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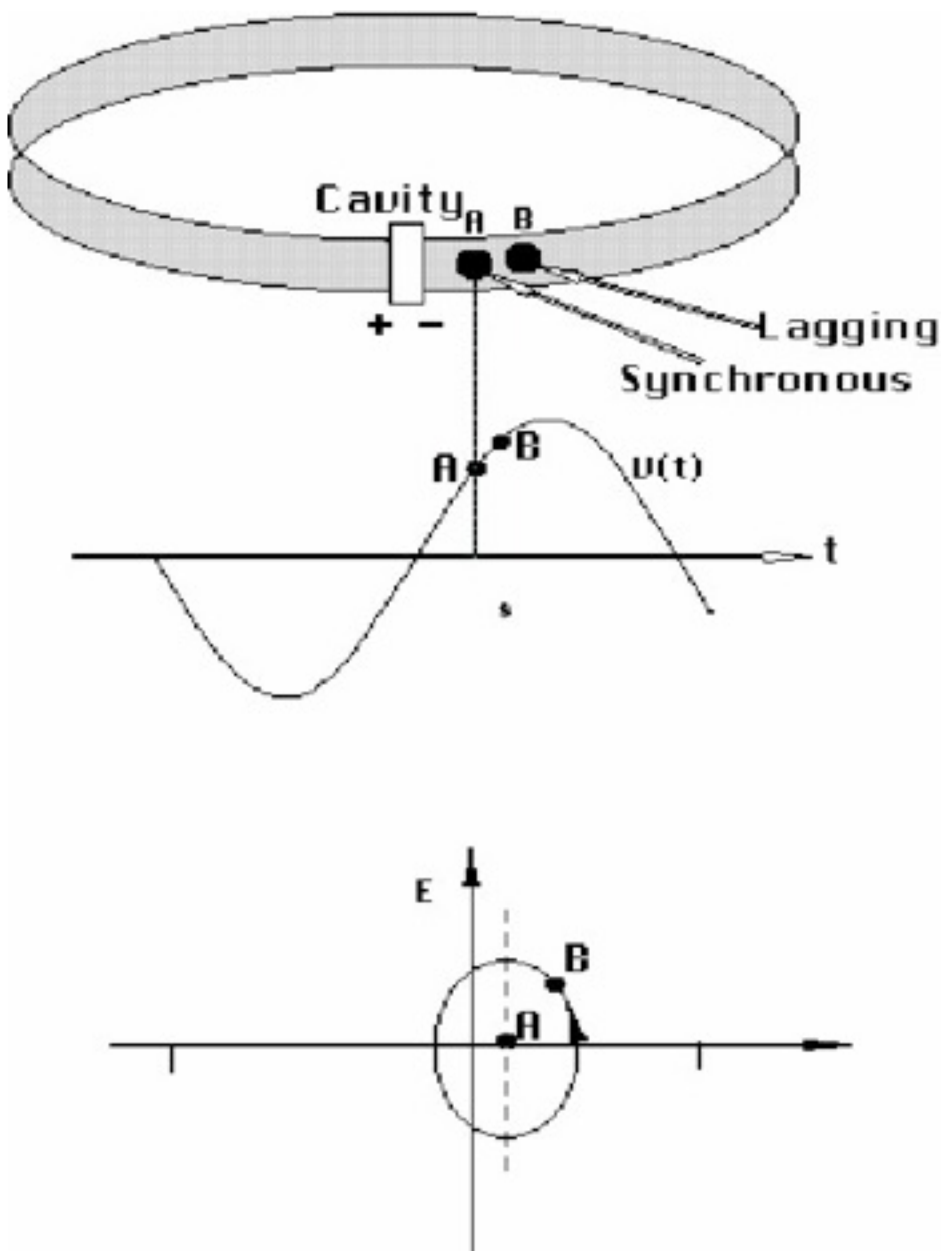
Beam Dynamics in Synchrotrons with Space-Charge

Basic Principles without space-charge

- RF resonant cavity providing accelerating voltage $V(t)$. Often $V = V_0 \sin(\phi_s + \omega_{rf} t)$, where ω_{rf} is the angular frequency synchronised with the arrival time of beam particles.

- Particle with phase $\phi = \phi_s$ at revolution period T_0 and momentum p_0 is called the *synchronous particle.*

- Synchronous particle synchronizes with rf wave with a frequency $\omega_{rf} = h\omega_0$, where $\omega_0 = \beta_0 c/R_0$ is revolution frequency and h is harmonic number. It encounters the rf voltage at phase angle ϕ_s on every revolution. The acceleration rate for synchronous particle is



$$\frac{\mathrm{d}\mathcal{E}_0}{\mathrm{d}t} = \frac{\omega_0}{2\pi} qV \sin\phi_s.$$

$\mathcal{E}_0$ is the synchronous energy

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Non-synchronous particles have small deviations of rf parameters:

$$\begin{aligned} \omega &= \omega_0 + \Delta\omega, & \phi &= \phi_s + \Delta\phi, & \theta &= \theta_s + \Delta\theta \\ p &= p_0 + \Delta p, & \mathcal{E} &= \mathcal{E}_0 + \Delta\mathcal{E}. \end{aligned}$$

θ is the azimuthal orbital angle, where $\Delta\phi = \phi - \phi_s = -h\Delta\theta$

$$\text{so} \quad \Delta\omega = \frac{\mathrm{d}}{\mathrm{d}t}\Delta\theta = -\frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}\Delta\phi = -\frac{1}{h}\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

Energy gain per revolution for non-synchronous particle is $qV\sin\phi$

$$\Longrightarrow \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \frac{\omega}{2\pi} qV\sin\phi$$

Thus

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = \frac{1}{2\pi} qV\big(\sin\phi - \sin\phi_s\big)}$$

Since $\quad \dfrac{\Delta p}{p_0} = \dfrac{1}{\beta^2}\dfrac{\Delta\mathcal{E}}{\mathcal{E}},$

an equivalent form is $\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\Delta p}{p_0} = \dfrac{\omega_0}{2\pi\beta^2\mathcal{E}} qV\big(\sin\phi - \sin\phi_s\big)$

$$\omega = \frac{\beta c}{R} \quad \Longrightarrow \quad \frac{\Delta\omega}{\omega_0} = \frac{\Delta\beta}{\beta_0} - \frac{\Delta R}{R_0}$$

$$= \left(\frac{1}{\gamma^2} - \alpha_p\right)\frac{\Delta p}{p_0} = \left(\frac{1}{\gamma^2} - \frac{1}{\gamma_t^2}\right)\frac{\Delta p}{p_0}$$

α_p is *momentum compaction*, $\alpha_p = \dfrac{1}{\gamma_t^2}$, where γ_t corresponds to the *transition energy* $m_0\gamma_t c^2$

$$\text{Write} \quad \eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} \qquad (\textit{slip factor})$$

Then

$$\boxed{\frac{\mathrm{d}\phi}{\mathrm{d}t} = -h\Delta\omega = h\omega_0\eta\frac{\Delta p}{p_0} = \frac{h\omega_0^2\eta}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = \frac{1}{2\pi}qV\left(\sin\phi - \sin\phi_s\right)}$$

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Small Amplitude Oscillations

Combined:
$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\phi - \phi_s) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{h\omega_0^2\eta}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)\right]$$
$$= \frac{h\omega_0^2 qV\eta}{2\pi\beta^2\mathcal{E}}\left[\sin\phi - \sin\phi_s\right]$$

linearised for $|\phi - \phi_s| \ll 1$
$$\approx \frac{h\omega_0^2 qV\eta\cos\phi_s}{2\pi\beta^2\mathcal{E}}(\phi - \phi_s)$$

$\Longrightarrow$ stable oscillations for $\eta\cos\phi_s < 0$ (McMillan & Veksler)

at an angular frequency $\Omega_s = \sqrt{-\dfrac{h\omega_0^2 qV\eta\cos\phi_s}{2\pi\beta^2\mathcal{E}}}$

Below transition, $\gamma < \gamma_t$, $\eta < 0$, so require $0 < \phi_s < \frac{1}{2}\pi$ $\boxed{\text{acceleration}}$

Above transition, $\gamma > \gamma_t$, $\eta > 0$, so shift synchronous phase to $\pi - \phi_s$

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Hamiltonian Formulation

$\left(\phi, \dfrac{\Delta\mathcal{E}}{\omega_0}\right)$ are conjugate coordinates in longitudinal phase-space

For a Hamiltonian system, require $\mathcal{H}\left(\phi, \dfrac{\Delta\mathcal{E}}{\omega_0}\right)$ such that

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial(\Delta\mathcal{E}/\omega_0)} = \frac{h\eta\omega_0^2}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = -\frac{\partial\mathcal{H}}{\partial\phi} = \frac{1}{2\pi}qV\left[\sin\phi - \sin\phi_s\right]$$

Suggests:

Hamiltonian $\boxed{\mathcal{H} = \frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)^2 + \frac{qV}{2\pi}\left[\cos\phi - \cos\phi_s + (\phi-\phi_s)\sin\phi_s\right]}$

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Synchrotron Mapping Equation

Hamiltonian formalism $\Longrightarrow$ uniform distribution of rf.

In reality, finite number of cavities, localised in short sections of synchrotrons.

Then use symplectic mapping equations:

After one revolution of ring, in time $T_0 = \dfrac{2\pi R_0}{\beta c} = \dfrac{2\pi}{\omega_0}$,

$$(\phi_n, \Delta\mathcal{E}_n) \to (\phi_{n+1}, \Delta\mathcal{E}_{n+1}), \quad \text{where}$$

$$\begin{aligned} \Delta\mathcal{E}_{n+1} &= \Delta\mathcal{E}_n + qV\big(\sin\phi_n - \sin\phi_s\big) \\ \phi_{n+1} &= \phi_n + \frac{2\pi h\eta}{\beta^2\mathcal{E}}\Delta\mathcal{E}_{n+1} \end{aligned}$$

Here $\mathcal{E} = \mathcal{E}_{0,n+1} = \mathcal{E}_0 + qV\sin\phi_s$, $\gamma = \mathcal{E}/m_0c^2$, $\beta = \sqrt{1 - 1/\gamma^2}$ and $\eta = \alpha_p - 1/\gamma^2$

Symplectic $\Longrightarrow$ Jacobian $\left|\dfrac{\partial(\Delta\mathcal{E}_{n+1}, \phi_{n+1})}{\partial(\Delta\mathcal{E}_n, \phi_n)}\right| = 1$

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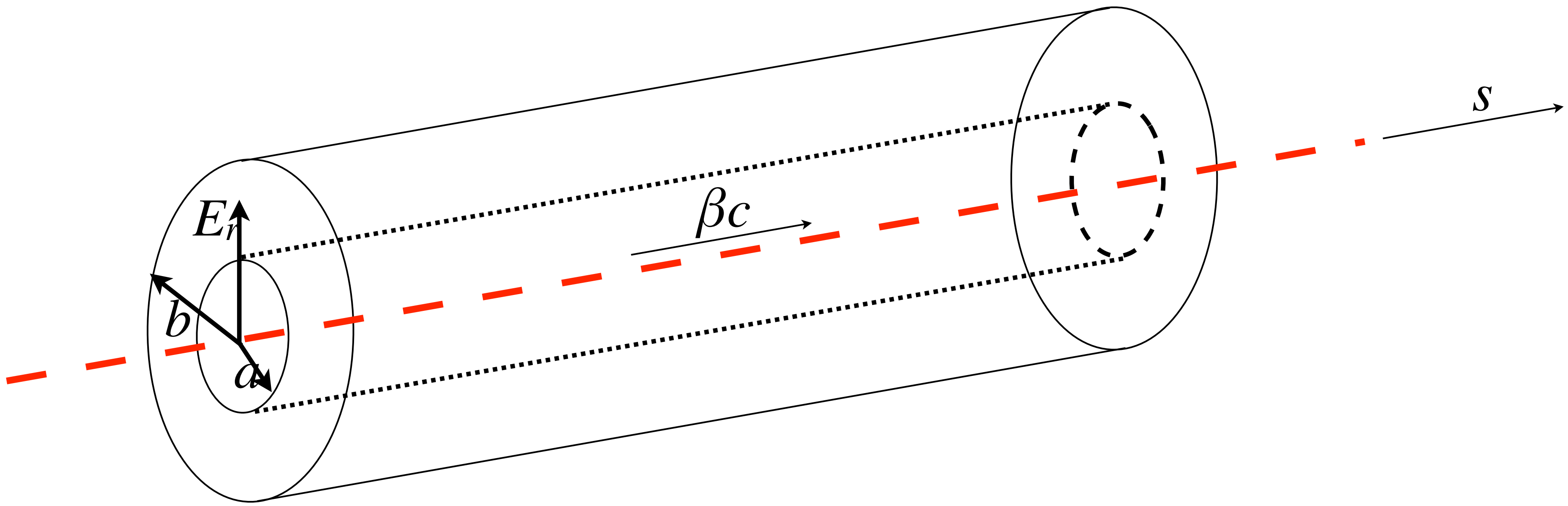
Fields created by an unbunched (coasting) beam

Assume a round beam of mean radius a in a circular pipe of radius b

Velocity of beam $= \beta c$

Line density (number of particles per unit length) $= \lambda(s)$



Fields are: $$E_r = \frac{q\lambda}{2\pi\epsilon_0}\frac{1}{r}; \qquad B_\theta = \frac{\mu_0 q\lambda\beta c}{2\pi}\frac{1}{r}; \qquad r \geqslant a$$

$$E_r = \frac{q\lambda}{2\pi\epsilon_0}\frac{r}{a^2}; \qquad B_\theta = \frac{\mu_0 q\lambda\beta c}{2\pi}\frac{r}{a^2}; \qquad r \leqslant a$$

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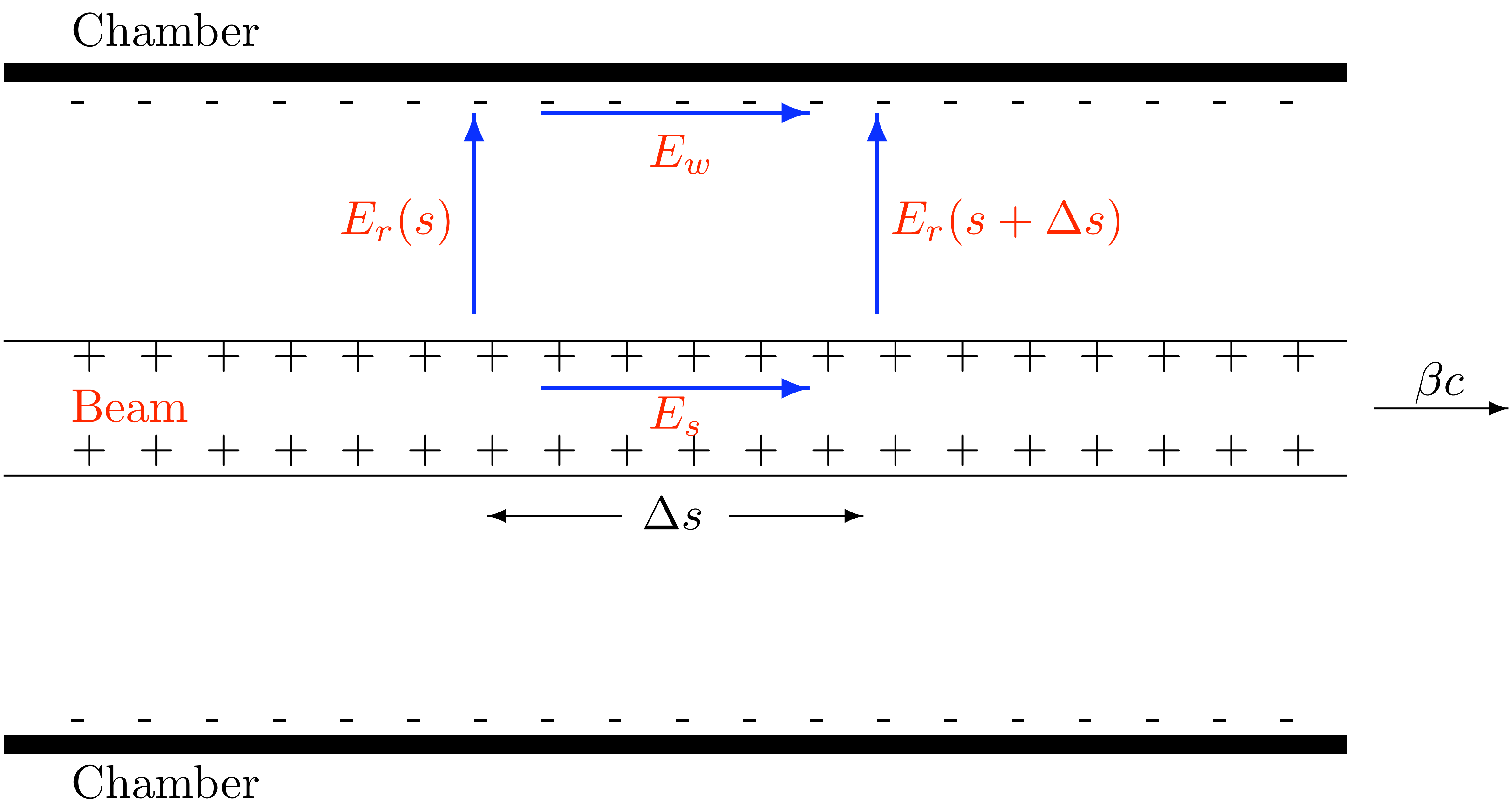
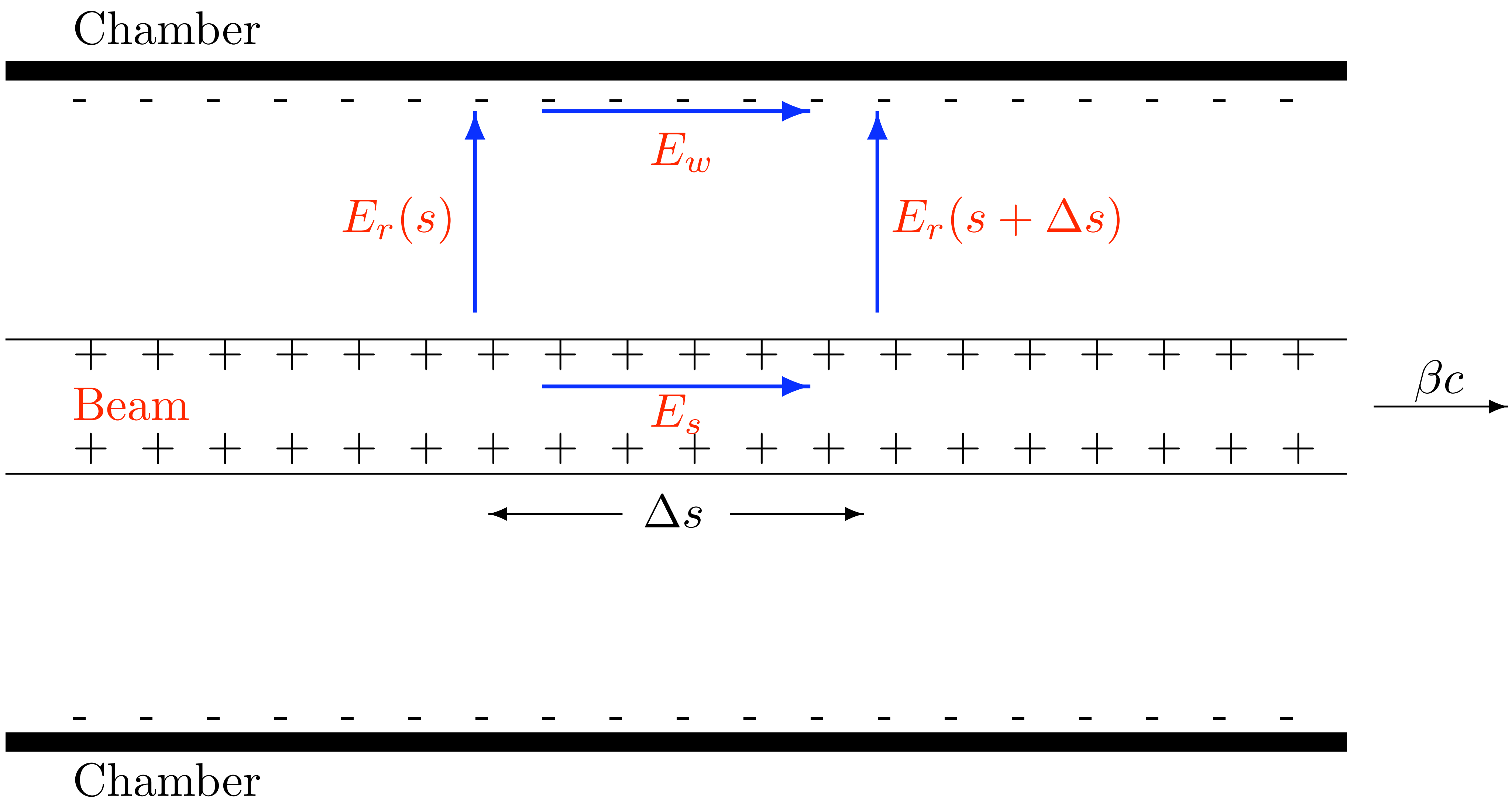


Beam induces charges on inner surface of chamber wall, which form a current I_w equal and opposite to the AC component of beam current.



$$\text{Faraday's Law:} \quad \oint \mathbf{E} \cdot \mathbf{dl} = -\frac{\partial}{\partial t} \iint \mathbf{B} \cdot \mathbf{dS}$$

$$\oint \mathbf{E} \cdot \mathbf{dl} = (E_s - E_w)\Delta s + \frac{q\lambda}{2\pi\epsilon_0} \left[\int_0^a \frac{r}{a^2}\, \mathrm{d}r + \int_a^b \frac{1}{r}\, \mathrm{d}r \right] \Bigg|_s^{s+\Delta s}$$

$$= (E_s - E_w)\Delta s + \frac{q}{4\pi\epsilon_0} \left(1 + 2 \ln \frac{b}{a} \right) \big(\lambda(s+\Delta s) - \lambda(s) \big)$$

$$\int \mathbf{B}\cdot\mathbf{dS} = \frac{\mu_0 q\beta\lambda c}{2\pi}\left[\int_0^a \frac{r}{a^2}\,\mathrm{d}r + \int_a^b \frac{1}{r}\,\mathrm{d}r\right]\Bigg|_s^{s+\Delta s}$$

$$= \frac{\mu_0 q\lambda\beta c}{4\pi}\left(1 + 2\ln\frac{b}{a}\right)$$

Faraday's Law: $\boxed{\oint \mathbf{E}\cdot\mathbf{dl} = -\frac{\partial}{\partial t}\iint \mathbf{B}\cdot\mathbf{dS}}$

$$\implies \quad (E_s - E_w)\Delta s + \frac{q}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\big(\lambda(s+\Delta s) - \lambda(s)\big) = \\ -\Delta s\left(1 + 2\ln\frac{b}{a}\right)\frac{\mu_0 q\beta^* c}{4\pi}\,\frac{\partial\lambda}{\partial t}$$

where β^* is speed of disturbance, maybe not the same as β but very close: $\beta^* \approx \beta$.

$$\implies \qquad (E_s - E_w) = -\frac{q}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\left[\frac{\partial\lambda}{\partial s} + \frac{\beta^*}{c}\frac{\partial\lambda}{\partial t}\right]$$

$$E_s = E_w - \frac{q}{4\pi\epsilon_0}\left(1 + 2\ln\frac{b}{a}\right)\left[\frac{\partial\lambda}{\partial s} + \frac{\beta^*}{c}\frac{\partial\lambda}{\partial t}\right]$$

Since $\dfrac{\partial\lambda}{\partial t} = -\beta^* c\dfrac{\partial\lambda}{\partial s} \approx -\beta c\dfrac{\partial\lambda}{\partial s},$

longitudinal field is

$$\begin{aligned} E_s &= -\frac{qg_0}{4\pi\epsilon_0}\left(1 - \beta^{*2}\right)\frac{\partial\lambda}{\partial s} + E_w \\ &\approx -\frac{qg_0}{4\pi\epsilon_0\gamma^2}\frac{\partial\lambda}{\partial s} + E_w \end{aligned}$$

where $g_0 = 1 + 2\ln\dfrac{b}{a}$

g_0 is geometry factor

Longitudinal space charge for a long bunched or coasting beam given by derivative of the line density and a geometry factor

Longitudinal Examples

1. Perfectly conducting smooth wall, $E_w = 0$,

$\Longrightarrow$ space charge field
$$\boxed{E_s = -\frac{q g_0}{4\pi\epsilon_0\gamma^2}\frac{\partial\lambda}{\partial s}}$$

2. Inductive wall (common in accelerators), inductance $L/2\pi R$ per unit length

$$\Longrightarrow \quad E_w = \frac{L}{2\pi R}\frac{\mathrm{d}I_w}{\mathrm{d}t} \approx q\beta^2c^2\frac{L}{2\pi R}\frac{\partial\lambda}{\partial s} \qquad Z_0 = \frac{1}{\epsilon_0 c} \approx 377\,\Omega$$

$$\Longrightarrow \quad E_s = -q\left[\frac{g_0}{4\pi\epsilon_0\gamma^2} - \frac{\beta^2c^2L}{2\pi R}\right]\frac{\partial\lambda}{\partial s} = -\frac{q\beta c}{2\pi}\left[\frac{g_0}{2\beta}\frac{Z_0}{\gamma^2} - \omega_0 L\right]\frac{\partial\lambda}{\partial s}$$

3. Smooth resistive wall $\Longrightarrow$ *skin effect*; gives an impedance

$$Z_{skin} = (1-i)\frac{\mu_0 R}{b}\sqrt{\frac{\omega}{2\mu\mu_0\sigma}}$$

μ = permeability
σ = conductivity

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Space-Charge and the Hamiltonian

Hamiltonian $$\mathcal{H} = \frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)^2 + \frac{qV}{2\pi}\left[\cos\phi - \cos\phi_s + (\phi - \phi_s)\sin\phi_s\right]$$

For general rf $$\mathcal{H} = \frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)^2 - \frac{q}{2\pi}U(\phi)$$

where $$U(\phi) = \int_{\phi_s}^{\phi}\left[V(\phi) - V(\phi_s)\right]\mathrm{d}\phi = \int_{\phi_s}^{\phi} V(\phi)\,\mathrm{d}\phi - V(\phi_s)(\phi - \phi_s)$$

Space-charge will alter the effective voltage, generally in a non-linear way.

But if the line density $\lambda \propto U$, the self forces ($\sim \frac{\partial\lambda}{\partial s} \propto \frac{\partial\lambda}{\partial\phi}$) caused by space-charge and inductive wall are proportional to the external force, giving a stationary distribution.

Local Elliptic Energy Distribution

Write $W = \dfrac{\Delta \mathcal{E}}{\omega_0}$, so that $\mathcal{H} = \dfrac{h\omega_0^2\eta}{2\beta^2\mathcal{E}}W^2 - \dfrac{q}{2\pi}U(\phi)$

A necessary and sufficient condition for a stationary particle distribution is that the phase-space density $f(W,\phi)$ can be written as a function of the Hamiltonian $f = f(\mathcal{H})$.

The Elliptic Distribution is given by

$$f(W,\phi) = \frac{\mathrm{d}^2 N}{\mathrm{d}W\,\mathrm{d}\phi} = f(\mathcal{H}) = c_1\sqrt{\mathcal{H}_0 - \mathcal{H}},$$

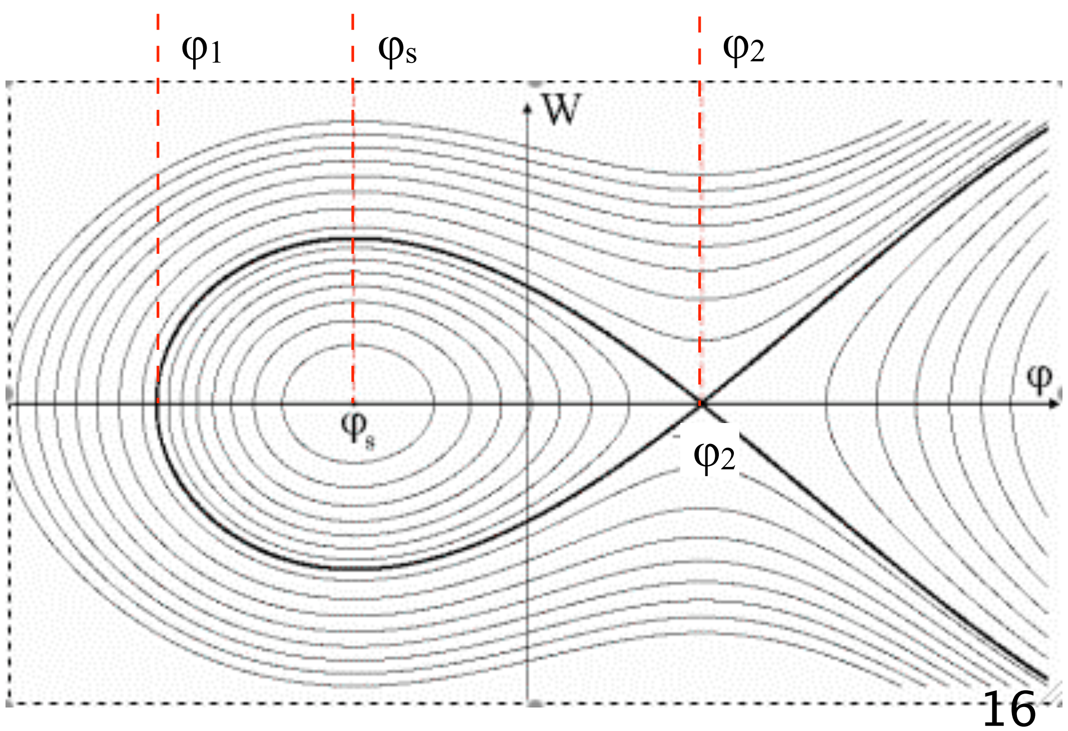
where $\mathcal{H}_0$ is the Hamiltonian of the extreme (boundary) particles.

Bucket extremities given by $W = 0$ or

$$U(\phi_1) = U(\phi_2) = -\frac{2\pi}{q}\mathcal{H}_0$$

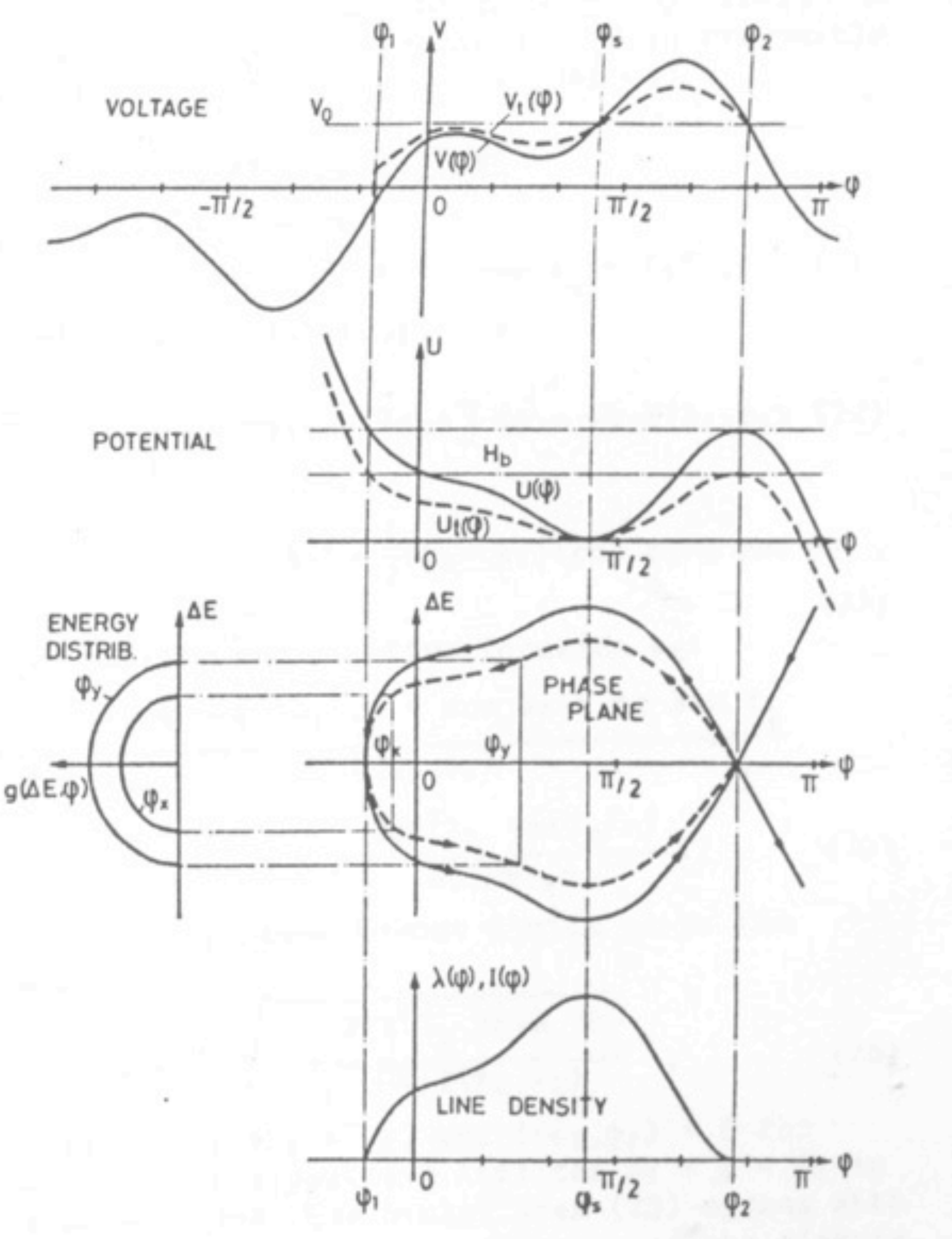
Then

$$\begin{aligned}\lambda(\phi) &= \frac{\mathrm{d}N}{\mathrm{d}\phi} = \int f(W,\phi)\,\mathrm{d}W \\ &\propto \mathcal{H}_0 + \frac{q}{2\pi}U(\phi) \propto U(\phi) - U(\phi_2)\end{aligned}$$



The line density is $\lambda(\phi) = c_2\big[U(\phi) - U(\phi_2)\big]$

and has the *same shape as the potential.*



More precisely:

$$\lambda(\phi) = N_b \frac{U(\phi) - U(\phi_2)}{u(\phi_1, \phi_2)}$$

where N_b is the number of particles in the bunch and

$$u(\phi_1, \phi_2) = \int_{\phi_1}^{\phi_2} \big[U(\phi) - U(\phi_2)\big] \, \mathrm{d}\phi$$

The bunch current is

$$I(\phi) = q \frac{\mathrm{d}N}{\mathrm{d}\phi} \frac{\mathrm{d}\phi}{\mathrm{d}t} = 2\pi h I_b \frac{U(\phi) - U(\phi_2)}{u(\phi_1, \phi_2)}.$$

where $I_b = qN_b\omega_0/2\pi$ is the mean current per bunch.

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Recall that the field in the beam from space-charge and inductance is

$$E_s = -\frac{q\beta c}{2\pi}\left[\frac{g_0}{2\beta}\,\frac{Z_0}{\gamma^2} - \omega_0 L\right]\frac{\partial\lambda}{\partial s}$$

corresponding to a voltage per turn

$$U_s = -q\beta c R\left[\frac{g_0}{2\beta}\,\frac{Z_0}{\gamma^2} - \omega_0 L\right]\frac{\partial\lambda}{\partial s}.$$

At low frequencies (long bunches), this gives a reactive coupling impedance

$$\frac{Z_e}{n} = i\left[\omega_0 L - \frac{g_0 Z_0}{2\beta\gamma^2}\right] = i\omega_0 L_e$$

where $n = \omega/\omega_0$ and L_e is the effective inductance.

So space-charge is equivalent to a negative, energy-dependent wall inductance.

	BNL-AGS	BNL-RHIC	Fermilab Booster	Fermilab MI	KEK-PS
γ_t	8.7	22.5	5.4	20.4	6.8
$Z_{sc}/n\,[\Omega]$	13	1.5	30	2.3	20

For the elliptic distribution, total voltage seen by the beam is

$$V_t(\phi) = \begin{cases} V(\phi) - 2\pi h^2 I_b \,\mathrm{Im}\,\{Z_e/n\}\dfrac{V(\phi) - V(\phi_s)}{u(\phi_1,\phi_2)}, & \phi_1 < \phi < \phi_2 \\ V(\phi) & \text{elsewhere} \end{cases}$$

The induced voltage has the same shape as the applied voltage

1. Below transition $\eta < 0$ and $u(\phi_1, \phi_2) < 0$, so a space-charge dominated beam with $\mathrm{Im}\,\{\frac{Z_e}{n}\} < 0$ leads to a reduced focusing force.
2. Above transition $\eta > 0$, $u(\phi_1, \phi_2) > 0$, so a dominating inductive wall impedance results in reduced focusing, while a space-charge dominated beam sees *enhanced* focusing.

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Voltage seen by a particle changes from $V(\phi) - V(\phi_s)$ to

$$V_t(\phi) - V(\phi_s) = \left(1 - \frac{2\pi h^2 I_b \mathrm{Im}\left\{Z_e/n\right\}}{u(\phi_1, \phi_2)}\right)\left[V(\phi) - V(\phi_s)\right]$$

and bucket area is changed relative to its low intensity value A_0 to

$$A_t = A_0 \sqrt{\frac{V_t(\phi) - V(\phi_s)}{V(\phi) - V(\phi_s)}} = A_0 \sqrt{1 - \frac{2\pi h^2 I_b \mathrm{Im}\left\{Z_e/n\right\}}{u(\phi_1, \phi_2)}}$$

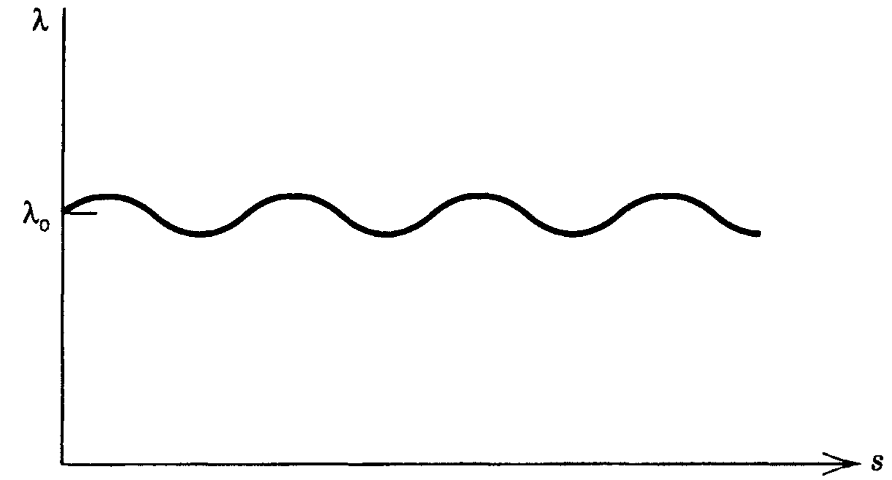
Gives limiting intensity:

$$\boxed{\hat{I}_b = \frac{u(\phi_1, \phi_2)}{2\pi h^2 \mathrm{Im}\left\{Z_e/n\right\}}.}$$

At this intensity, induced voltage cancels the applied voltage, bucket area goes to zero and phase-space density is infinite.

Instability

Space charge field $\boxed{E_s = -\frac{q g_0}{4\pi\epsilon_0\gamma^2}\frac{\partial\lambda}{\partial s}}$



Suppose there is a small disturbance in the line density.

In regions where $\frac{\partial\lambda}{\partial s} > 0$ space charge field is negative.

In regions where $\frac{\partial\lambda}{\partial s} < 0$, the field is positive $\Longrightarrow$

(i) below transition ($\eta < 0$) particles will speed up in regions where $\frac{\partial\lambda}{\partial s} < 0$ and increase their revolution frequency. Hence move towards the trough in the wave.

Similarly particles in regions where $\frac{\partial\lambda}{\partial s} > 0$ will slow down and again fill the trough.

$\Longrightarrow$ the disturbance is damped.

(ii) Above transition, reverse holds and the disturbance will grow.

INSTABILITY

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Negative Mass (microwave) Instability

Assume a (wakefield) disturbance to the distribution of the form

$\lambda = \lambda_0 + \lambda_1 e^{i(\Omega t - n\theta)}$ $\quad$ ($\theta = s/R =$ orbiting angle, $n =$ mode number).

Satisfying Vlasov's equation gives: $\left(\frac{\Omega}{n\omega}\right)^2 = -i\frac{qI_0 Z_e/n}{2\pi\beta^2\mathcal{E}}\eta$

Stability requires real Ω $\iff$ $-i\frac{Z_e}{n}\eta = \left[\omega_0 L - \frac{g_0 Z_0}{2\beta\gamma^2}\right]\eta > 0$

For space-charge, Z_e/n is capacitive (negative inductive), so require $\eta < 0$ for stability. There is an effective frequency shift without producing damaging collective effects. Above transition, $\eta > 0$, there is a space-charge driven instability, known as the *negative mass or microwave instability*

$$c.f.\quad \mathcal{H} \propto -\frac{1}{2}\frac{h\eta\omega_0^2}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right)^2 + \frac{q}{2\pi}U(\phi)$$

	Z_e/n	capacitive	inductive	resistive
Below transition	$\eta < 0$	stable	unstable	unstable
Above transition	$\eta > 0$	unstable	stable	unstable

Theory shows that the microwave instability can be avoided for reasonably long bunches (e.g. SNS) provided

$$I_b \lesssim 0.4\hat{I}_b.$$

The induced (space-charge) voltage should never exceed 40% of the applied voltage.
The corresponding reduction in bucket area is 23%.

Sinusoidal Voltage

Sinusoidal voltage, $V(\phi) = V_0 \sin\phi$ gives

$$U(\phi) = V_0\big[\cos\phi_s - \cos\phi - (\phi - \phi_s)\sin\phi_s\big]$$

$$\Longrightarrow \qquad u(\phi_1,\phi_2) = \int_{\phi_1}^{\phi_2} \big[U(\phi) - U(\phi_2)\big]\,\mathrm{d}\phi = -V_0 f(\phi_1,\phi_2)$$

where $f(\phi_1,\phi_2) = \sin\phi_2 - \sin\phi_1 - \dfrac{1}{2}(\phi_2 - \phi_1)(\cos\phi_1 + \cos\phi_2)$.

Note: uses $U(\phi_1) = U(\phi_2)$ to eliminate ϕ_s.

Line density is $\quad \lambda(\phi) = \dfrac{N_b}{f(\phi_1,\phi_2)}\big[\cos\phi + \phi\sin\phi_s - \cos\phi_2 - \phi_2\sin\phi_s\big]$.

For short bunches, $\phi_l = \phi_2 - \phi_1$ and $f(\phi_1,\phi_2) \approx \dfrac{1}{12}\phi_l^3\cos\phi_s$. Then

$$\left.\begin{aligned} \text{potential:} \qquad U(\phi) &\approx \frac{1}{2}(\phi - \phi_s)^2 V_0\cos\phi_s \\ \text{line density:} \qquad \lambda(\phi) &\approx \frac{6N_b}{\phi_l^3}(\phi_2 - \phi)(\phi - \phi_1) \end{aligned}\right\}$$

quadratic, resulting in linear fields.

Longitudinal Envelope Equation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{h\omega_0^2\eta}{\beta^2\mathcal{E}}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = \frac{qV_0}{2\pi}\left(\sin\phi - \sin\phi_s\right) + \frac{q}{2\pi}U_s(\phi)$$

$$U_s(\phi) = -\frac{qRg_0}{2\epsilon_0\gamma^2}\frac{\partial\lambda}{\partial s} = \frac{qh^2g_0}{2\epsilon_0\gamma^2 R}\frac{\partial\lambda}{\partial\phi} \approx -\frac{qh^2g_0}{2\epsilon_0\gamma^2 R} \times \frac{12N_b}{\phi_l^3}(\phi - \phi_s)$$

In terms of distance from the synchronous particle,

$$\frac{\mathrm{d}z}{\mathrm{d}t} = R\frac{\mathrm{d}\Delta\theta}{\mathrm{d}t} = -\frac{R}{h}\frac{\mathrm{d}\Delta\phi}{\mathrm{d}t} = -\frac{R\omega_0^2\eta}{\beta^2\mathcal{E}}\frac{\Delta\mathcal{E}}{\omega_0}$$

$$\Longrightarrow \quad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{R\omega_0^2\eta}{\beta^2\mathcal{E}}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Delta\mathcal{E}}{\omega_0}\right) = -\frac{qR\omega_0^2\eta}{2\pi\beta^2\mathcal{E}}\left\{V_0\cos\phi_s - \frac{6qN_bh^2g_0}{\epsilon_0 R\gamma^2\phi_l^3}\right\}(\phi - \phi_s)$$

$$\Longrightarrow \quad \boxed{\frac{\mathrm{d}^2 z}{\mathrm{d}s^2} + k_z z - \frac{\mathcal{K}_l}{z_m^3}z = 0}$$

where $k_z = -\dfrac{qh\eta V_0}{2\pi R^2\beta^2\mathcal{E}}\cos\phi_s$

and $\mathcal{K}_l = -\dfrac{3}{2}\dfrac{r_0 g_0 N_b\eta}{\beta^2\gamma^3}$

Bunch has an approximately parabolic line density between $[-z_m, z_m]$

Corresponding envelope equation is $$\boxed{z_m'' + k_z z_m - \frac{\mathcal{K}_l}{z_m^2} - \frac{\epsilon_{zz'}^2}{z_m^3} = 0}$$

$\epsilon_{zz'}$ is the total (unnormalised) emittance of the bunch in the moving frame.

Note:

(i) Longitudinal equations for a straight channel can be obtained by setting $\alpha_p = 0$, $\eta = -1/\gamma^2$

(ii) Below transition, $\eta < 0$, so for $\phi_s < \frac{1}{2}\pi$, $k_z > 0$ and $\mathcal{K}_l > 0$; analogous to a normal linear accelerator.

(iii) Above transition, $\eta > 0$, so longitudinal focusing requires a change $\phi_s \rightarrow \pi - \phi_s$. Now the perveance $\mathcal{K}_l < 0$ and space-charge is focusing. Space-charge electric field increases the energy of a particle at the front of the bunch. This increases orbit radius, which slows down angular motion and hence reduces distance z from bunch centre.

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(iv) Synchrotron tune is $Q_z^2 = Q_{z0}^2 - \frac{R^2 \mathcal{K}_l}{z_m^2} = Q_{z0}^2 + \frac{3}{2} \frac{r_0 g_0 N_b R^2 \eta}{\beta^2 \gamma^3 z_m^2}$.

For small differences, this gives

$$\Delta Q_z = \frac{3}{4} \frac{r_0 g_0 N_b R^2 \eta}{\beta^2 \gamma^3 z_m^2 Q_{z0}}$$

Longitudinal tune shift is negative below transition and positive above transition.

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Acceleration in a Synchrotron

Magnets ramped so that magnetic field $B(t)$ matches the acceleration from the cavities for synchronous particle. Suppose $B(t) = B_0 - B_1 \cos(2\pi f t)$.

"Rigidity" $\quad \frac{p}{q} = B\rho \quad \Longrightarrow \quad \frac{\mathrm{d}p}{\mathrm{d}t} = q\rho\dot{B}$

$$\mathcal{E}^2/c^2 = p^2 + m_0^2 c^2 \quad \Longrightarrow \quad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = \frac{pc^2}{\mathcal{E}}\frac{\mathrm{d}p}{\mathrm{d}t}$$

Energy gain per revolution $\quad V(\phi_s) = \frac{2\pi R}{\beta c}\frac{1}{q}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = 2\pi R\rho\dot{B}$

Assume $\quad V(\phi, t) = V_0(t)\sin\phi, \quad$ with V_0 modulated in time.

$$\Longrightarrow \quad \sin\phi_s = \frac{2\pi R}{V_0(t)}\rho\dot{B} = \frac{2\pi R\rho}{V_0(t)} \times 2\pi f B_1 \sin(2\pi f t)$$

Example (ISIS): $R = 26\,\mathrm{m},\ \rho = 7\,\mathrm{m},\ f = 50\,\mathrm{Hz},\ B_1 = 0.2604\,\mathrm{T}$

$$\Longrightarrow \quad \sin\phi_s \approx \frac{93.5\,\mathrm{kV}}{V_0(t)}\sin(2\pi f t)$$

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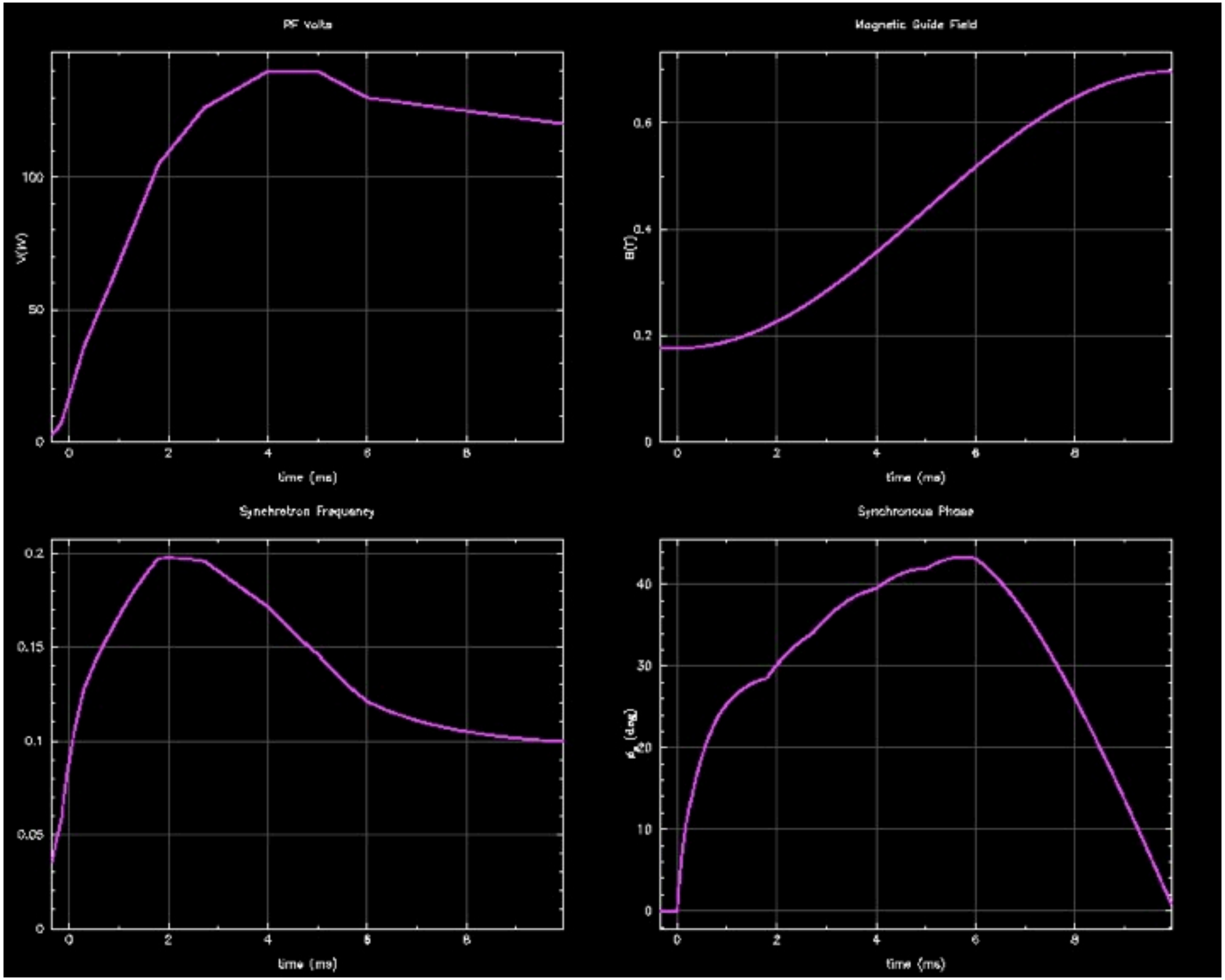
$$\sin\phi_s \approx \frac{93.5\,\mathrm{kV}}{V_0(t)}\sin(2\pi f t) \qquad 0 \le t \le 10\,\mathrm{ms}$$

Peak voltage $V_0(t)$ has to be modulated so that RHS is less than 1.



How to Overcome Space-Charge?

Dual harmonic rf scheme:

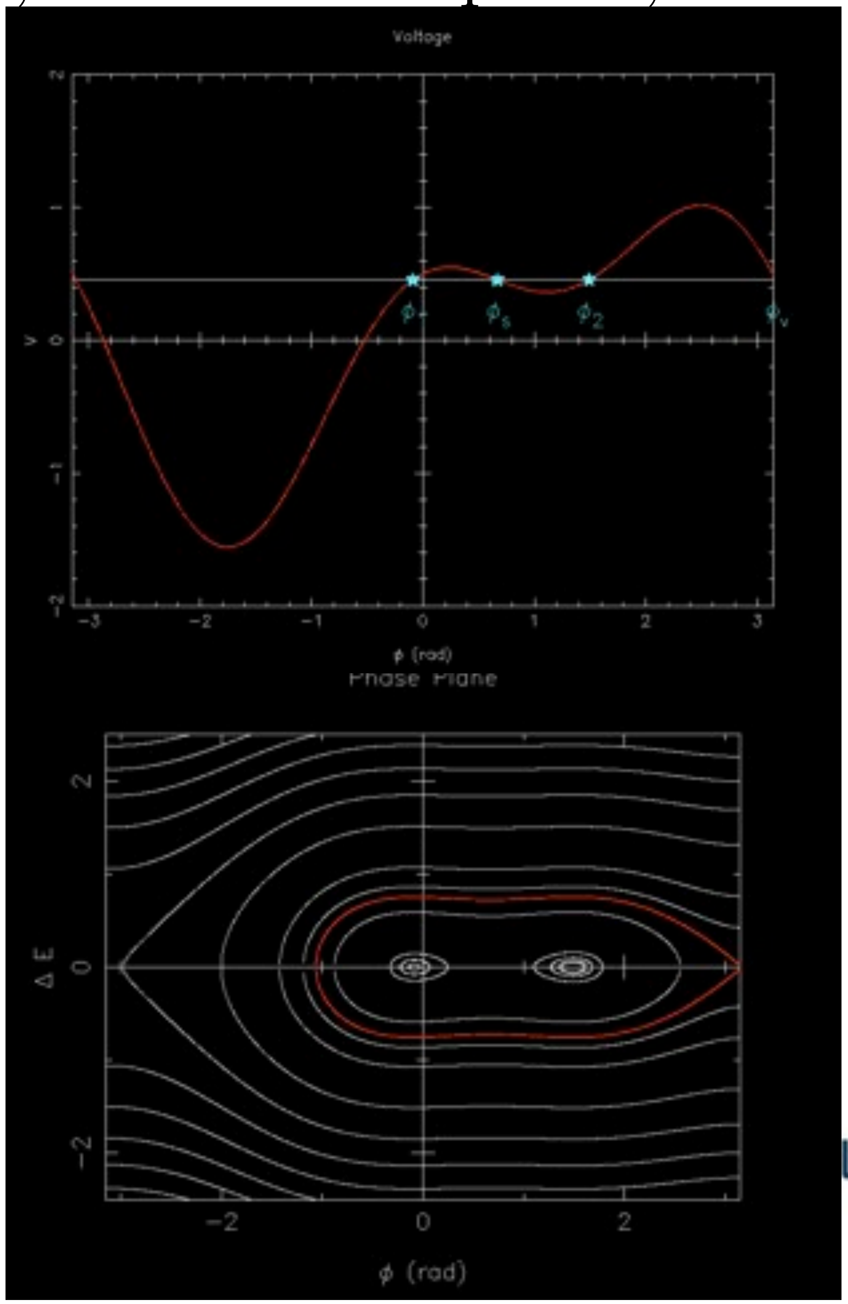
$$V(\phi,t) = V_0(t)\left[\sin\phi - \delta\sin(2\phi+\theta)\right]$$

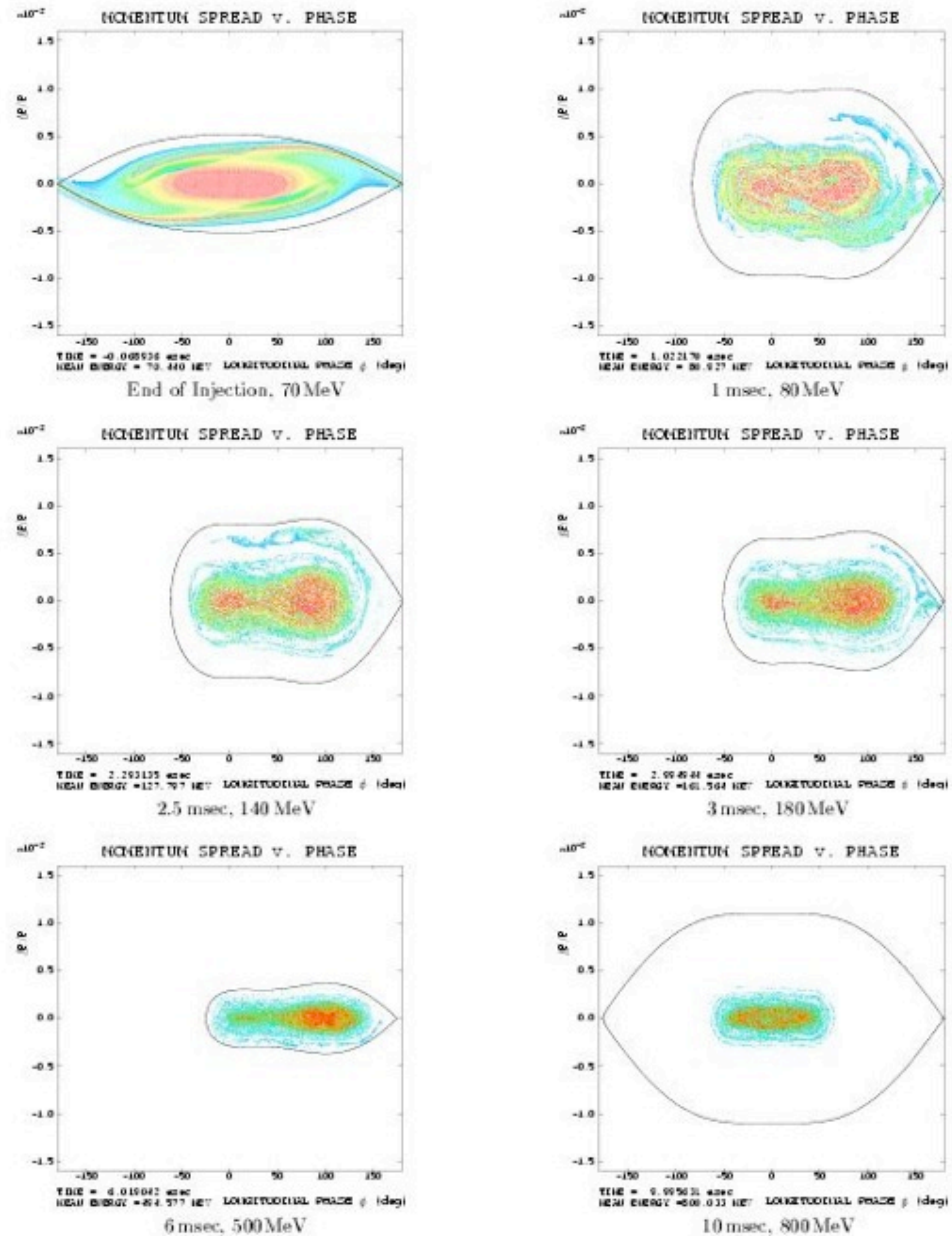
where δ, the ratio of the $h:2h$ voltages, and θ, the relative phase, are functions of time.

$$\sin\phi_s - \delta\sin(2\phi_s+\theta) = \frac{2\pi R\rho}{V_0(t)}\dot{B}$$

Double oscillation centre, causes bunches to lengthen, flattening the line-density, and reducing the peak current; and gives

- improved bunching factor $B_f = \dfrac{\bar{I}}{\hat{I}}$.
- reduced transverse space-charge forces





Longitudinal phase space plots for acceleration of 3×10^{13} protons in the ISIS synchrotron from 70 MeV to 800 MeV with dual harmonic RF system.

Simulation of the ISIS acceleration cycle, showing the formation of the double oscillation centre and merging to give the final beam on target.