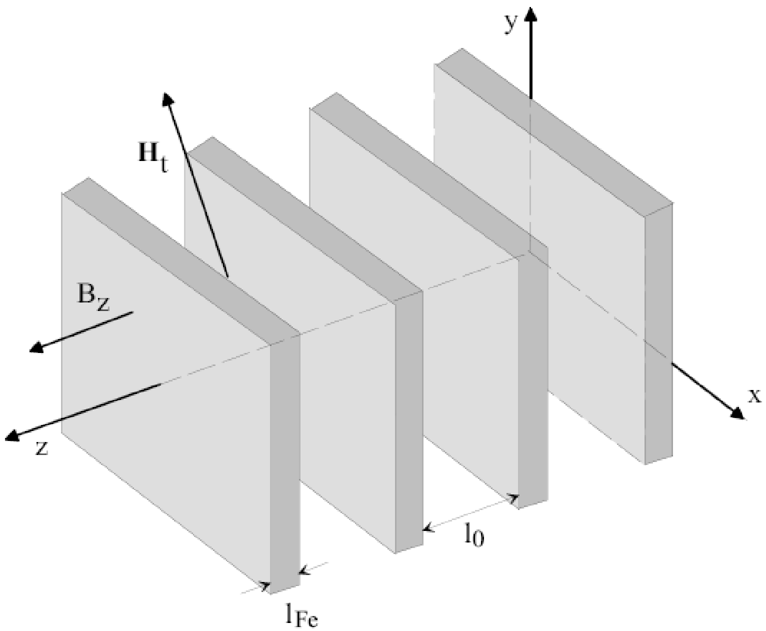
$\alpha_1 \approx 0$ or $\alpha_2 \approx \pi/4$





Stephan Russenschuck, CERN-AT-MEL

Stacking Factor



$$\mathbf{H}_t^0 = \mathbf{H}_t^{Fe} = \overline{\mathbf{H}}_t$$

$$\overline{\mathbf{B}}_t = \frac{1}{l_{Fe} + l_0}\left(l_{Fe}\mu\overline{\mathbf{H}}_t + l_0\mu_0\overline{\mathbf{H}}_t\right)$$

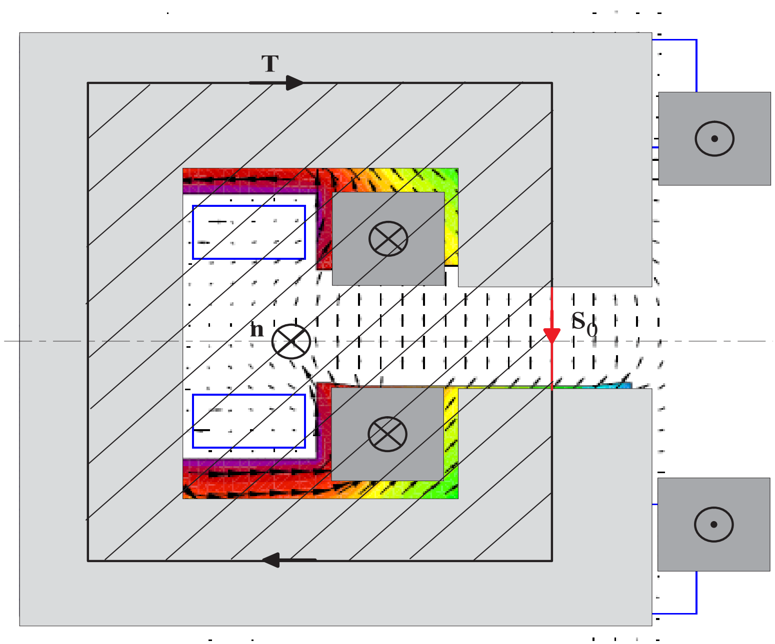
$$B_z^0 = B_z^{Fe} = \overline{B}_z$$

$$\overline{H}_z = \frac{1}{l_{Fe} + l_0}\left(l_{Fe}\frac{\overline{B}_z}{\mu} + l_0\frac{\overline{B}_z}{\mu_0}\right)$$

$$\lambda = \frac{l_{Fe}}{l_{Fe} + l_0}$$

$$\overline{\mu}_t = \lambda\mu + (1 - \lambda)\mu_0$$

$$\overline{\mu}_z = \left(\frac{\lambda}{\mu} + \frac{1 - \lambda}{\mu_0}\right)^{-1}$$



$$\int_{\partial a} \mathbf{H} \cdot \mathrm{d}\mathbf{s} = \int_a \mathbf{J} \cdot \mathrm{d}\mathbf{a}$$

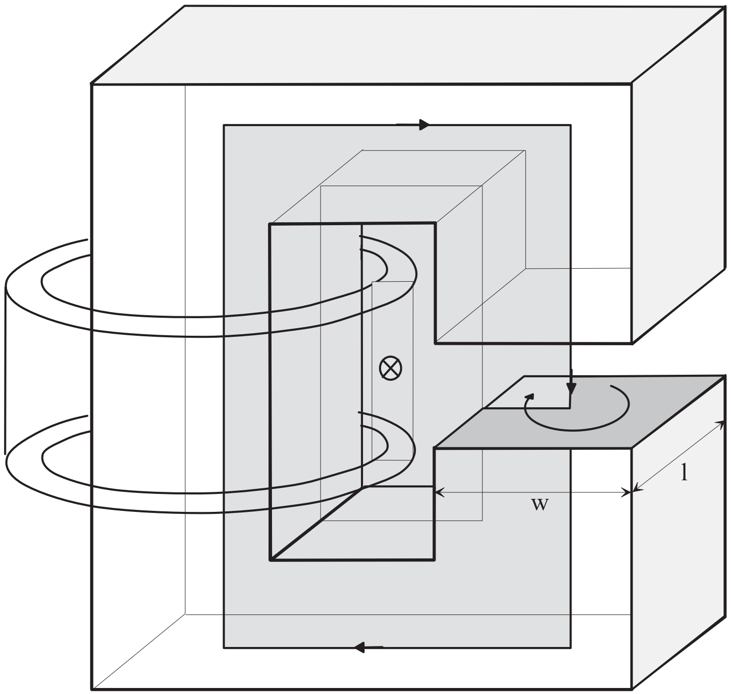
$$\int_{\partial a} \mathbf{H} \cdot \mathbf{T}\, \mathrm{d}s = \int_a \mathbf{J} \cdot \mathbf{n}\, \mathrm{d}a$$

$$H_{\mathrm{i}}\, s_{\mathrm{i}} + H_0\, s_0 = N\, I$$

$$\frac{1}{\mu_0 \mu_{\mathrm{r}}} B_{\mathrm{i}}\, s_{\mathrm{i}} + \frac{1}{\mu_0} B_0\, s_0 = N\, I$$

$$\boxed{B_0 = \frac{\mu_0 N\, I}{s_0}}$$

Dipole with Varying Cut-Section



$$\sum_{i=0}^{n} H_i s_i = N\, I$$

$$H_i = \frac{B_i}{\mu_i} = \frac{\Phi}{a_i\, \mu_i}$$

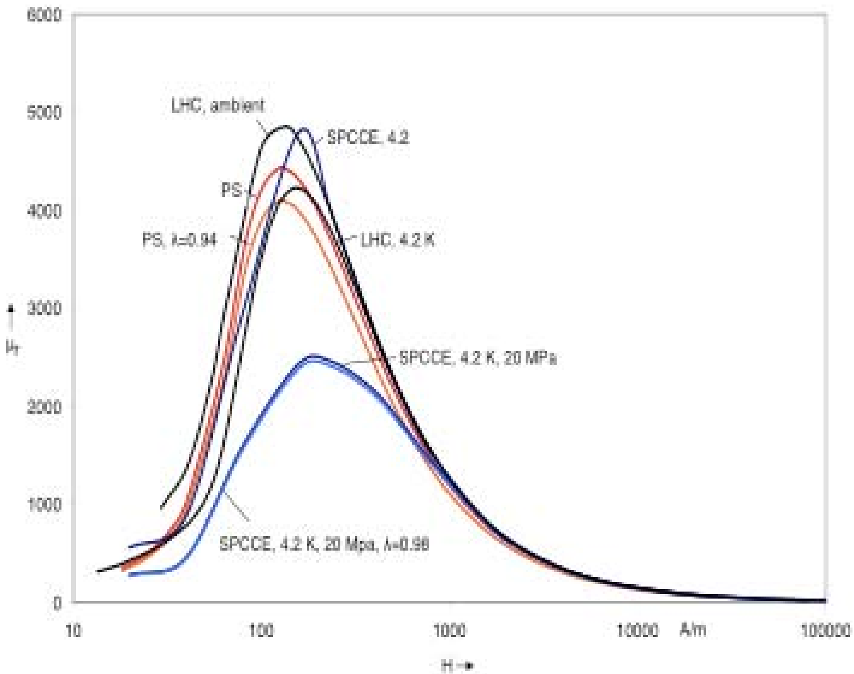
$$\Phi \sum_{i=0}^{n} \frac{s_i}{a_i\, \mu_i} = N\, I = V_{\mathrm{m}}$$

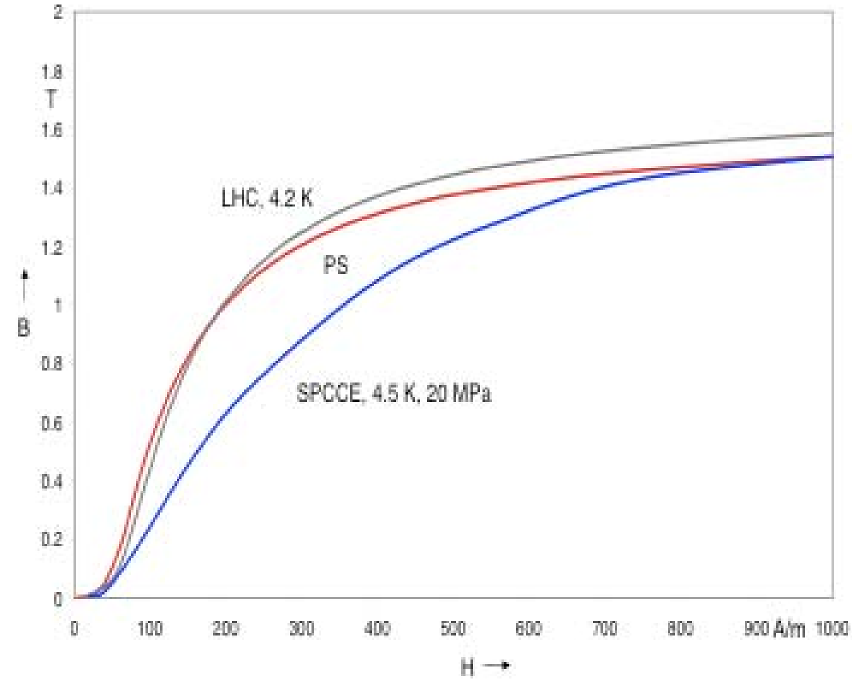
Ohm's law: $I \sum_{i=0}^{n} \frac{s_i}{a_i\, \kappa_i} = U$

$$N\, I = \Phi \sum_{i=0}^{n} \frac{s_i}{a_i\, \mu_i} = \Phi \left(\frac{s_0}{a_0\, \mu_0} + \sum_{i=1}^{n} \frac{s_i}{a_i\, \mu_i} \right)$$

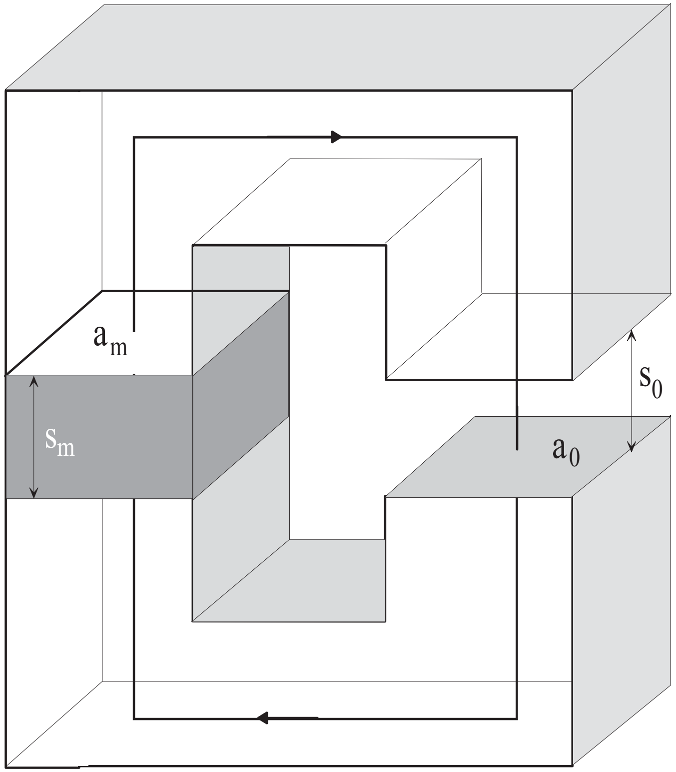
Conclusion: Magnet with big air-gap is stablized against variations in permeability

LHC and PS Yoke Material





Permanent Magnet Excitation

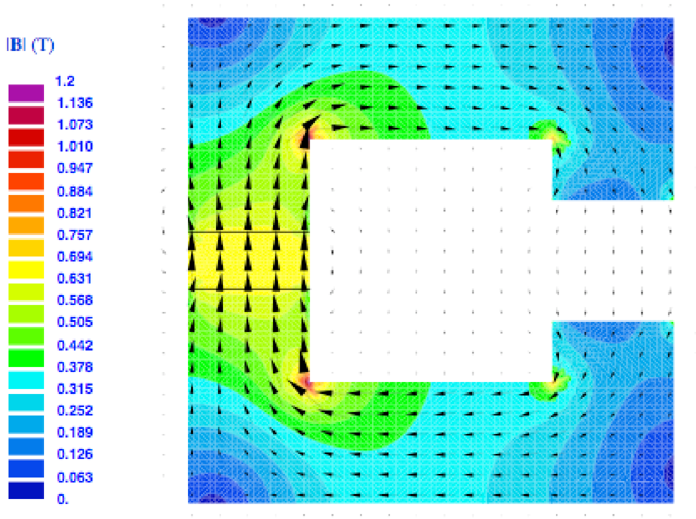


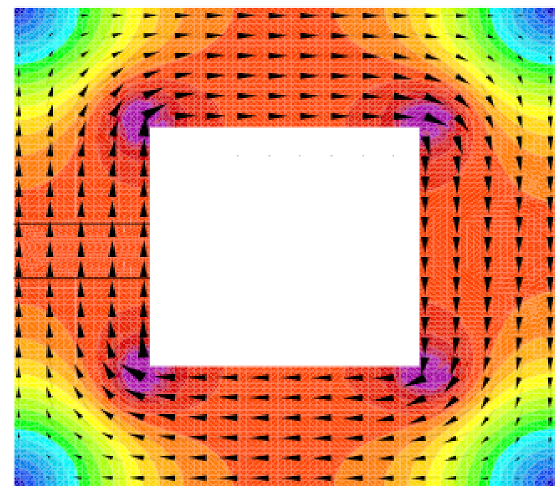
$$H_0\, s_0 + H_m\, s_m = 0$$

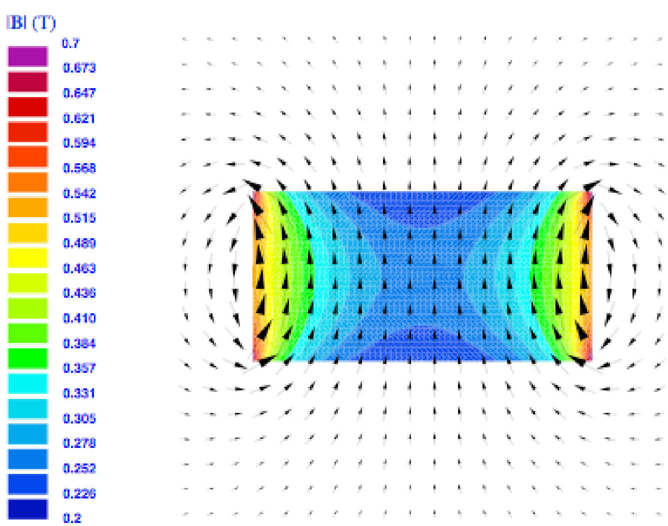
$$B_m a_m = B_0 a_0 = \mu_0 H_0 a_0$$

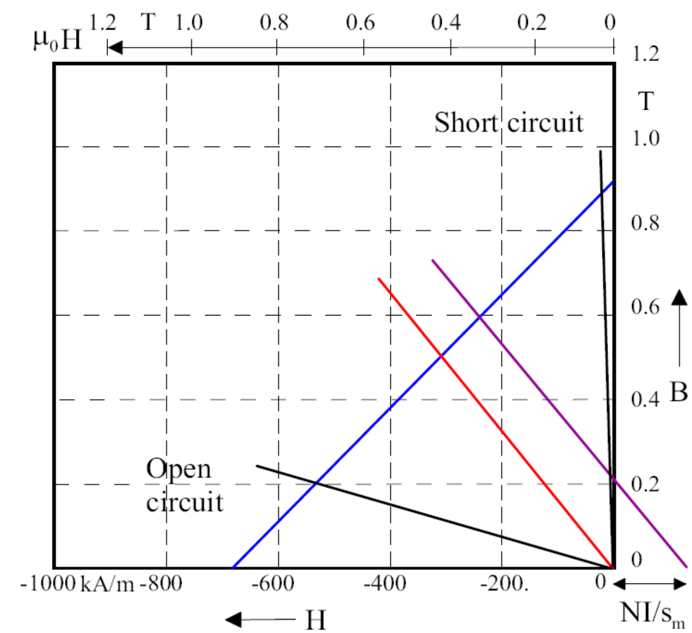
$$\begin{aligned} H_0 s_0 &= -H_m s_m, \\ \frac{1}{\mu_0} B_m \frac{a_m}{a_0} s_0 &= -H_m s_m, \\ B_m &= -\mu_0 \frac{s_m}{s_0}\frac{a_0}{a_m} H_m, \\ \frac{B_m}{\mu_0 H_m} &= -\frac{s_m}{s_0}\frac{a_0}{a_m} = P \end{aligned}$$

$$s = \frac{B_m}{H_m} = \mu_0 P = -\mu_0 \frac{s_m}{s_0}\frac{a_0}{a_m} = \mu_0 \frac{M\,(1-N)}{H_m - N\,M}$$



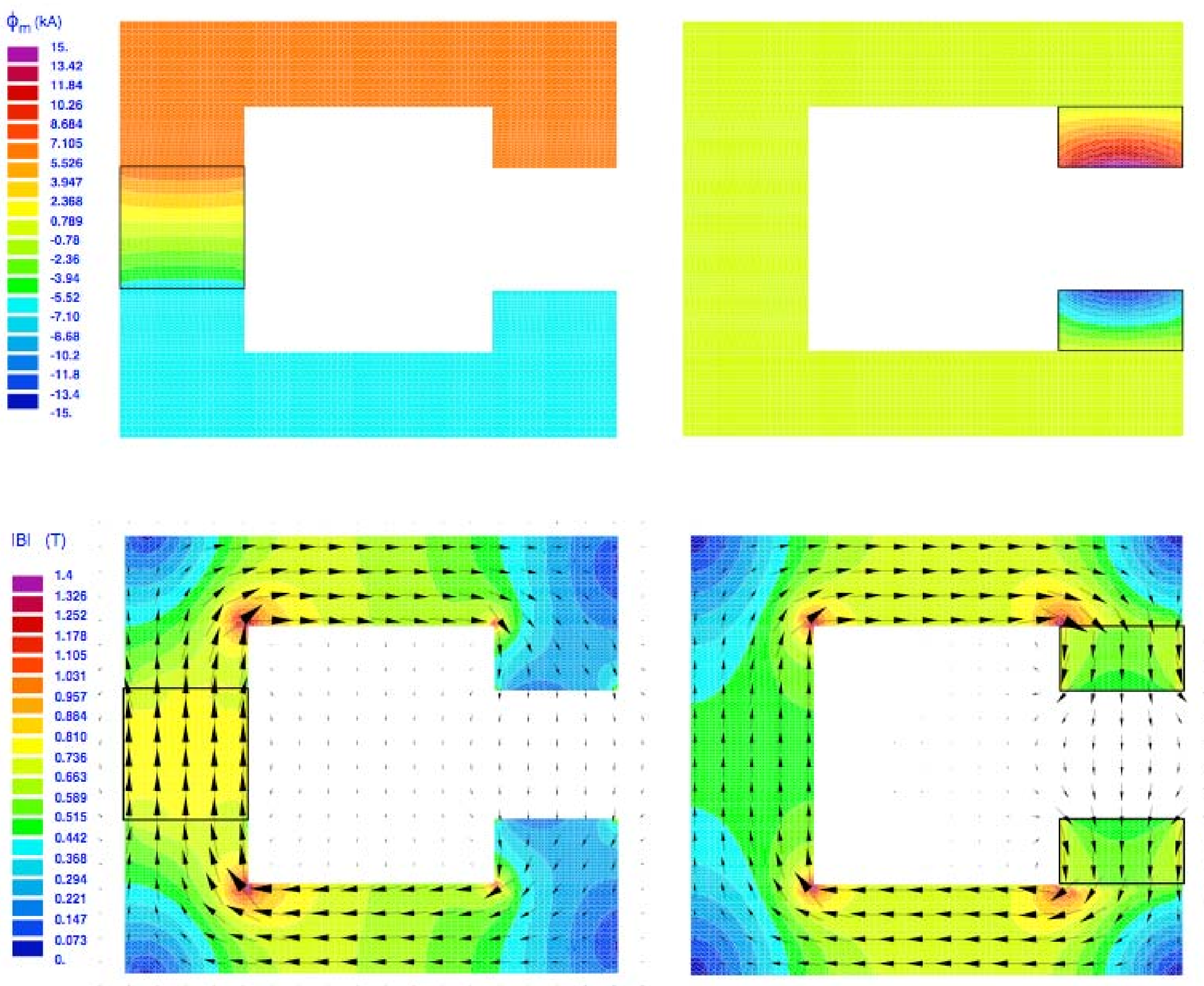




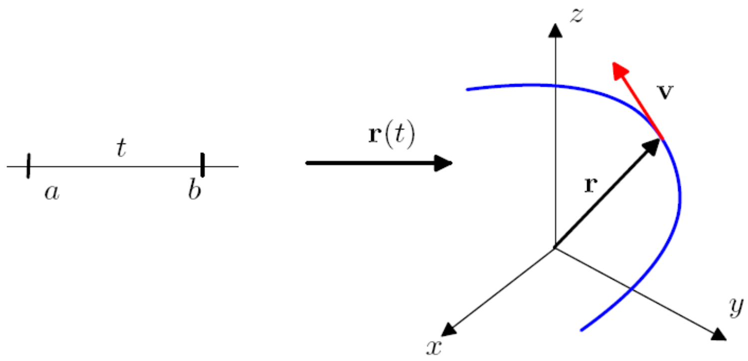


$$s = \frac{B_m}{H_m} = \mu_0 P = -\mu_0 \frac{s_m}{s_0} \frac{a_0}{a_m}$$

Optimal Position of Permanent Magnet



Space curve with $\mathbf{r}(t) = (x(t), y(t), z(t))$ parametrized such that $\mathbf{r}(0) = P$.



1-smooth scalar field $\phi : E_3 \to R : \mathbf{r} \mapsto \phi(\mathbf{r})$ expressed as $\phi(x, y, z)$, then $\phi(\mathbf{r}(t))$ at parameter (time) t.

$$\partial_{\mathbf{v}}\phi = \frac{\partial \phi}{\partial v} = \frac{\mathrm{d}}{\mathrm{d}t}[\phi(\mathbf{r} + t\mathbf{v})]_{t=0} = \lim_{t \to 0} \frac{\phi(\mathbf{r} + t\mathbf{v}) - \phi(\mathbf{r})}{t}$$

$$\partial_{\mathbf{e_x}}\phi = \frac{\partial \phi(x, y, z)}{\partial x} = \lim_{\Delta x \to 0} \frac{\phi(x + \Delta x, y, z) - \phi(x, y, z)}{\Delta x}$$

$$\partial_{\mathbf{v}}\phi = \frac{\mathrm{d}}{\mathrm{d}t}\phi(\mathbf{r}(t)) = \frac{\partial \phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} = \mathrm{grad}\,\phi \cdot \mathbf{v}$$

$$\mathrm{grad}\phi = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

The Differential Operators

$$\nabla = \frac{\partial}{\partial x}\mathbf{e}_x + \frac{\partial}{\partial y}\mathbf{e}_y + \frac{\partial}{\partial z}\mathbf{e}_z$$

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla \phi = \text{grad}\,\phi = \frac{\partial \phi}{\partial x}\mathbf{e}_x + \frac{\partial \phi}{\partial y}\mathbf{e}_y + \frac{\partial \phi}{\partial z}\mathbf{e}_z$$

$$\nabla \cdot \mathbf{a} = \text{div}\,\mathbf{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

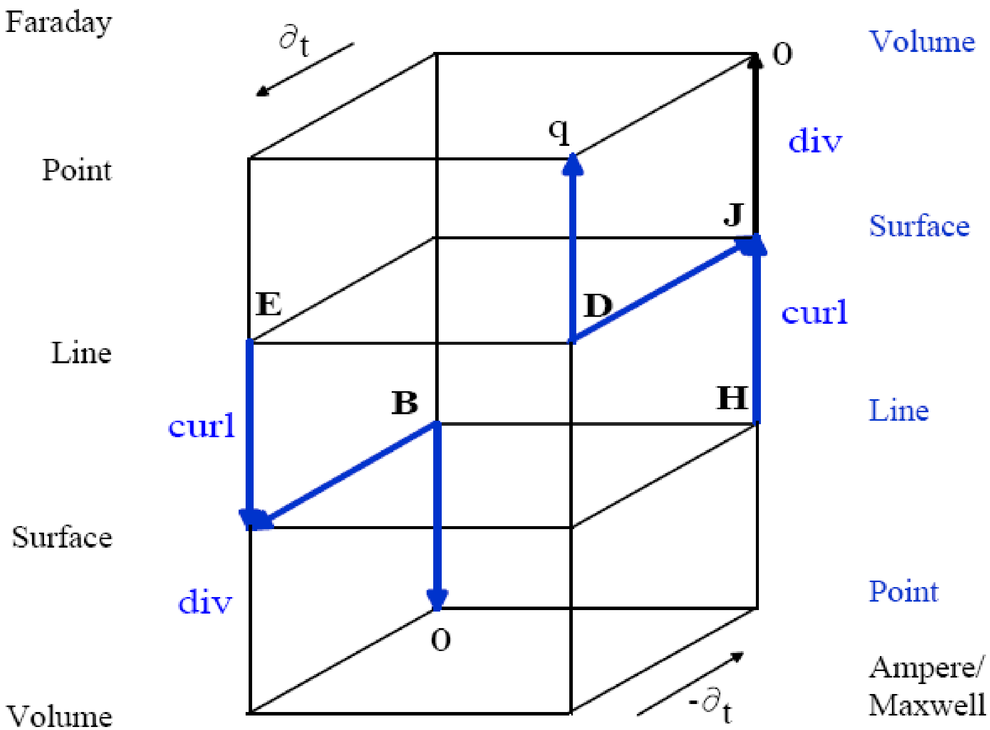
$$\nabla \times \mathbf{a} = \text{curl}\,\mathbf{a} = (\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z})\mathbf{e}_x + (\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x})\mathbf{e}_y + (\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y})\mathbf{e}_z$$

$$\nabla^2 \mathbf{A} = (\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2})\mathbf{e}_x + (\frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_y}{\partial z^2})\mathbf{e}_y + (\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2})\mathbf{e}_z = (\nabla^2 A_x)\mathbf{e}_x + (\nabla^2 A_y)\mathbf{e}_y + (\nabla^2 A_z)\mathbf{e}_z$$

Conclusion: This is horrible, so let's try the geometric approach

Maxwell's House

$$\mathbf{v} \cdot \operatorname{grad} \phi = \lim_{s \to 0} \frac{\phi(P_2) - \phi(P_1)}{s}$$



$$\operatorname{div} \mathbf{g} = \lim_{V \to 0} \frac{\int_{\partial V} \mathbf{g} \cdot \mathbf{da}}{V}$$

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{g} = \lim_{a \to 0} \frac{\int_{\partial a} \mathbf{g} \cdot \mathrm{ds}}{a}$$

Maxwell's Equations in Differential Form

$$\int_a \mathrm{curl}\vec{g}\cdot \mathrm{d}\vec{a} = \int_{\partial a} \vec{g}\cdot \mathrm{d}\vec{s}$$

$$\int_V \mathrm{div}\,\vec{g}\,\mathrm{d}V = \int_{\partial V} \vec{g}\cdot \mathrm{d}\vec{a}$$

$$\int_{\partial a} \vec{H}\cdot \mathrm{d}\vec{s} = \int_a \vec{J}\cdot \mathrm{d}\vec{a} + \frac{\mathrm{d}}{\mathrm{d}t}\int_a \vec{D}\cdot \mathrm{d}\vec{a}$$

$$\int_{\partial a} \vec{E}\cdot \mathrm{d}\vec{s} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_a \vec{B}\cdot \mathrm{d}\vec{a}$$

$$\int_{\partial V} \vec{B}\cdot \mathrm{d}\vec{a} = 0$$

$$\int_{\partial V} \vec{D}\cdot \mathrm{d}\vec{a} = \int_V \rho\,\mathrm{d}V$$

$$\int_a \mathrm{curl}\vec{H}\cdot \mathrm{d}\vec{a} = \int_a (\vec{J} + \frac{\partial}{\partial t}\vec{D})\cdot \mathrm{d}\vec{a}$$

$$\int_a \mathrm{curl}\vec{E}\cdot \mathrm{d}\vec{a} = -\int_a \frac{\partial}{\partial t}\vec{B}\cdot \mathrm{d}\vec{a}$$

$$\int_V \mathrm{div}\,\vec{B}\,\mathrm{d}V = 0$$

$$\int_V \mathrm{div}\,\vec{D}\,\mathrm{d}V = \int_V \rho\,\mathrm{d}V$$

$$\mathrm{curl}\vec{H} = \vec{J} + \partial_t\vec{D}$$

$$\mathrm{curl}\vec{E} = -\partial_t\vec{B}$$

$$\mathrm{div}\,\vec{B} = 0$$

$$\mathrm{div}\,\vec{D} = \rho$$

Lemmata of Poincare

$$\int_V \operatorname{div}\operatorname{curl}\mathbf{g}\,\mathrm{d}V = \int_{\partial V}\operatorname{curl}\mathbf{g}\cdot\mathrm{d}\mathbf{a} = \int_{\partial(\partial V)}\mathbf{g}\cdot\mathrm{d}\mathbf{s} = 0$$

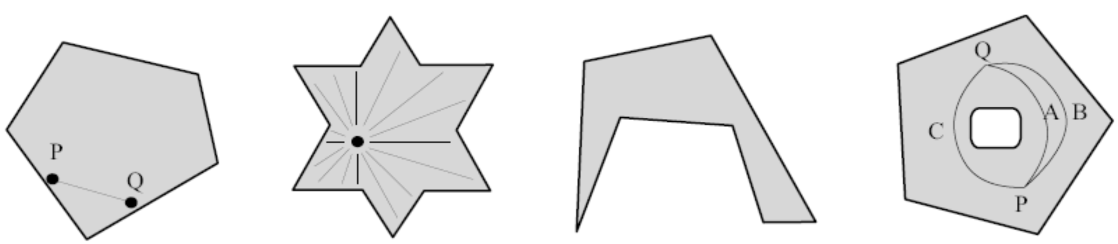
$$\int_a \operatorname{curl}\operatorname{grad}\phi\cdot\mathrm{d}\mathbf{a} = \int_{\partial a}\operatorname{grad}\phi\cdot\mathrm{d}\mathbf{s} = \phi|_{\partial(\partial a)} = 0$$

A curl free smooth vector field $\mathbf{h} \in \mathcal{V}(\Omega)$ over an open star shaped domain $\Omega \subset \mathbb{R}^3$ can always be expressed by a (smooth) scalar potential $\phi \in \mathcal{S}(\Omega)$.

$$\operatorname{curl}\mathbf{h} = 0 \qquad \rightarrow \qquad \mathbf{h} = \operatorname{grad}\phi.$$

A source free, smooth vector field $\mathbf{b} \in \mathcal{V}(\Omega)$ over an open star shaped domain $\Omega \subset \mathbb{R}^3$ can always be expressed by a (smooth) vector potential $\mathbf{a} \in \mathcal{V}(\Omega)$.

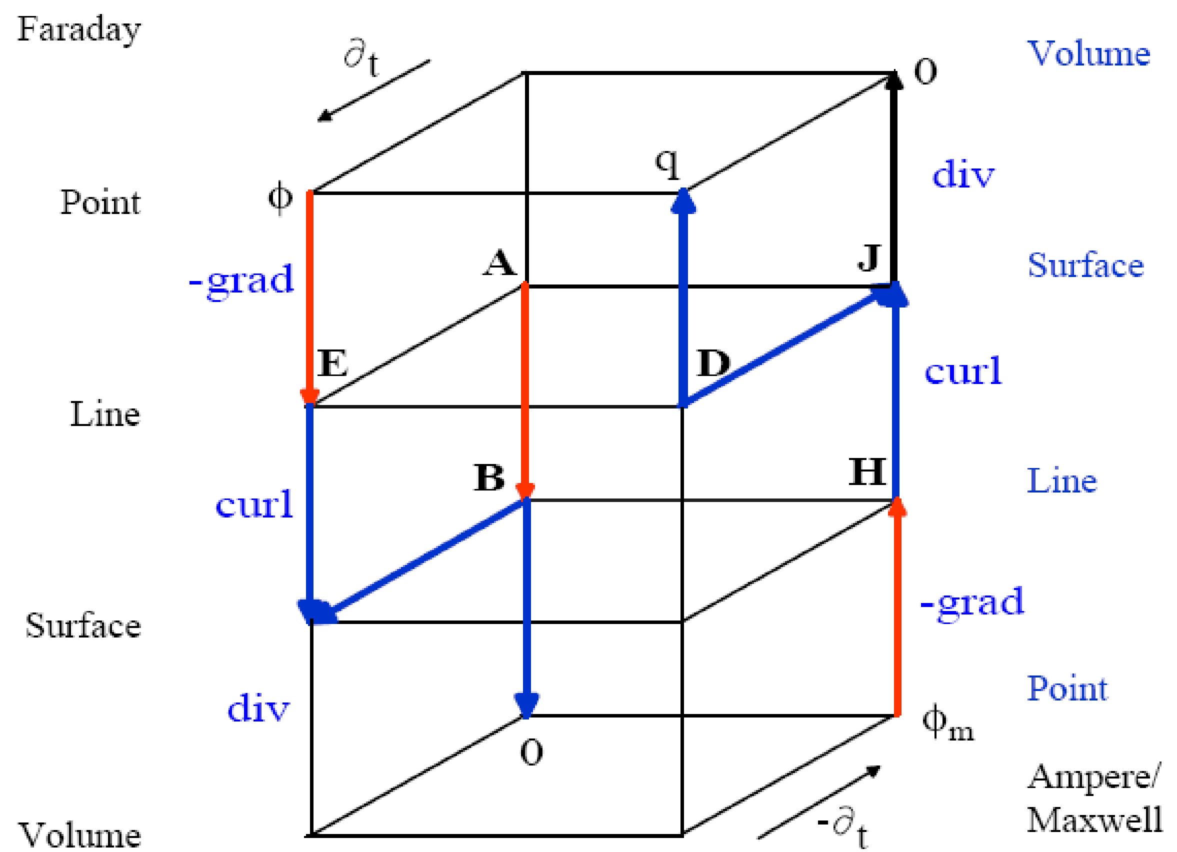
$$\operatorname{div}\mathbf{b} = 0 \qquad \rightarrow \qquad \mathbf{b} = \operatorname{curl}\mathbf{a}.$$



In 3D: also connected boundaries

Maxwell's House

Inner oriented



Outer oriented

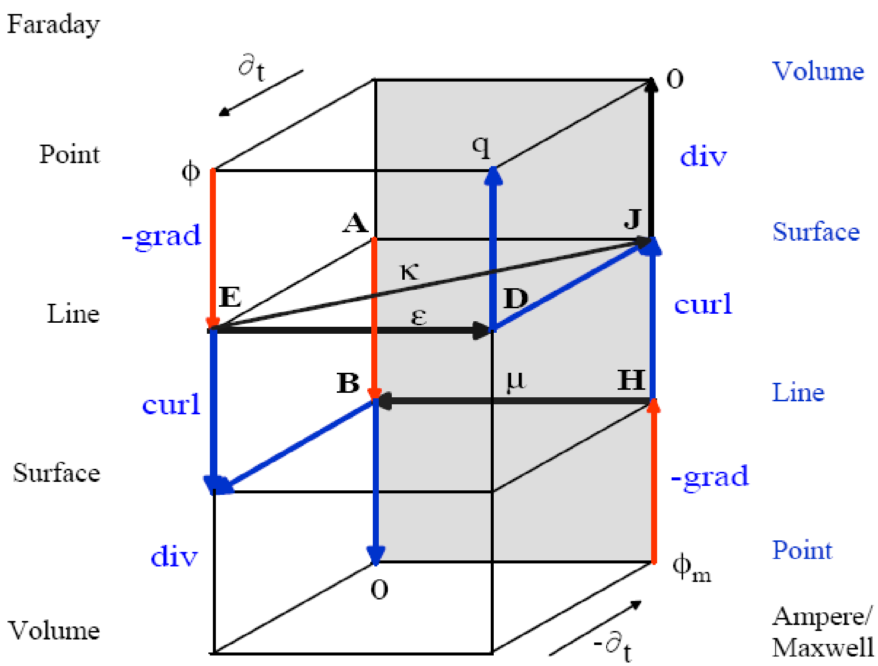
Would be even more symmetric with magnetic monopoles

Maxwell's Facade

$$\text{curl}\,\frac{1}{\mu}\,\text{curl}\,\mathbf{A} = \mathbf{J} \qquad \frac{1}{\mu}\,\text{curl curl}\,\mathbf{A} = 0 \qquad \nabla^2\mathbf{A} - \text{grad div}\,\mathbf{A} = 0 \qquad \nabla^2 A_z = 0$$

Constant permeablity and no sources

Only for Cartesian components



$$\text{div}\,\mu\,\text{grad}\,\phi_\text{m} = 0 \qquad \mu_0\,\text{div grad}\,\phi_\text{m} = 0 \qquad \nabla^2\phi_\text{m} = 0$$

No sources

Method of Separation

$$\nabla^2 A_z = -\mu_0 J_z \quad \text{in } \Omega_\text{a} \qquad \nabla^2 A_z = 0 \quad \text{in } \Omega_\text{a}, \mathbf{J} = 0$$

$$r^2 \frac{\partial^2 A_z}{\partial r^2} + r \frac{\partial A_z}{\partial r} + \frac{\partial^2 A_z}{\partial \varphi^2} = 0 \quad \text{in } \Omega_\text{a}, \mathbf{J} = 0$$

$$A_z = R(r)\phi(\varphi)$$

$$\frac{\partial A_z}{\partial r} = \frac{\partial R(r)}{\partial r}\phi(\varphi),$$

$$\frac{\partial^2 A_z}{\partial r^2} = \frac{\partial^2 R(r)}{\partial r^2}\phi(\varphi),$$

$$\frac{\partial^2 A_z}{\partial \varphi^2} = \frac{\partial^2 \phi(\varphi)}{\partial \varphi^2} R(r).$$

$$\underbrace{\frac{1}{R(r)}\left(r^2 \frac{\partial^2 R(r)}{\partial r^2} + r \frac{\partial R(r)}{\partial r}\right)}_{n^2} = \underbrace{-\frac{1}{\phi(\varphi)} \frac{\partial^2 \phi(\varphi)}{\partial \varphi^2}}_{n^2}$$

$$r^2 \frac{\mathrm{d}^2 R(r)}{\mathrm{d}r^2} + r \frac{\mathrm{d}R(r)}{\mathrm{d}r} - n^2 R(r) = 0$$

$$\frac{\mathrm{d}^2 \phi(\varphi)}{\mathrm{d}\varphi^2} + n^2 \phi(\varphi) = 0$$

How do you solve differential equations: Look them up in a book

Solution of Laplace's Equation

$$R(r) = \mathcal{E}\, r^n + \mathcal{F}\, r^{-n},$$

$$\phi(\varphi) = \mathcal{G}\, \sin n\varphi + \mathcal{H}\, \cos n\varphi.$$

$$A_z(r,\varphi) = \sum_{n=1}^{\infty} (\mathcal{E}_n r^n + \mathcal{F}_n r^{-n})(\mathcal{G}_n \sin n\varphi + \mathcal{H}_n \cos n\varphi)$$

$$= \sum_{n=1}^{\infty} r^n (\mathcal{C}_n \sin n\varphi - \mathcal{D}_n \cos n\varphi)$$

$$B_r(r,\varphi) = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} = \sum_{n=1}^{\infty} n r^{n-1} (\mathcal{C}_n \cos n\varphi + \mathcal{D}_n \sin n\varphi),$$

$$B_\varphi(r,\varphi) = -\frac{\partial A_z}{\partial r} = -\sum_{n=1}^{\infty} n r^{n-1} (\mathcal{C}_n \sin n\varphi - \mathcal{D}_n \cos n\varphi).$$

What have we won? If we know the field at a reference radius, we know it everywhere inside

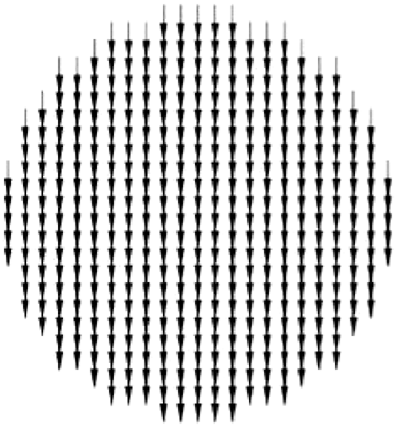
$$A_n = n r_0^{n-1} \mathcal{C}_n \qquad \text{and} \qquad B_n = n r_0^{n-1} \mathcal{D}_n$$

$$B_r(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n \sin n\varphi + A_n \cos n\varphi) = B_N \sum_{n=1}^{\infty} (b_n \sin n\varphi + a_n \cos n\varphi)$$

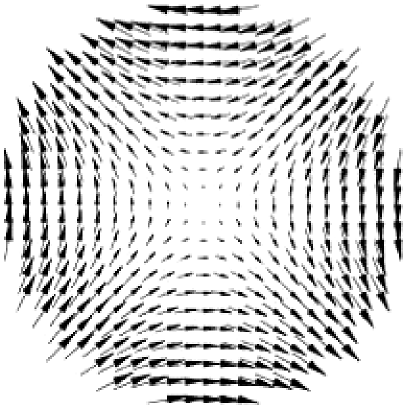
$$B_\varphi(r_0, \varphi) = \sum_{n=1}^{\infty} (B_n \cos n\varphi - A_n \sin n\varphi) = B_N \sum_{n=1}^{\infty} (b_n \cos n\varphi - a_n \sin n\varphi)$$

$$A_n(r_1) = (\frac{r_1}{r_0})^{n-1} A_n(r_0), \qquad B_n(r_1) = (\frac{r_1}{r_0})^{n-1} B_n(r_0).$$

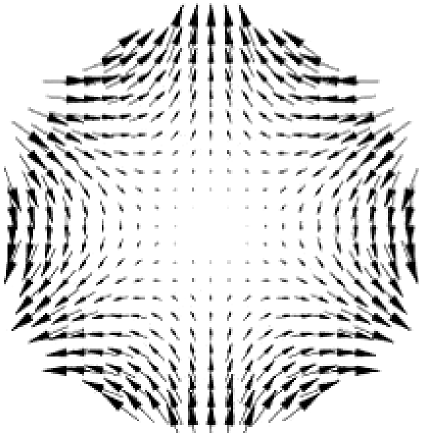
$$b_n(r_1) = \frac{B_n(r_1)}{B_N(r_1)} = \frac{(\frac{r_1}{r_0})^{n-1} B_n(r_0)}{(\frac{r_1}{r_0})^{N-1} B_N(r_0)} = (\frac{r_1}{r_0})^{n-N} b_n(r_0),$$



$$B_r = \mathcal{C}_1 \cos\varphi + \mathcal{D}_1 \sin\varphi$$
$$B_\varphi = -\mathcal{C}_1 \sin\varphi + \mathcal{D}_1 \cos\varphi$$
$$B_x = \mathcal{C}_1$$
$$B_y = \mathcal{D}_1$$

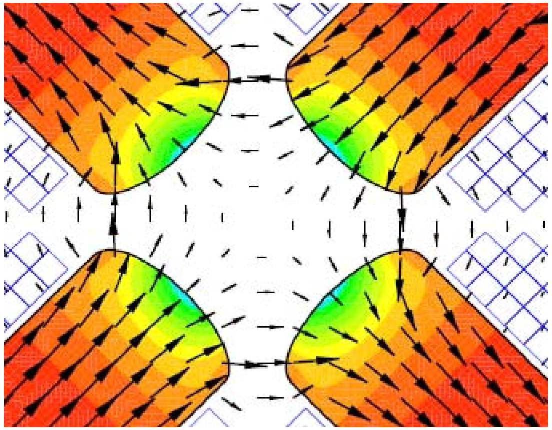


$$B_r = 2\mathcal{C}_2 r \cos 2\varphi + 2\mathcal{D}_2 r \sin 2\varphi$$
$$B_\varphi = -2\mathcal{C}_2 r \sin 2\varphi + 2\mathcal{D}_2 r \cos 2\varphi$$
$$B_x = 2\mathcal{C}_2 x + 2\mathcal{D}_2 y$$
$$B_y = -2\mathcal{C}_2 y + 2\mathcal{D}_2 x$$

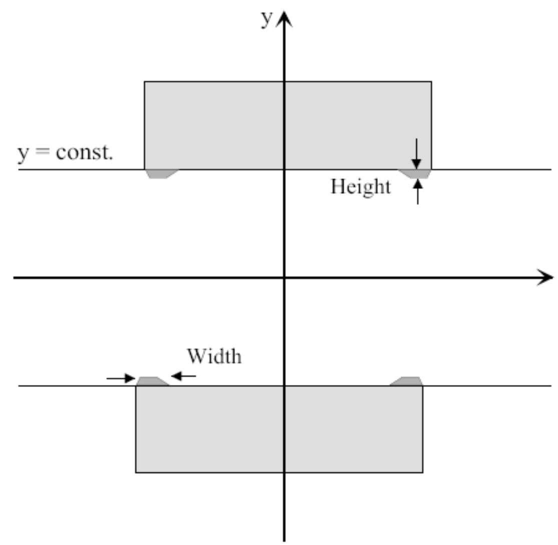


$$B_r = 3\mathcal{C}_3 r^2 \cos 3\varphi + 3\mathcal{D}_3 r^2 \sin 3\varphi$$
$$B_\varphi = -3\mathcal{C}_3 r^2 \sin 3\varphi + 3\mathcal{D}_3 r^2 \cos 3\varphi$$
$$B_x = 3\mathcal{C}_3 (x^2 - y^2) + 6\mathcal{D}_3 xy$$
$$B_y = -6\mathcal{C}_3 xy + 3\mathcal{D}_3 (x^2 - y^2)$$

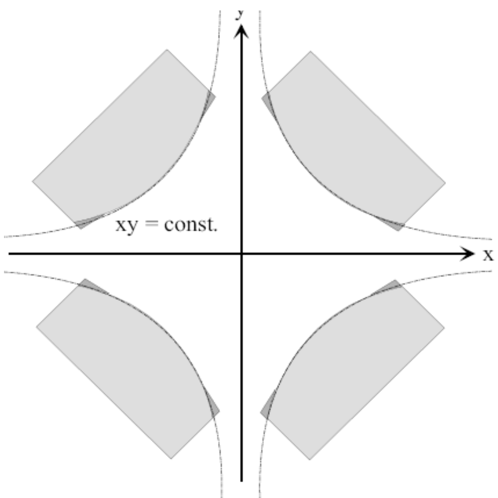
Ideal Pole Shape of Conventional Magnets

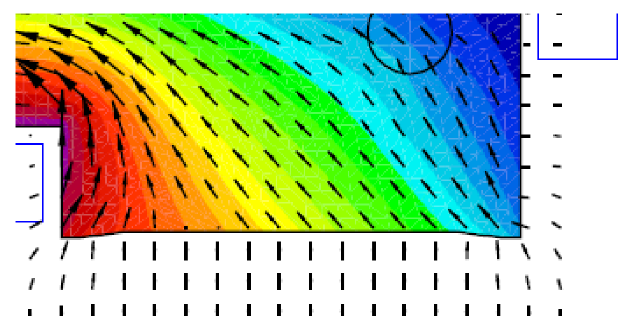


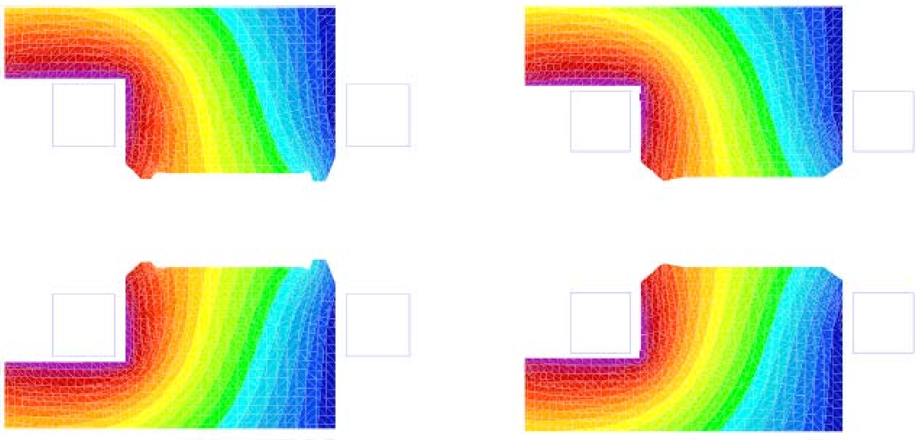
$$\phi_\mathrm{m} = \mathcal{C}_1 x + \mathcal{D}_1 y$$



$$\phi_\mathrm{m} = \mathcal{C}_2(x^2 - y^2) + 2\mathcal{D}_2 xy$$

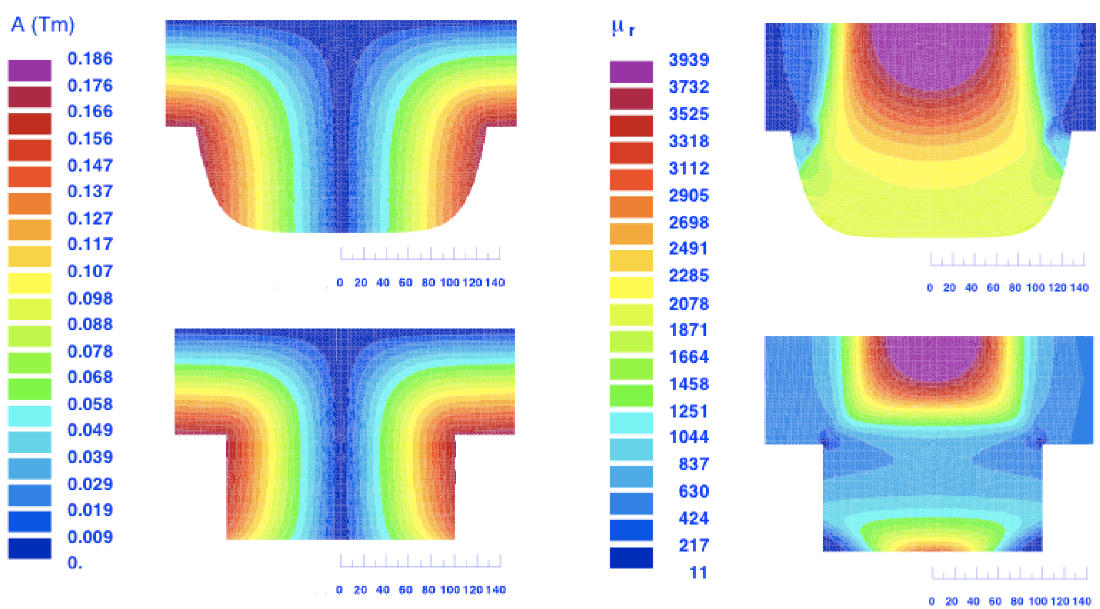






Pole shimming

Rogowski profiles



Integrated Multipole Field Errors

$$\nabla^2\Phi_m(x,y,z) = \frac{\partial^2\Phi_m(x,y,z)}{\partial x^2} + \frac{\partial^2\Phi_m(x,y,z)}{\partial y^2} + \frac{\partial^2\Phi_m(x,y,z)}{\partial z^2} = 0$$

$$\overline{\Phi}_m(x,y) = \int_{-z_0}^{z_0} \Phi_m(x,y,z)\,dz$$

$$\nabla^2\overline{\Phi}_m(x,y) = \frac{\partial^2\overline{\Phi}_m(x,y)}{\partial x^2} + \frac{\partial^2\overline{\Phi}_m(x,y)}{\partial y^2} = 0$$

Sufficient condition: Integration path is extended far enough away from (into) the magnet so that the axial component of the field has dropped to zero.

$$\frac{\partial^2\overline{\Phi}_m}{\partial x^2} + \frac{\partial^2\overline{\Phi}_m}{\partial y^2} = \int_{-z_0}^{z_0} \left(\frac{\partial^2\Phi_m}{\partial x^2} + \frac{\partial^2\Phi_m}{\partial y^2}\right) dz = \int_{-z_0}^{z_0} \left(-\frac{\partial^2\Phi_m}{\partial z^2}\right) dz = -\left.\frac{\partial\Phi_m}{\partial z}\right|_{-z_0}^{z_0} = H_z|_{-z_0} - H_z|_{z_0}$$

Caution: Always use long measurement coils, or don’t apply the scaling laws