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Beam Dynamics with Space-Charge

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Lecture Plan (1)

- Review of particle equations of motion in 2D without space-charge
  - Courant-Snyder parameters, envelope equation
- Examples of space-charge
- Particle and envelope equations with linear space-charge
  - –Space-charge (Laslett) Tune shift
  - –Image effects
- Envelope oscillations, resonances

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Lecture Plan (2)

- General particle equations under space-charge
- Non-linear beams
  - rms beam sizes, rms emittance
  - rms envelope equations
  - evolution of rms emittance
- Examples of 2D distributions
  - KV, waterbag, Gaussian
  - concept of stationary distributions
- Beam halo
  - causes, measurement (kurtosis)

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Lecture Plan (3)

- Longitudinal space-charge
- Self-consistent distributions
  - Hofmann-Pederson model
- Microwave instability
- Acceleration cycle in a synchrotron
- Bunch compression
- Long and short bunches

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Reading

- E.J.N. Wilson: *Introduction to Accelerators*
- S.Y. Lee: *Accelerator Physics*
- M. Reiser: *Theory and Design of Charged Particle Beams*
- D. Edwards & M. Syphers: *An Introduction to the Physics of High Energy Accelerators*
- M. Conte & W. MacKay: *An Introduction to the Physics of Particle Accelerators*
- R. Dilao & R. Alves-Pires: *Nonlinear Dynamics in Particle Accelerators*
- *R. Davidson & Hong Qin: Physics of Intense Charged Particle Beams in High Energy Accelerators*

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Notation and Basic Formulae

Velocity of light	$c = 2.99792458 \times 10^8 \text{ m/sec}$
Relative velocity	$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}, \quad \mathbf{v} = \boldsymbol{\beta} c$
Relativistic gamma	$\gamma = \frac{1}{\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}} = \frac{1}{\sqrt{1 - \boldsymbol{\beta}^2}}$
Rest mass	m_0
Relativistic mass	$m = m_0 \gamma$
Momentum	$\mathbf{p} = m\mathbf{v} = m_0 \gamma \mathbf{v} = m_0 \gamma \boldsymbol{\beta} c$
Energy	$\mathcal{E} = mc^2 = m_0 \gamma c^2$
Kinetic energy	$T = \mathcal{E} - m_0 c^2 = m_0 (\gamma - 1) c^2$

Note: $$\frac{\mathcal{E}^2}{c^2} = \mathbf{p}^2 + m_0^2 c^2$$

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Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

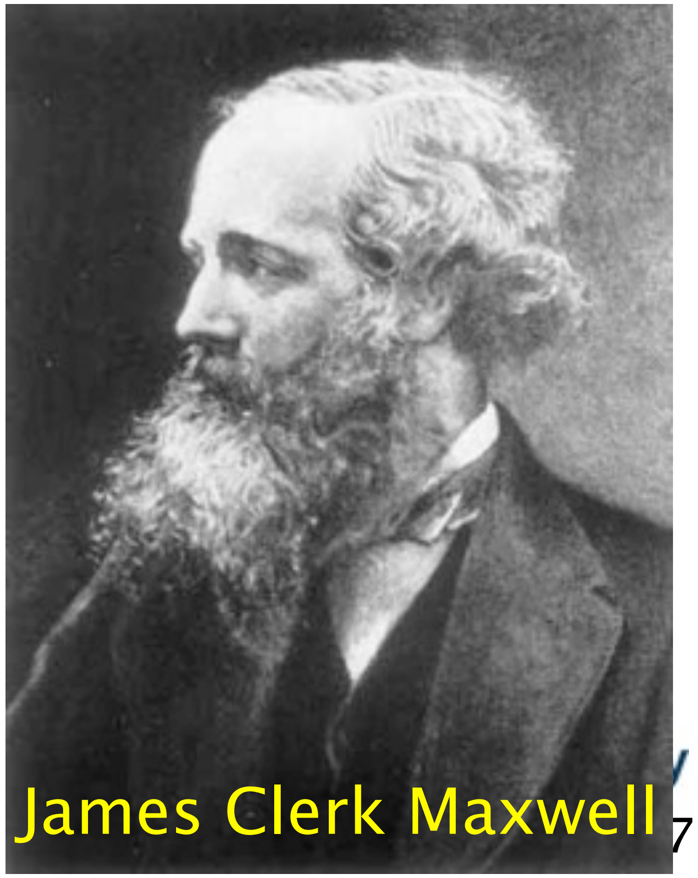
$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$\mathbf{E}$	Electric field
$\mathbf{B}$	Magnetic flux density
ρ	Charge density
$\mathbf{j}$	Current density
μ_0	Permeability of free space, $4\pi \times 10^{-7}$
ϵ_0	Permittivity of free space, 8.854×10^{-12}

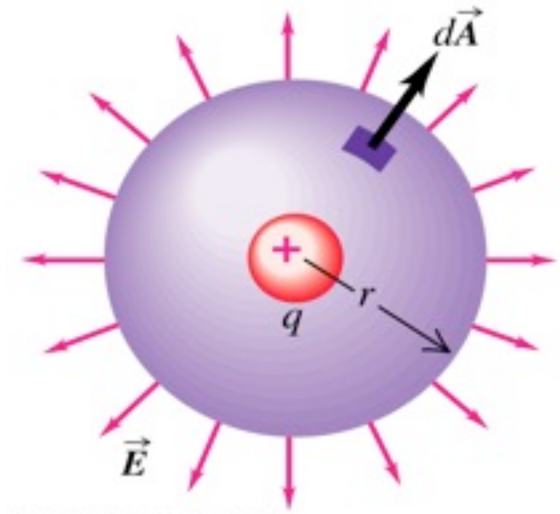
$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$



James Clerk Maxwell

Gauss' Flux Theorem

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$



Equivalent to Gauss' Flux Theorem:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\iff \quad \iiint_V \boldsymbol{\nabla} \cdot \mathbf{E}\, \mathrm{d}V = \iint_S \mathbf{E} \cdot \, \mathrm{d}\mathbf{S} = \frac{1}{\epsilon_0} \iiint \rho \, \mathrm{d}V = \frac{Q}{\epsilon_0}$$

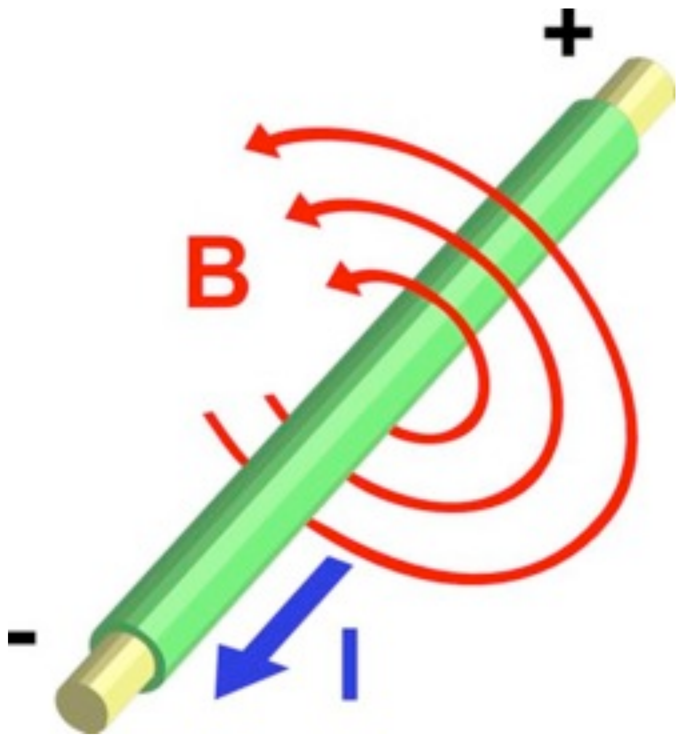
The flux of electric field out of a closed region is proportional to the total electric charge Q enclosed within the surface.

Ampère's Circuital Law

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{j}$$

$$\iff \quad \iint_S \nabla \wedge \mathbf{B} \cdot \mathrm{d}\mathbf{S} = \oint \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 \iint_S \mathbf{j} \cdot \mathrm{d}\mathbf{S} = \mu_0 I$$



For a straight line current:

$$B_\theta = \frac{\mu_0 I}{2\pi r}$$

Review of Simple Particle Dynamics

Equation of motion in electromagnetic fields:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q\Big(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}\Big)$$

Lorentz Force

produces acceleration

produces bending

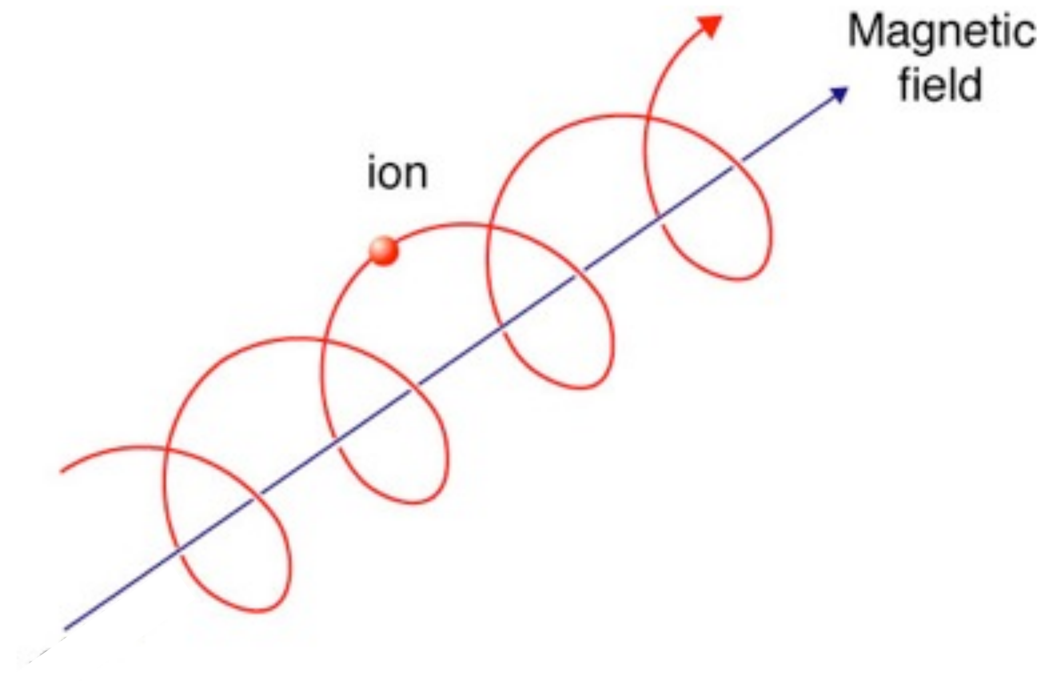
Total energy $\mathcal{E} = mc^2$ and $\mathcal{E}^2 = \mathbf{p}^2c^2 + m_0^2c^4$

$$\Longrightarrow \qquad \mathcal{E}\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = c^2\mathbf{p}\cdot\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = qc^2\mathbf{p}\cdot\mathbf{E}$$

$$\Longrightarrow \qquad \frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = q\mathbf{v}\cdot\mathbf{E}$$

Energy change only from Electric fields

Motion in Constant Magnetic Field



$$\frac{\mathrm{d}}{\mathrm{d}t}(m_0\gamma\mathbf{v}) = q\mathbf{v}\wedge\mathbf{B}$$

$$\Longrightarrow \qquad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{m_0\gamma}\mathbf{v}\wedge\mathbf{B}$$

$$\Longrightarrow \qquad \frac{v_\perp^2}{\rho} = \frac{q}{m_0\gamma}v_\perp B$$

$$\Longrightarrow \qquad \text{circular motion with radius} \quad \rho = \frac{m_0\gamma v_\perp}{qB}$$

$$\Longrightarrow \qquad \text{at an angular frequency} \quad \omega = \frac{v_\perp}{\rho} = \frac{qB}{m_0\gamma} = \frac{qB}{m}$$

Constant magnetic field gives uniform spiral about B with constant energy.

$$B\rho = \frac{m_0\gamma v}{q} = \frac{p}{q}$$

Magnetic Rigidity

External Forces

- Maxwell's time-independent equations, no sources:

$$\nabla \wedge \vec{B} = 0 \quad \Longrightarrow \quad \exists \phi \text{ such that } \vec{B} = \nabla \phi \quad \text{Scalar magnetic potential}$$

$$\nabla \cdot \vec{B} = 0 \quad \Longrightarrow \quad \nabla^2 \phi = 0, \quad \text{Laplace's equation}$$

- Simplest solutions, with $z = x + iy \quad (i = \sqrt{-1})$ are

$$\boxed{\phi = K z^n, \;\; K \text{ constant}}$$

$$\begin{array}{lll} n = 1 & \phi \propto x + iy & \text{dipole} \\ n = 2 & \phi \propto (x^2 - y^2) + 2ixy & \text{quadrupole} \\ n = 3 & \phi \propto x(x^2 - 3y^2) + iy(3x^2 - y^2) & \text{sextupole} \end{array}$$

- Then $\boxed{\vec{B} = nK(1, i, 0) z^{n-1} \qquad \text{(real part understood)}}$

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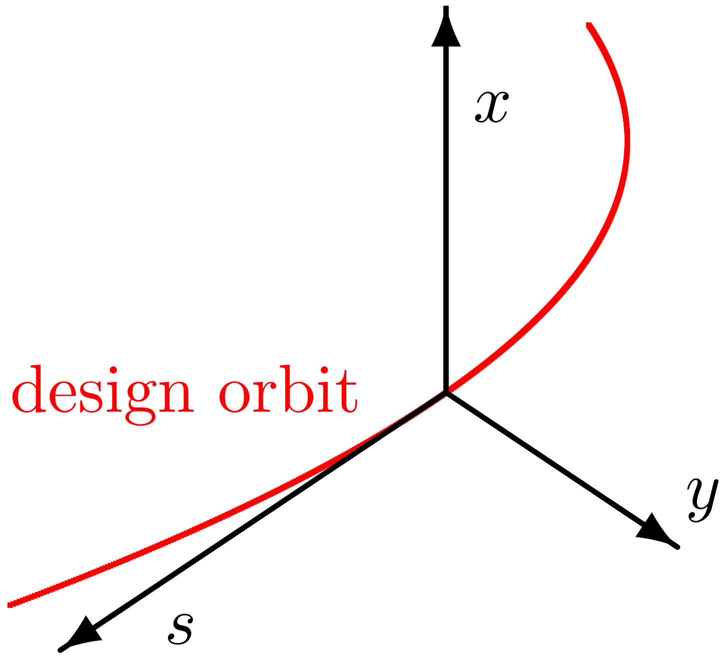
Equations of Motion



No bends

$$m_0 \frac{\mathrm{d}}{\mathrm{d}t}(\gamma \vec{v}) = q\vec{v} \wedge \vec{B} = q \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{s} \end{bmatrix} \wedge Kn \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} z^{n-1}$$

$$\Longrightarrow \qquad m_0 \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \gamma\dot{x} \\ \gamma\dot{y} \\ \gamma\dot{s} \end{bmatrix} = Knq \begin{bmatrix} -i\dot{s} \\ \dot{s} \\ i\dot{x} - \dot{y} \end{bmatrix} z^{n-1}$$

Write $x' = \frac{\dot{x}}{\dot{s}}, y' = \frac{\dot{y}}{\dot{s}}$, and note in an accelerator $\dot{x}, \dot{y} \ll \dot{s}$

so that $\mathbf{v} = (x', y', 1)\dot{s}, \quad x', y' \ll 1$ and $\frac{\mathrm{d}}{\mathrm{d}t} = \dot{s}\frac{\mathrm{d}}{\mathrm{d}s}$

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Paraxial Equations

From above, $\frac{\mathrm{d}}{\mathrm{d}t}(\gamma \dot{s}) \sim 0$

$$\gamma = \left(1 - \frac{\dot{x}^2 + \dot{y}^2 + \dot{s}^2}{c^2}\right)^{-\frac{1}{2}} \approx \left(1 - \beta^2(x'^2 + y'^2 + 1)\right)^{-\frac{1}{2}}$$
$$= (1 - \beta^2)^{-\frac{1}{2}} + \text{second order terms}$$

Ignoring second and higher velocity terms, equations of motion are

$$m_0 \gamma \beta^2 c^2 \begin{bmatrix} x'' \\ y'' \end{bmatrix} = K n q \beta c \begin{bmatrix} -i \\ 1 \end{bmatrix} z^{n-1}$$

$n = 2$ (K imag)	quadrupole	$\begin{bmatrix} x'' + kx \\ y'' - ky \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$n = 2$ (K real)	skew quadrupole	$\begin{bmatrix} x'' + ky \\ y'' + kx \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$n = 3$ (K imag)	sextupole	$\begin{bmatrix} x'' + k(x^2 - y^2) \\ y'' - 2kxy \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$n = 3$ (K real)	skew sextupole	$\begin{bmatrix} x'' + 2kxy \\ y'' + k(x^2 - y^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$k = q \times \frac{\text{Field strength}}{m_0 \gamma \beta c}$$
$$= \text{Field strength}/B\rho$$

Equations of Motion in Dipoles

Need to take into account the change in direction of the axes $\implies \dot{s} \to \dot{s}(1+\kappa x)$ where $\kappa = \dfrac{1}{\rho}$ is the curvature of the design orbit.

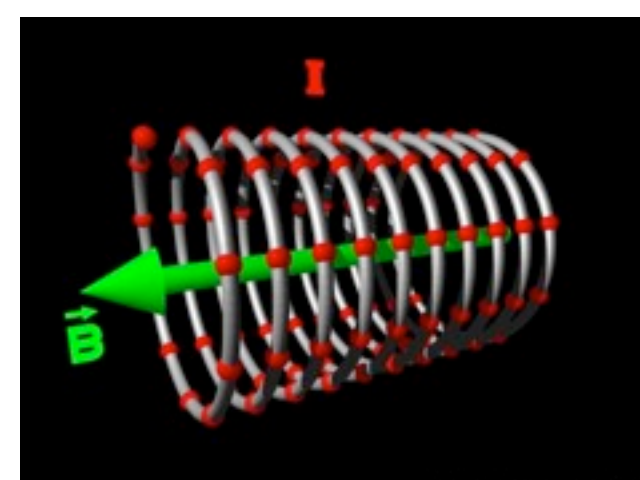
Equations become

$$\begin{bmatrix} x'' + \kappa^2 x \\ y'' \end{bmatrix} = \begin{bmatrix} x'' + \dfrac{1}{\rho^2} x \\ y'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So dipole acts like a focusing quadrupole in the plane of the bend.

Also note solenoids:

$$\begin{bmatrix} x'' + 2ky' + k'y \\ y'' - 2kx' - k'x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



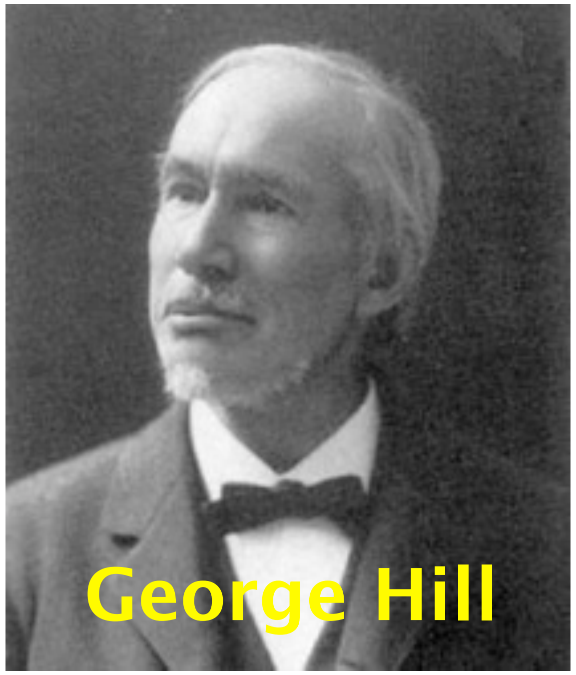
Mathieu-Hill Equations

Paraxial equation of motion in periodic systems:

$$\begin{aligned} x''(s) + k_x(s)x &= 0 \\ y''(s) + k_y(s)y &= 0 \end{aligned}$$

where s is distance along beam axis

$k_x(s)$, $k_y(s)$ periodic focusing functions, $k(s+L) = k(s)$



Floquet's Theorem confirms two independent solutions:

$$u = w(s)\mathrm{e}^{i\psi(s)}, \quad v = w(s)\mathrm{e}^{-i\psi(s)}$$

The Wronskian is $W(u,v) = uv' - vu' = -2iw^2\psi' = C$, a constant

$$\text{Choose } C = -2i \quad \Longrightarrow \quad \frac{\mathrm{d}\psi}{\mathrm{d}s} = \psi' = \frac{1}{w^2}.$$

Then, substitute u or v into Mathieu-Hill equation:

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$$u' = (w' + iw\psi')\mathrm{e}^{i\psi} = (w' + i/w)\mathrm{e}^{i\psi},$$

$$u'' = (w'' - iw'/w^2 + iw'\psi' - \psi'/w)\mathrm{e}^{i\psi} = (w'' - 1/w^3)\mathrm{e}^{i\psi}$$

$$u'' + ku = 0 \quad \Longrightarrow \quad \boxed{w'' + kw - \frac{1}{w^3} = 0}$$

Any solution of Matthieu-Hill is a linear combination of u, v,

$$\text{so set} \qquad x = A\,w(s)\,\cos(\psi(s) + \phi).$$

$$\frac{x}{w} = A\cos(\psi + \phi), \qquad \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{x}{w}\right) = -\frac{A}{w^2}\sin(\psi + \phi)$$

$$\Longrightarrow \qquad \frac{x^2}{w^2} + \Big(wx' - w'x\Big)^2 = A^2.$$

$$\text{Or:} \qquad \boxed{\hat\gamma x^2 + 2\hat\alpha x x' + \hat\beta x'^2 = A^2}$$

$$\text{where} \quad \hat\beta = w^2, \quad \hat\alpha = -ww' = -\frac{1}{2}\hat\beta', \quad \hat\gamma = \frac{1}{w^2} + w'^2 = \frac{1 + \hat\alpha^2}{\hat\beta}$$

Phase-Space Ellipse; Emittance

$$\hat{\gamma}(s)x^2 + 2\hat{\alpha}(s)xx' + \hat{\beta}(s)x'^2 = A^2$$

Area of ellipse is $\pi A^2(\hat{\beta}\hat{\gamma} - \hat{\alpha}^2) = \pi A^2$

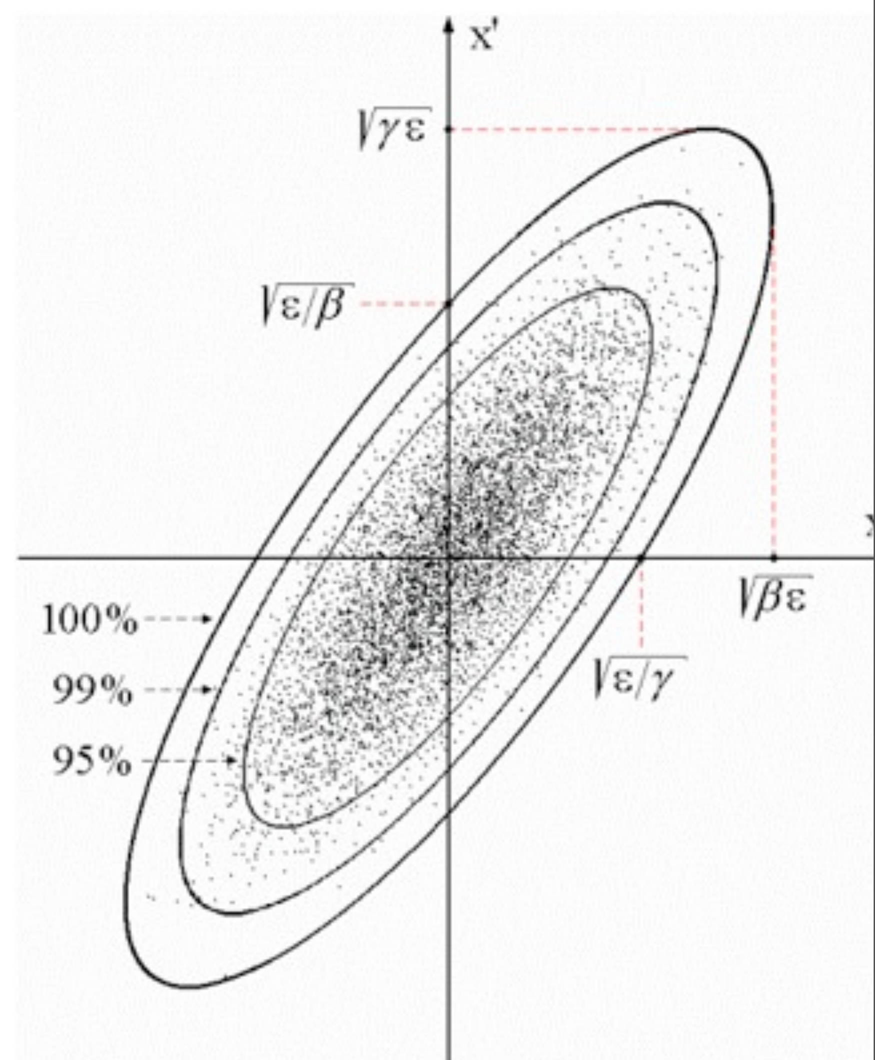
Area of largest ellipse for **all particles in beam** is denoted $\pi\epsilon$

$\hat{\gamma}x^2 + 2\hat{\alpha}xx' + \hat{\beta}x'^2 \leq \epsilon$ is beam ellipse in x-x' phase space and ϵ is called the **beam emittance**

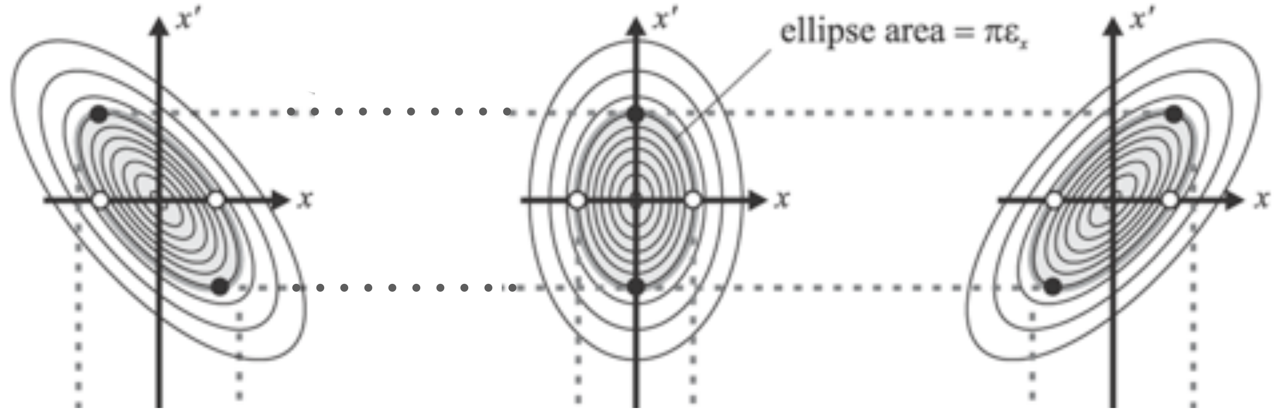
$\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ are Courant-Snyder parameters

Beam size (half-width) is $a(s) = \sqrt{\epsilon\hat{\beta}(s)}$

Phase advance $\psi = \int \psi' \, \mathrm{d}s = \int \frac{\mathrm{d}s}{w^2} = \int \frac{\mathrm{d}s}{\hat{\beta}}$



- ϵ is a constant of the motion, independent of s
- $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ determine the shape of the ellipse in x-x phase space
- For a ring $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are periodic functions, so that the ellipse rotates and shears with position s, while its area $\pi\epsilon$ is conserved (no acceleration)



- Phase advance around a ring gives the tune (number of oscillations per revolution) $Q = \dfrac{1}{2\pi} \oint \dfrac{ds}{\hat{\beta}(s)}$
- Beam envelope given by maximum value of x:

$$a = A_{max} w(s) = \sqrt{\epsilon}\, w(s)$$

Equation for w then gives the *envelope equation*:

$$w'' + kw - \frac{1}{w^3} = 0 \quad \Longrightarrow \quad \boxed{a'' + ka - \frac{\epsilon^2}{a^3} = 0}$$

Space-Charge: simple idea

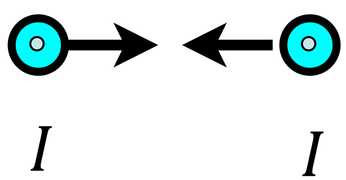
2D point charges q experience repulsive electrostatic force of magnitude

$$F_e = \frac{q^2}{2\pi\epsilon_0 r}.$$

Coulomb repulsion

Particles moving with speed v equivalent to two current wires $I = qv$.

Magnetostatic force between two current wires is attractive of magnitude



$$F_m = \frac{\mu_0 I^2}{2\pi r} = \frac{\mu_o q^2 v^2}{2\pi r} = \frac{v^2}{c^2} F_e.$$

Magnetic attraction

Combined force is a repulsive self-field $\left(1 - \dfrac{v^2}{c^2}\right) F_e.$

For electrons travelling at or close to c, space-charge forces can be negligible.
For proton or ion machines, where $\dfrac{v}{c} = \beta \sim 0.5$, effects are important.
Note: other factors come in, e.g. intensity.

Relativistic Transformation of Fields

Consider a single particle, charge q, moving with velocity $\mathbf{v}$.

In rest-frame, electrostatic field $\mathbf{E_0}$, and $\mathbf{B_0} = 0$.

In lab-frame, transformation equations are

$$\mathbf{E}_\perp = \gamma\Big(\mathbf{E_0}_\perp - \mathbf{v}\times\mathbf{B_0}\Big) \qquad \mathbf{E}_\parallel = \mathbf{E_0}_\parallel$$

$$\mathbf{B}_\perp = \gamma\Big(\mathbf{B_0}_\perp + \frac{1}{c^2}\mathbf{v}\times\mathbf{E_0}\Big) \qquad \mathbf{B}_\parallel = \mathbf{B_0}_\parallel$$

Transverse Lorentz force is

$$\begin{aligned}
\mathbf{F}_\perp &= q\Big(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}\Big) = \gamma q\Big(\mathbf{E_0}_\perp + \frac{1}{c^2}\mathbf{v}\times(\mathbf{v}\times\mathbf{E_0})\Big) \\
&= \gamma q\Big(1 - \frac{\mathbf{v}\cdot\mathbf{v}}{c^2}\Big)\mathbf{E_0}_\perp \\
&= q\frac{1}{\gamma^2}\mathbf{E}_\perp
\end{aligned}$$

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Illustration of Space-Charge; Tune Shift

Consider a 2D axisymmetric beam with charge density $\rho(r) = qn(r)$.

Electric field is radial and inside the beam is given by Gauss' Flux Theorem.
$\{\text{Flux of } \mathbf{E} \text{ through circle of radius } r\} = \dfrac{1}{\epsilon_0} \times \{\text{charge enclosed}\}$.

$$\Longrightarrow \quad 2\pi r E_r(r) = \frac{1}{\epsilon_0}\int \rho \, \mathrm{d}V = \frac{1}{\epsilon_0}\int_0^r 2\pi r \rho(r) \, \mathrm{d}r$$

$$\Longrightarrow \quad E_r(r) = \frac{q}{\epsilon_0}\frac{1}{r}\int_0^r r n(r) \, \mathrm{d}r$$

Magnetic field is angular, from Ampère's Law $\int \mathbf{B} \cdot \mathbf{dl} = \mu_0 \times \{\text{current flowing through loop}\}$

$$\Longrightarrow \quad 2\pi r B_\theta = \mu_0 \int_0^r \beta c \rho(r) 2\pi r \, \mathrm{d}r$$

$$\Longrightarrow \quad B_\theta = \frac{q\beta}{c\epsilon_0}\frac{1}{r}\int_0^r r n(r) \, \mathrm{d}r$$

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Space charge force on a particle is

$$F(r) = q\Big(E_r - \beta c B_\theta\Big) = \frac{q^2}{\epsilon_0}\frac{1}{r}(1-\beta^2)\int_0^r rn(r)\,\mathrm{d}r$$

Consider a *Gaussian distribution* $\quad n(r) = A\exp\left(-\frac{r^2}{2\sigma^2}\right)$

where $A = \dfrac{N}{2\pi\sigma^2}$ and N is the number of particles per unit length.

Then $\qquad F(r) = \dfrac{Nq^2}{2\pi\epsilon_0\gamma^2}\dfrac{1}{r}\left(1-\exp\left(-\dfrac{r^2}{2\sigma^2}\right)\right)$

For betatron oscillations at angular frequency w, equation of particle motion is

$$\frac{\mathrm{d}^2 r}{\mathrm{d}t^2} + \omega^2 r = \frac{F(r)}{m_0\gamma} = \frac{Nq^2}{2\pi\epsilon_0 m_0\gamma^3}\frac{1}{r}\left(1-\exp\left(-\frac{r^2}{2\sigma^2}\right)\right)$$

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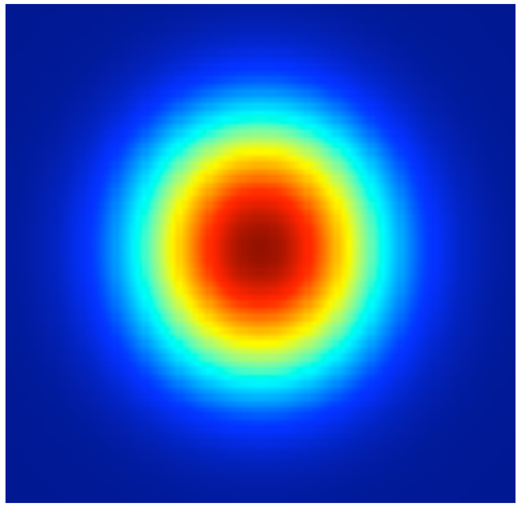
Consider the Gaussian beam transported round a ring of mean radius R and let ϕ be the azimuthal angle round the ring.

Then $\beta c\,\mathrm{d}t = R\,\mathrm{d}\phi$ and the equation of motion becomes



$$\begin{aligned}\frac{\mathrm{d}^2 r}{\mathrm{d}\phi^2} + Q^2 r &= \frac{Nq^2R^2}{2\pi\epsilon_0 m_0 \beta^2\gamma^3 c^2}\frac{1}{r}\left(1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right) \\ &= \frac{2Nr_0R^2}{\beta^2\gamma^3}\frac{1}{r}\left(1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right),\end{aligned}$$

where $r_0 = \dfrac{q^2}{4\pi\epsilon_0 m_0 c^2}$ is the *classical radius* and Q is the *tune*.

Now $$\begin{aligned}\frac{1}{r}\left\{1 - \exp\left(-\frac{r^2}{2\sigma^2}\right)\right\} &= \frac{1}{r}\left\{1 - \left(1 - \frac{r^2}{2\sigma^2} + \ldots\right)\right\} \\ &= \frac{r}{2\sigma^2} + \ldots\end{aligned}$$

Equation of motion becomes: $$\frac{\mathrm{d}^2 r}{\mathrm{d}\phi^2} + \left(Q^2 - \frac{N r_0 R^2}{\sigma^2 \beta^2 \gamma^3}\right) r = 0$$

Equivalent to $Q \to Q + \Delta Q$, where

$$\begin{aligned} Q^2 - \frac{N r_0 R^2}{\sigma^2 \beta^2 \gamma^3} &= (Q + \Delta Q)^2 \\ &\approx Q^2 + 2Q\Delta Q \end{aligned}$$

Space-charge tune shift $\Longrightarrow$ $$\boxed{\Delta Q = -\frac{N r_0 R^2}{2Q\sigma^2 \beta^2 \gamma^3}}$$

Most pronounced a low energies when $\beta^2\gamma^3$ is small, particularly for bright, intense beams.

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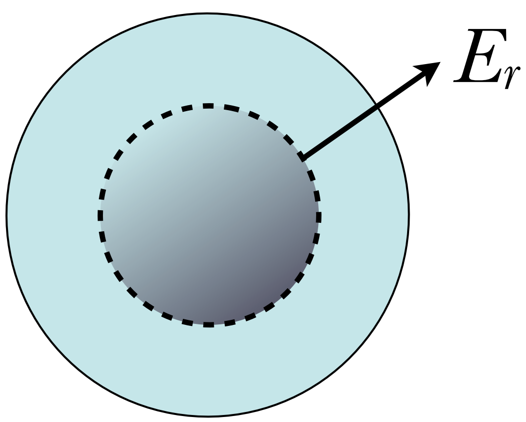
Equations of Motion with Space-Charge

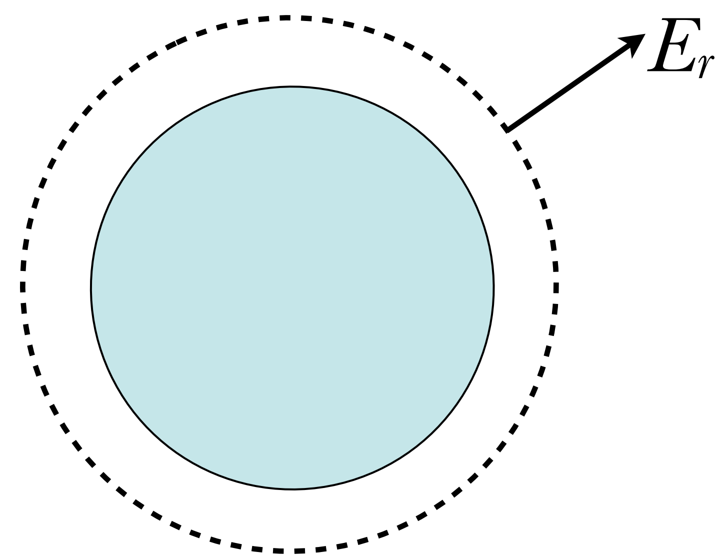
Consider a paraxial beam with uniform circular cross-section of radius a.

Gauss' Flux Theorem gives

$$2\pi r E_r = \frac{1}{\epsilon_0} \times \{\text{charge enclosed}\} = \begin{cases} \rho\pi r^2/\epsilon_0 & r < a \\ \rho\pi a^2/\epsilon_0 & r > a \end{cases}$$



$$E_r = \begin{cases} \dfrac{Nqr}{2\pi\epsilon_0 a^2} & r \leqslant a \\ \\ \dfrac{Nq}{2\pi\epsilon_0 r} & r > a \end{cases}$$



Within the beam, space-charge forces are:

$$E_x = \frac{x}{r} E_r, \quad E_y = \frac{y}{r} E_r \qquad \Longrightarrow \qquad \mathbf{E} = \frac{Nq}{2\pi\epsilon_0 a^2}(x, y).$$

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Equations of Motion are:

For round beam, need axisymmetric focusing, $k_x=k_y=k$

$$
\begin{aligned}
x'' + k(s)x &= \frac{q}{m_0\gamma^3\beta^2c^2}E_x \\
y'' + k(s)y &= \frac{q}{m_0\gamma^3\beta^2c^2}E_y
\end{aligned}
$$

or:

$$
\begin{aligned}
x'' + \left(k(s) - \frac{K}{a^2}\right)x &= 0 \\
y'' + \left(k(s) - \frac{K}{a^2}\right)y &= 0
\end{aligned}
$$

$$K = \frac{I}{I_0}\frac{2}{\beta^3\gamma^3}$$

perveance

I is the beam current $\quad I = Nq\beta c$

I_0 is the *characteristic current*

$$
\begin{aligned}
I_0 &= \frac{4\pi\epsilon_0 m_0 c^3}{q} \approx \frac{1}{30}\frac{m_0c^2}{q} \\
&\approx \begin{cases} 17\,\mathrm{kA} & \text{for electrons} \\ 31(A/Z)\,\mathrm{MA} & \text{for ions} \end{cases}
\end{aligned}
$$

Corresponding envelope equation is:

$$a'' + ka - \frac{\epsilon^2}{a^3} - \frac{K}{a} = 0.$$

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Incoherent Tune Shift

Assume an unbunched beam, uniform density, circular cross-section in a ring, mean radius R.

$$x'' + \left(k(s) + k_{\rm sc}(s)\right) x = 0$$

$$\text{where } k_{\rm sc} = -\frac{K}{a^2} = -\frac{I}{I_0}\frac{2}{\beta^3\gamma^3}\frac{1}{a^2} = -\frac{2Ir_0}{q\beta^3\gamma^3 c}\frac{1}{a^2}$$

$$\Delta Q_x = \frac{1}{4\pi}\oint K_x(s)\beta_x(s)\,\mathrm{d}s = \frac{1}{4\pi}\oint k_{\rm sc}(s)\beta_x(s)\,\mathrm{d}s$$

$$\Delta Q_x = -\frac{1}{4\pi}\int_0^{2\pi R}\frac{2r_0 I}{q\beta^3\gamma^3 c}\frac{\beta_x(s)}{a^2}\,\mathrm{d}s = -\frac{r_0 R I}{q\beta^3\gamma^3 c}\left\langle\frac{\beta_x(s)}{a^2(s)}\right\rangle = \frac{1}{\epsilon_x}$$

$$\Delta Q_{x,y} = -\frac{r_0 R I}{q\beta^3\gamma^3 c\epsilon_{x,y}}, \quad \text{with } I = \frac{Nq\beta c}{2\pi R}$$

$$\boxed{\Delta Q_{x,y} = -\frac{r_0 N}{2\pi\beta^2\gamma^3\,\epsilon_{x,y}}}$$

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$$\Delta Q_{x,y} = -\frac{r_0 N}{2\pi\beta^2\gamma^3\,\epsilon_{x,y}}$$

- N = total number of particles in the ring
- $\varepsilon_{x,y}$ = horizontal (vertical) 100% emittance
- Direct space charge effect
- Does not depend on machine size $2\pi R$
- Vanishes for $\gamma >> 1$
- Important effect in low energy machines
- *Incoherent* tune shift
  - particle moves within the beam

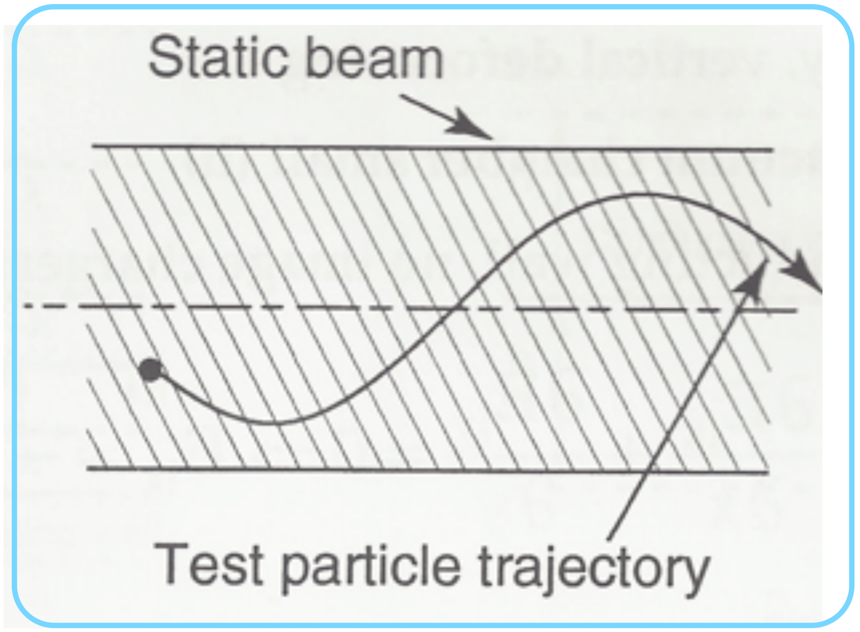
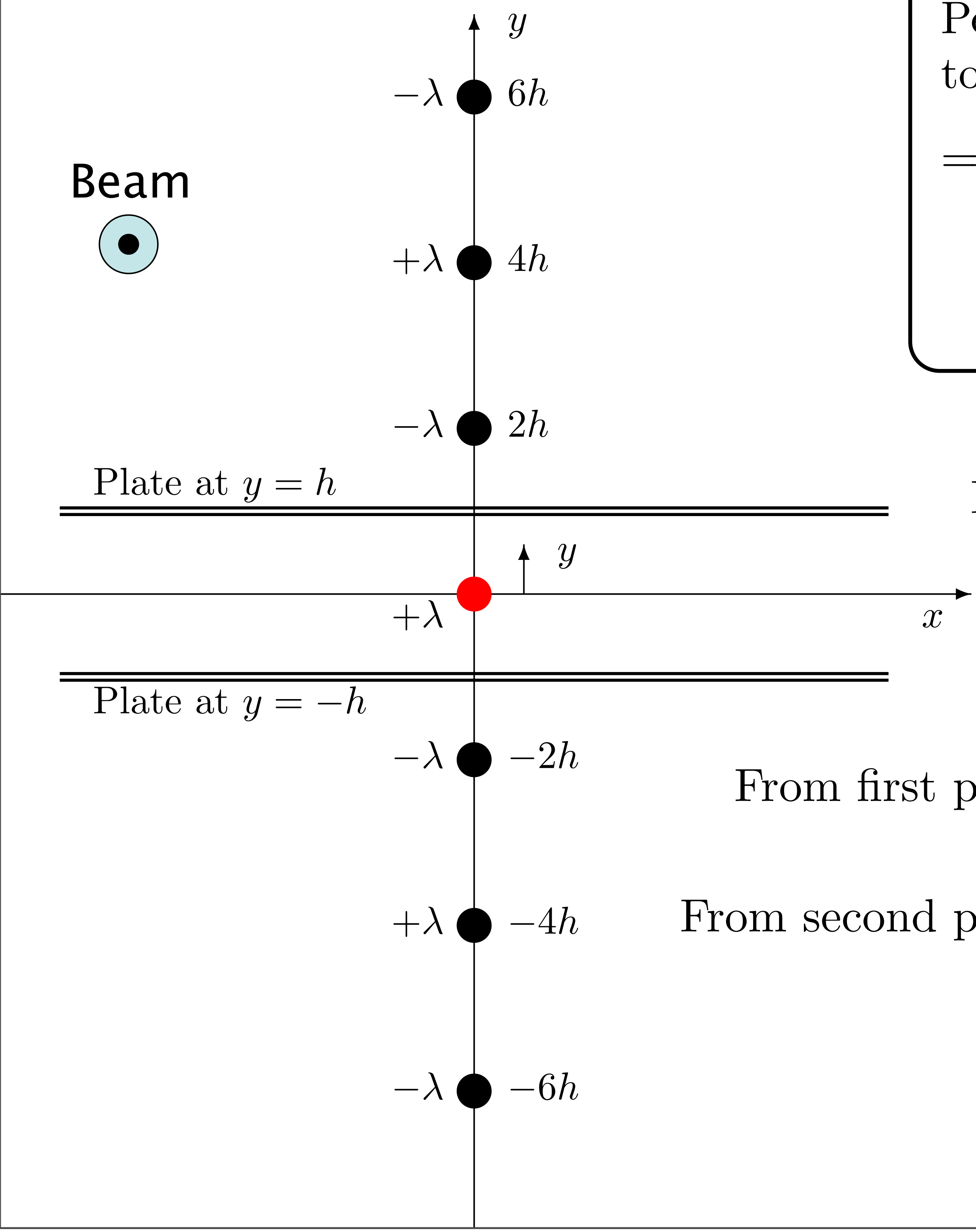


Image Effects - Parallel Conducting Plates



Perfectly conducting collector plates parallel to beam pipe

$\Longrightarrow$ infinite system of images

$$
\begin{array}{llll}
+\lambda & \text{at} & \pm 4nh & n = 1, 2, \ldots \\
-\lambda & \text{at} & \pm 2(2n-1)h & n = 1, 2, \ldots
\end{array}
$$

Field created by line charge at distance d is

$$E_y = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{d}$$

From first pair of images $\quad E_{1y} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{2h-y} - \frac{1}{2h+y}\right)$

From second pair of images $\quad E_{2y} = \frac{\lambda}{2\pi\epsilon_0}\left(\frac{1}{4h+y} - \frac{1}{4h-y}\right)$

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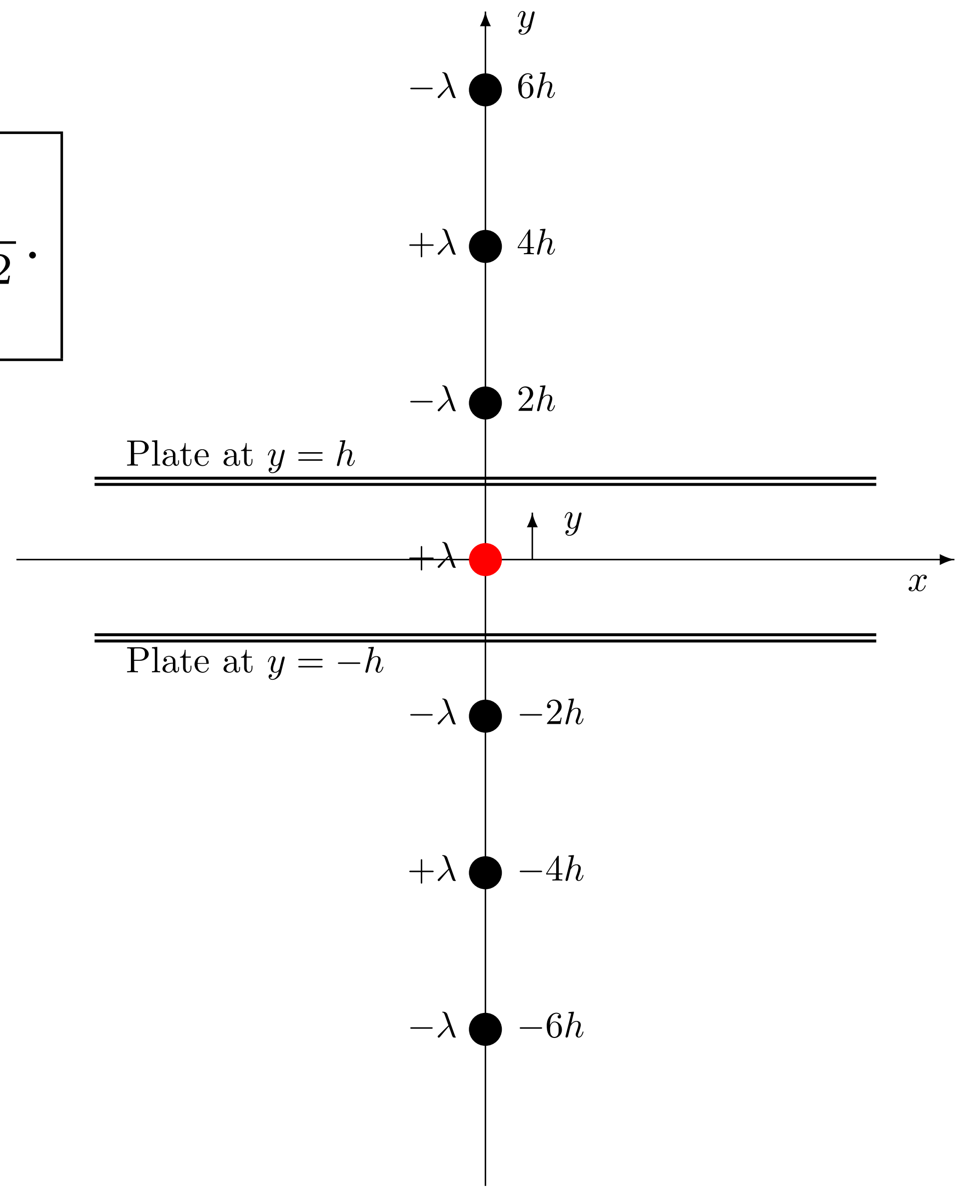
Images produce an extra electrostatic field given by

$$E_y^i = \frac{\lambda}{2\pi\epsilon_0}\sum_{n=1}^{\infty}\left[\frac{1}{2(2n-1)h-y} - \frac{1}{2(2n-1)h+y}\right] + \frac{\lambda}{2\pi\epsilon_0}\sum_{n=1}^{\infty}\left[\frac{1}{4nh+y} - \frac{1}{4nh-y}\right]$$

$$= \frac{\lambda}{2\pi\epsilon_0}\sum_{n=1}^{\infty}\left[\frac{2y}{4(2n-1)^2h^2-y^2} - \frac{2y}{4(2n)^2-y^2}\right]$$

$$\Longrightarrow \quad \boxed{E_y^i = \frac{\lambda y}{\pi\epsilon_0}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2h^2-y^2}.}$$



For small y, $$E_y^i \approx \frac{\lambda y}{4\pi\epsilon_0 h^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\Longrightarrow \quad \boxed{E_y^i \approx \frac{\lambda\pi y}{48\epsilon_0 h^2}}$$

- Vertical image field vanishes at $y = 0$
- Field linear in y, vertically defocusing
- Field large if vacuum chamber is small

$$E_y^i \approx \frac{\lambda\pi}{48\epsilon_0 h^2} y$$

There are no image charges between the conducting walls (i.e. in the vacuum chamber), so

$$\operatorname{div} \mathbf{E}^i = 0 \quad \Longrightarrow \quad \frac{\partial E_x^i}{\partial x} + \frac{\partial E_y^i}{\partial y} = 0$$

$$\Longrightarrow \quad E_x^i = -\frac{\lambda\pi}{48\epsilon_0 h^2} x$$

$$\Longrightarrow \quad \text{image forces } F_y^i = \frac{q\lambda}{\pi\epsilon_0 h^2}\frac{\pi^2}{48} y, \quad F_x^i = -\frac{q\lambda}{\pi\epsilon_0 h^2}\frac{\pi^2}{48} x$$

Incoherent tune shift for round beam between parallel conducting walls

$$\Delta Q_x = -\frac{2r_0 IR\langle\beta_x\rangle}{qc\beta^3\gamma}\left[\frac{1}{2\langle a^2\rangle\gamma^2} - \frac{\pi^2}{48h^2}\right]$$

direct image

$$\Delta Q_y = -\frac{2r_0 IR\langle\beta_y\rangle}{qc\beta^3\gamma}\left[\frac{1}{2\langle a^2\rangle\gamma^2} + \frac{\pi^2}{48h^2}\right]$$

- Image effects $\sim \dfrac{1}{\gamma}$
- (Electrical) image effects normally focusing in horizontal, defocusing in vertical planes.

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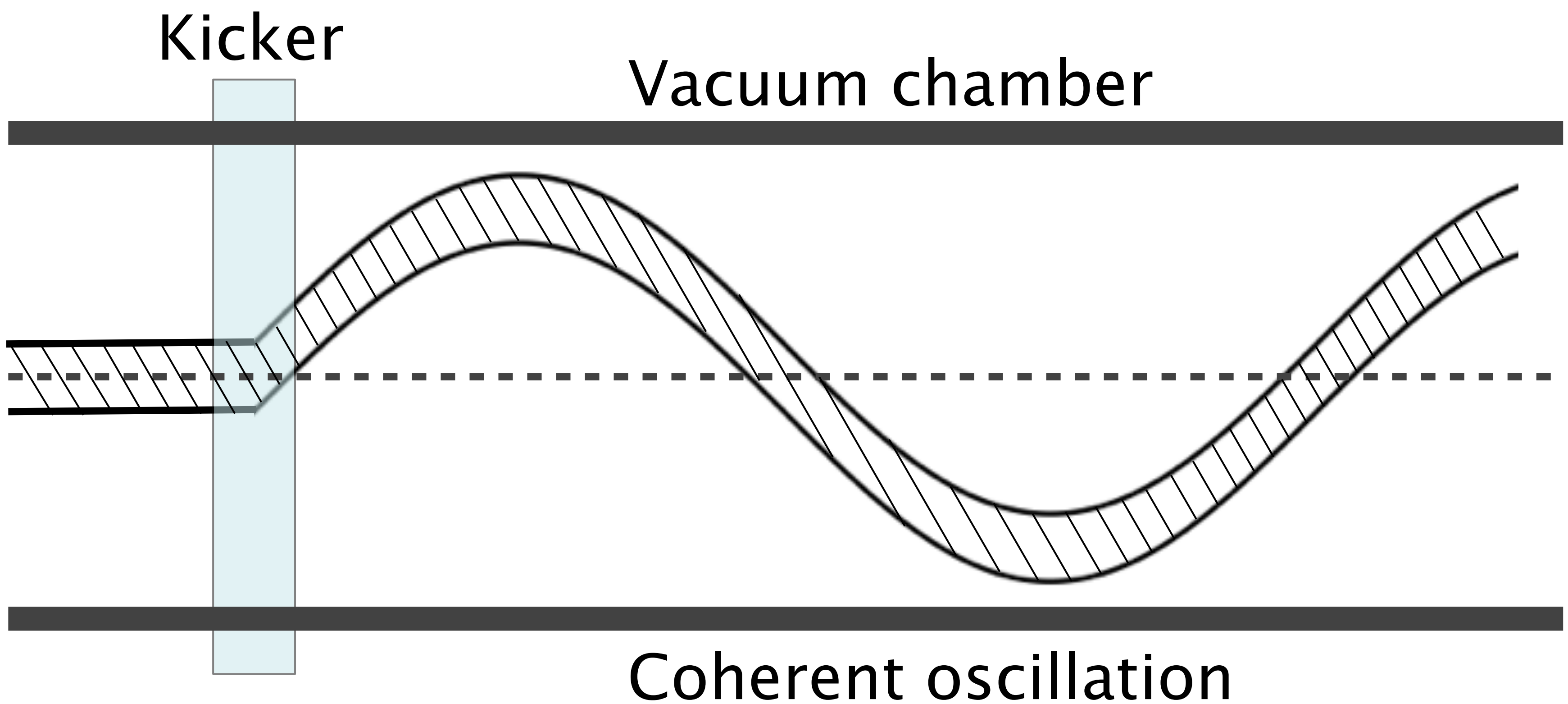
Coherent Tune Shift



The centre of mass moves, performing betatron oscillations as a whole. The beam environment influences the coherent tune ⇒ coherent tune shift

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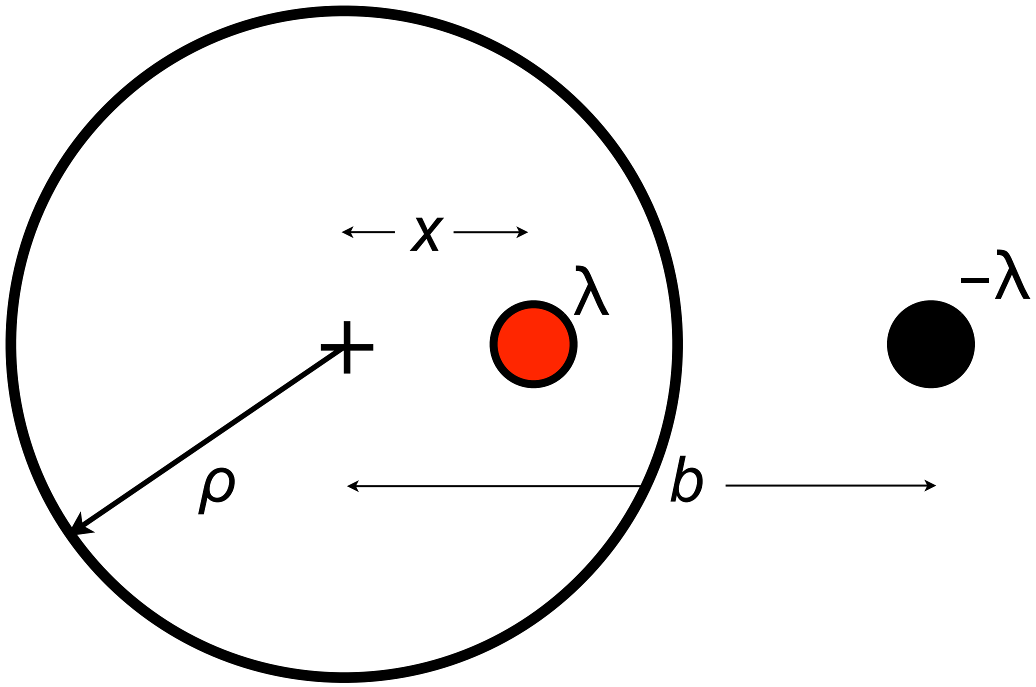
Image Effects - Circular Pipe

x = mean position of beam
beam radius $\ll \rho$



Single image line charge $-\lambda$ at inverse point b where

$$bx = \rho^2$$

Beam experiences a field

$$\begin{aligned} E_x^i &= \frac{\lambda}{2\pi\epsilon_0}\frac{1}{b-x} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{\rho^2/x - x} \\ &= \frac{\lambda}{2\pi\epsilon_0}\frac{x}{\rho^2 - x^2} \\ &\approx \frac{\lambda x}{2\pi\epsilon_0\rho^2} \qquad \text{for small } x \end{aligned}$$

$$\Delta Q_{x,y,\mathrm{coh}} = -\frac{r_0 R\langle\beta_{x,y}\rangle I}{ec\beta^3\gamma\rho^2} = -\frac{r_0\langle\beta_{x,y}\rangle}{2\pi\beta^2}\frac{N}{\gamma\rho^2}$$

- Only weak dependence on γ
- Coherent ΔQ never positive

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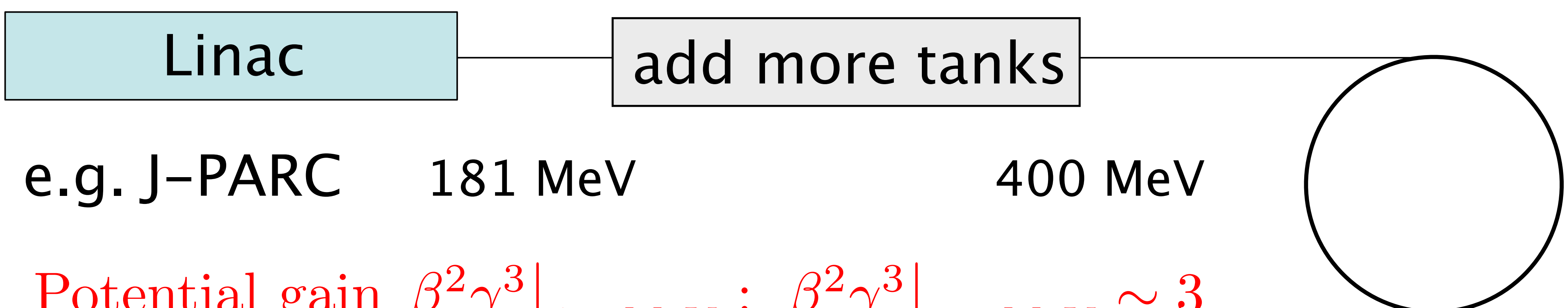
Space-Charge Limit

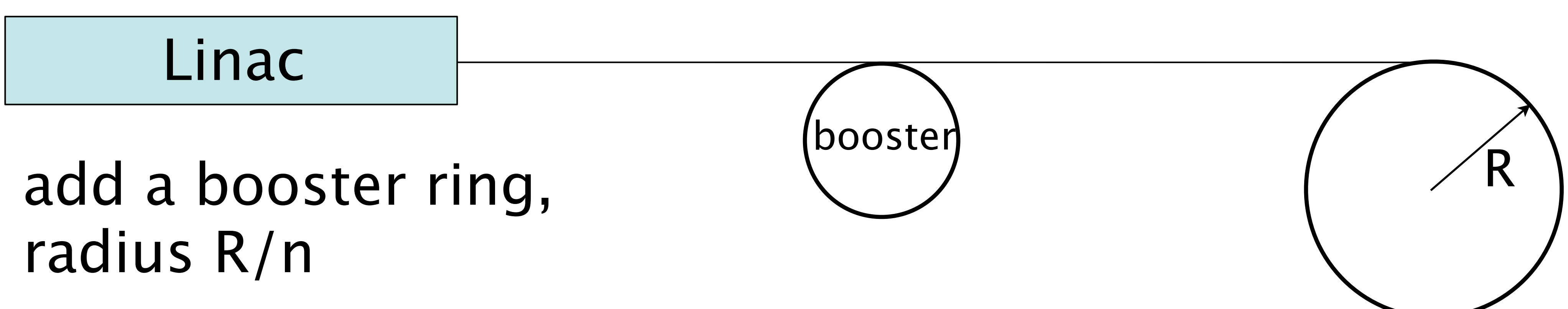
$$\Delta Q \propto \frac{N}{\epsilon \beta^2 \gamma^3}$$

Proton/ion machines will be limited in N because ΔQ will cross resonances when filling the acceptance.

N can be increased by increasing the injection energy and hence $\beta^2\gamma^3$ without changing ΔQ.



$$\text{Potential gain } \beta^2\gamma^3\big|_{400\,\mathrm{MeV}} : \beta^2\gamma^3\big|_{181\,\mathrm{MeV}} \sim 3$$



add a booster ring,
radius R/n

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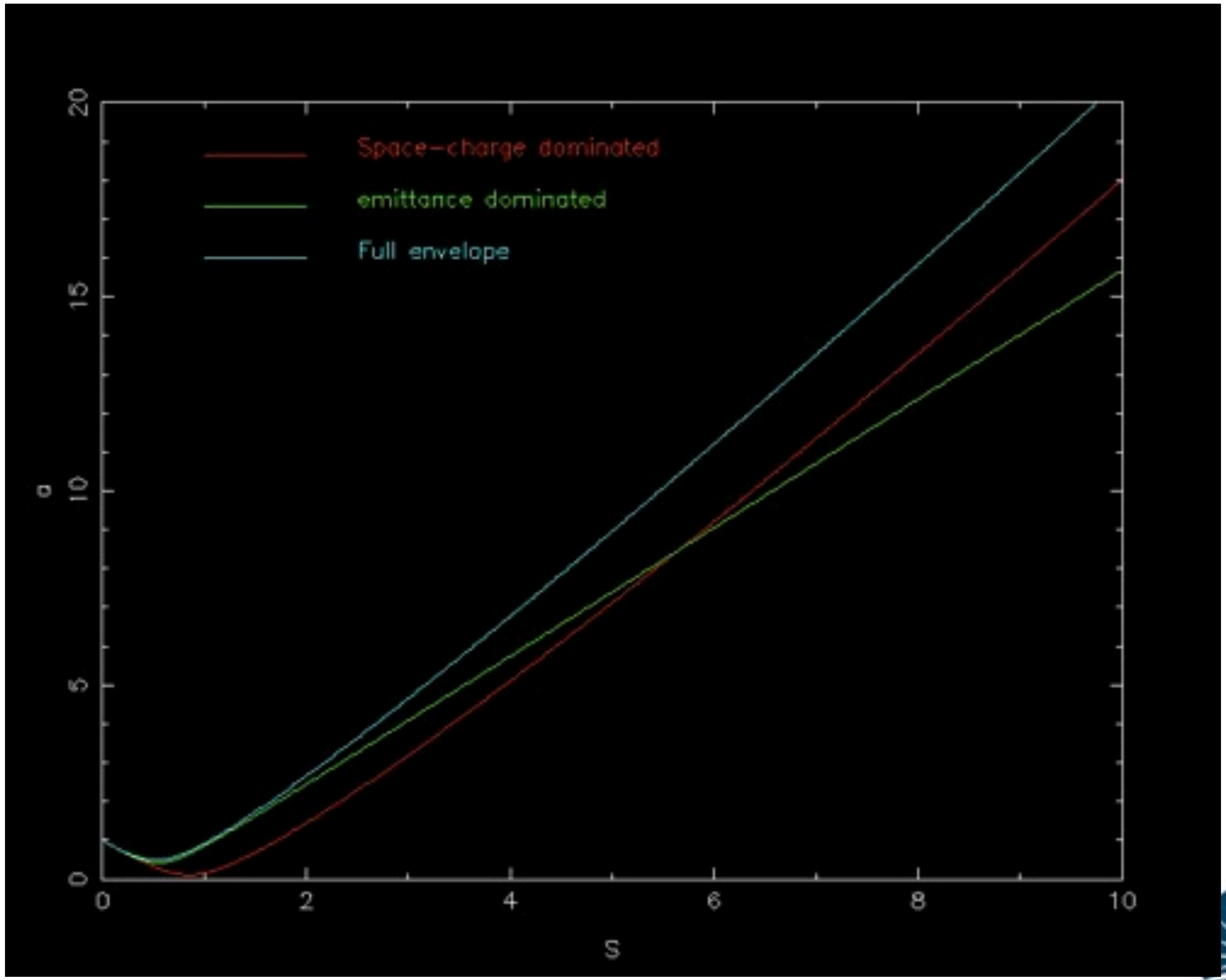
Envelope Equation

$$a'' + k(s)a - \frac{\epsilon^2}{a^3} - \frac{K}{a} = 0$$

$$\epsilon^2 \gg Ka^2 \quad \Longrightarrow \quad Emittance\ dominated$$
$$\epsilon^2 \ll Ka^2 \quad \Longrightarrow \quad Space\text{-}charge\ dominated$$



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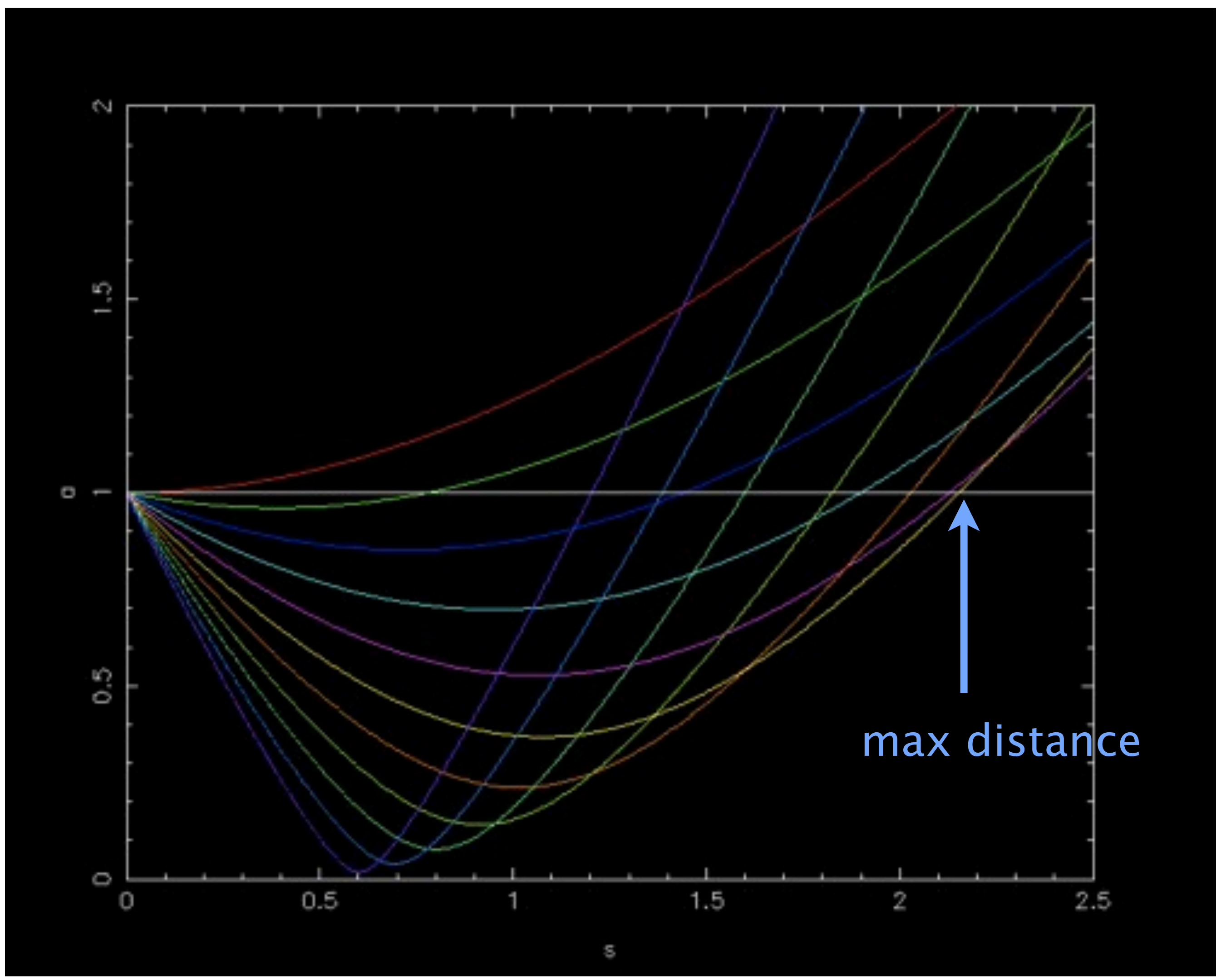
For space-charge dominated beams: $a'' = \dfrac{K}{a}$ with initial values a_0, a_0'

Use normalised coordinates $\mathcal{A} = \dfrac{a}{a_0}$, $\mathcal{S} = \sqrt{2K}\dfrac{s}{a_0}$

$$\Longrightarrow \qquad \ddot{\mathcal{A}} \equiv \frac{\mathrm{d}^2\mathcal{A}}{\mathrm{d}\mathcal{S}^2} = \frac{1}{2\mathcal{A}}, \text{which integrates to} \quad \dot{\mathcal{A}}^2 - \dot{\mathcal{A}}_0^2 = \ln \mathcal{A}$$



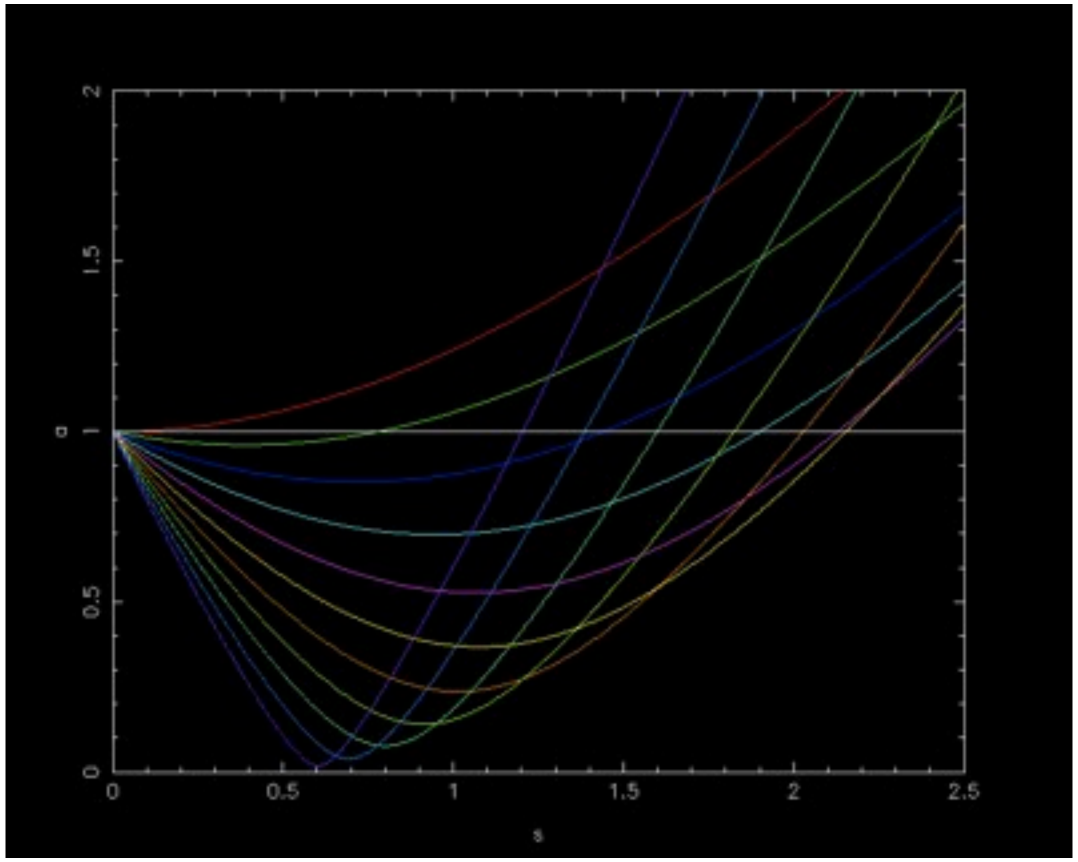
Beam size v. distance (normalised units)

In some high current applications, need to transport beam through tube of diameter D and length L with help of a focussing lens. Require waist at centre, equal diameters at ends.
Graphs show a maximum value S=2.16, hence a maximum current that can be transported for a given tube diameter.

$$L = 2.16\frac{a_0}{\sqrt{2K}} = 2.16\frac{D/2}{\sqrt{2K}}$$

$$\Longrightarrow \qquad I_{max} = \frac{1}{4}I_0(\beta\gamma)^3\left(\frac{1.08D}{L}\right)^2$$

Consider upper curve, which has $a_0'=0$.



Beam radius doubles in (normalised) distance $\mathcal{S} \approx 2.12$

Then
$$\dot{\mathcal{A}}^2 = \ln \mathcal{A} \implies \frac{1}{\mathcal{A}}\,\mathrm{d}\mathcal{A} = 2\dot{\mathcal{A}}\,\mathrm{d}\dot{\mathcal{A}}$$
$$\implies \quad \mathrm{d}\mathcal{S} = \frac{\mathrm{d}\mathcal{A}}{\dot{\mathcal{A}}} = 2\mathcal{A}\,\mathrm{d}\dot{\mathcal{A}}$$
$$\implies \quad \mathcal{S} = 2\int_0^{[\ln \mathcal{A}]^{\frac{1}{2}}} \mathrm{e}^{\dot{\mathcal{A}}^2}\,\mathrm{d}\dot{\mathcal{A}}$$

For $1 \leqslant \mathcal{A} \leqslant 2$, this can be approximated to $\sim 3\%$ by $\mathcal{S} \approx 2(\mathcal{A}-1)^{\frac{1}{2}}$.

Then beam radius is
$$\mathcal{A} = \frac{a}{a_0} = 1 + \tfrac{1}{4}\mathcal{S}^2 = 1 + \tfrac{1}{2}K\left(\frac{s}{a_0}\right)^2$$
or
$$\frac{a}{a_0} = 1 + \frac{I}{I_0}\frac{1}{(\beta\gamma)^3}\left(\frac{s}{a_0}\right)^2$$

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For an electron beam $I_0 \sim 1.7 \times 10^4$

Beam doubles in size when $s = 1.31 \times 10^2 \times (\gamma^2 - 1)^{3/4} I^{-1/2} a_0$

If kinetic energy is equal to rest energy 511 keV, and current is 200 A, then

$$s = 21.0a_0, \quad \text{or} \quad 52.5\,\text{cm for a } 2.5\,\text{cm beam.}$$

For a proton beam, $I_0 \sim 3.13 \times 10^7$

so doubling occurs when $s = 5.59 \times 10^3 \times (\gamma^2 - 1)^{3/4} I^{-1/2} a_0$

For $T = T_{\text{rest}} = 938\,\text{MeV}$, $I \sim 45\,\text{A}$, $s \sim 200a_0$.

For the complete envelope equation (round beam, no focusing),

$$a'' = \frac{\epsilon^2}{a^3} + \frac{K}{a}$$

Multiply by $2a'$ and integrate:

$$a'^2 = -\frac{\epsilon^2}{a^2} + 2K \ln a + \text{constant}$$

$$\Longrightarrow \qquad a' = \pm \left[a_0'^2 + \epsilon^2 \left(\frac{1}{a_0^2} - \frac{1}{a^2} \right) + 2K \ln \frac{a}{a_0} \right]^{\frac{1}{2}}$$

and

$$s = \int_{a_0}^{a} \left[a_0'^2 + \epsilon^2 \left(\frac{1}{a_0^2} - \frac{1}{a^2} \right) + 2K \ln \frac{a}{a_0} \right]^{-\frac{1}{2}} \, \mathrm{d}a$$

If $K \neq 0$ this has to be evaluated numerically.

Beam Transport in a Uniform Focusing Channel

Assume no applied accelerating force and canonical angular momentum $p_\theta = 0$

Paraxial ray equation for axisymmetric beam of radius a is

$$r'' + k_0^2 r - \frac{K}{a^2} r = 0$$

where k_0^2 represents linear external focusing force, K is the perveance.

Corresponding envelope equation is

$$a'' + k_0^2 a - \frac{K}{a} - \frac{\epsilon^2}{a^3} = 0$$

Best known example is the long solenoid channel, where

$$k_0 = \frac{\omega_L}{\beta c} = \frac{|qB|}{2m_0\gamma\beta c}, \qquad \omega_L = \text{Larmor frequency}$$

For microwave sources, using time t as independent variable, ray equation is

$$\ddot{r} + \omega_0^2 r - \tfrac{1}{2}\omega_p^2 r = 0, \quad \text{where } \omega_p \text{ is the plasma frequency, } \omega_p^2 = \frac{2K\beta^2 c^2}{a^2}$$

Matched Beams

For a constant focusing channel, there is a special solution with a =constant. Beam envelope is a straight line.

$$a'' = 0 \quad \Longrightarrow \quad k_0^2 a - \frac{K}{a} - \frac{\epsilon^2}{a^3} = 0$$

Introduce the *wave number* k defined by $k^2 = \left(\frac{2\pi}{\lambda}\right)^2 = k_0^2 - \frac{K}{a^2}$

Then $$ka^2 = \epsilon$$

In terms of frequencies, $\omega^2 = \omega_0^2 - \frac{1}{2}\omega_p^2$

k and ω define the wavelength $\lambda = 2\pi/k$ and oscillation frequency due to the action of both the applied force and the space-charge force.

Note $\omega < \omega_0$. ω/ω_0 is the *tune depression* due to space-charge.

Special Cases

1. Laminar flow, $\epsilon = 0$.

   Beam radius is $\quad a_B = \dfrac{K^{\frac{1}{2}}}{k_0}$ *Brillouin flow*

2. Negligible space-charge, $K \approx 0$.

   Beam radius is $\quad a_0 = \left(\dfrac{\epsilon}{k_0}\right)^{\frac{1}{2}}$ c.f. Twiss $\hat{\beta} = 1/k_0$, constant.

$$**************$$

Introduce dimensionless parameter $u = \dfrac{K}{2k_0\epsilon}$ into $\quad k_0^2 a - \dfrac{K}{a} - \dfrac{\epsilon^2}{a^3} = 0.$

Then $\quad \left(\dfrac{a}{a_0}\right)^4 - 2u\left(\dfrac{a}{a_0}\right)^2 - 1 = 0 \quad \Longrightarrow \quad a = a_0\left[u + \sqrt{1+u^2}\right]^{\frac{1}{2}}$

Equivalently $\quad a = \frac{1}{2} a_B \left[1 + \sqrt{1 + u^{-2}}\right]^{\frac{1}{2}}$

Without space-charge, a beam of zero emittance has radius a_0.
As current increases, beam radius expands and diameter of beam pipe needs to be large enough to accommodate.

Suppose maximum beam size is $a_0 = a_m$; then *acceptance* is $\boldsymbol{\alpha} = k_0 a_m^2$.

Maximum beam current that can be transported follows from the maximum perveance:

$$K = k_0^2 a_m^2 - \frac{\epsilon^2}{a_m^2} = k_0 \boldsymbol{\alpha} \left[1 - \left(\frac{\epsilon}{\boldsymbol{\alpha}} \right)^2 \right]$$

$$\Longrightarrow \qquad I = \tfrac{1}{2} I_0 \beta^3 \gamma^3 k_0 \boldsymbol{\alpha} \left[1 - \left(\frac{\epsilon}{\boldsymbol{\alpha}} \right)^2 \right]$$

Observe:

- transportable current increases rapidly with particle energy
- acceptance $\boldsymbol{\alpha}$ must exceed the emittance ϵ of the beam.
- Maximum current when $\epsilon/\boldsymbol{\alpha} \to 0$ (laminar beam limit).
  In this case, $K = k_0^2 a^2$ or $\omega_0^2 = \frac{1}{2}\omega_p^2$ (well known result for nonrelativistic ideal Brillouin flow)

Tune Depression

Zero space-charge particle equation $\quad r'' + k_0^2 r = 0$
Space-charge particle equation $\quad r'' + k^2 r = 0$

Frequencies $\quad \omega_0 = k_0 \beta c, \;\; \omega = k \beta c$

Recall $\quad ka^2 = \epsilon, \;\; k_0 a^2 = \boldsymbol{\alpha}, \;\; u = \dfrac{K}{2k_0\epsilon}$ and $\quad k_0^2 a - \dfrac{K}{a} - \dfrac{\epsilon^2}{a^3} = 0$

$$\Longrightarrow \qquad 0 = k_0^2 - \frac{K}{a^2} - \left(\frac{\epsilon}{a^2}\right)^2 = k_0^2 - \frac{2k_0\epsilon u}{a^2} - k^2 = k_0^2 - 2kk_0 u - k^2.$$

Therefore tune depression is $\dfrac{\omega}{\omega_0} = \dfrac{k}{k_0} = \dfrac{\epsilon}{\boldsymbol{\alpha}} = \sqrt{1+u^2} - u$

Limit between space-charge and emittance dominated beams is $Ka^2 = \epsilon^2$

$$\Longrightarrow \qquad \frac{K}{k_0} = \frac{\epsilon^2}{\boldsymbol{\alpha}} \Longrightarrow \quad u = \frac{K}{2k_0\epsilon} = \frac{\epsilon}{2\boldsymbol{\alpha}} = \frac{\omega}{2\omega_0}; \text{ then find } u = \frac{1}{2\sqrt{2}}$$

$$\Longrightarrow \qquad \frac{\omega}{\omega_0} = \frac{1}{\sqrt{2}} = 0.71$$

For tune shifts below 0.71, beam is dominated by space-charge; above 0.71, emittance dominates

gy

Mismatched Beams

Matched beam radius $\bar{a}$: $$k_0^2\bar{a} - \frac{K}{\bar{a}} - \frac{\epsilon^2}{\bar{a}^3} = k^2\bar{a} - \frac{\epsilon^2}{\bar{a}^3} = 0$$

Mismatched beam: put $a = \bar{a} + X$ in envelope equation, $|X| \ll \bar{a}$

$$\text{Then}\quad X'' + k_0^2(\bar{a}+X) - \frac{K}{\bar{a}}\left(1+\frac{X}{\bar{a}}\right)^{-1} - \frac{\epsilon^2}{\bar{a}^3}\left(1+\frac{X}{\bar{a}}\right)^{-3} = 0$$

$$\Longrightarrow \text{(to first-order)} \quad X'' + \left(k_0^2 + \frac{K}{\bar{a}^2} + 3\frac{\epsilon^2}{\bar{a}^4}\right) X = 0$$

So envelope oscillations have the form $X'' + k_e^2 X = 0$

$$\begin{aligned} \text{where}\quad k_e^2 &= k_0^2 + \frac{K}{\bar{a}^2} + 3\frac{\epsilon^2}{\bar{a}^4} \\ &= k_0^2 + \left(k_0^2 - k^2\right) + 3k^2 = 2k_0^2 + 2k^2 \end{aligned}$$

Single particles oscillate with frequency ω while the envelope oscillates

with frequency $$\omega_e = \left[2\omega_0^2 + 2\omega^2\right]^{\frac{1}{2}} = \sqrt{2}\omega_0\left[1 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{\frac{1}{2}}$$

In terms of the plasma frequency $\quad \omega_p^2 = \frac{2K}{\bar{a}^2} = 2\left(\omega_0^2 - \omega^2\right),$

$$\omega_e = \left[4\omega_0^2 - \omega_p^2\right] = 2\omega_0 \left[1 - \frac{1}{4}\left(\frac{\omega_p}{\omega_0}\right)^2\right]$$

Known as *in-phase mode* for an axisymmetric beam. Solutions for the quadrupole (elliptical) case give the *out-of-phase mode*

$$\omega_e = \sqrt{2}\omega_0 \left[1 + \left(\frac{\omega}{\omega_0}\right)^2\right]^{\frac{1}{2}} = \begin{cases} 2\omega_0 & K = 0 \\ \sqrt{2}\omega_0 & \epsilon = 0 \end{cases}$$

For a long solenoid channel and zero intensity ($K = 0$), particles oscillate at the Larmor frequency while the envelope of the mismatched beam oscillates at the cyclotron frequency.

For ideal Brillouin flow ($\epsilon = 0$), $\omega = \sqrt{2}\omega_0 = \omega_p$ and envelope oscillates with the plasma frequency.

Note: valid only for small mismatch $|X| \ll \bar{a}$.

Beams with Elliptical Cross-section

Space-charge potential for a uniform round beam is ($\mathbf{E} = -\nabla\phi$):

$$\phi(x, y, s) = -\frac{Nq}{4\pi\epsilon_0 a^2}\left(x^2 + y^2\right).$$

For a uniform *elliptical* beam with semi-axes a, b,

$$\rho(x, y) = \begin{cases} \dfrac{Nq}{\pi ab}, & \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

The electric field within the beam, from Poisson's equation $\nabla^2\phi = -\rho/\epsilon_0$, is

$$\mathbf{E} = \frac{Nq}{\pi\epsilon_0(a+b)}\left(\frac{x}{a}, \frac{y}{b}\right).$$

corresponding to a potential

$$\phi(x, y, s) = -\frac{Nq}{2\pi\epsilon_0(a+b)}\left(\frac{x^2}{a} + \frac{y^2}{b}\right)$$

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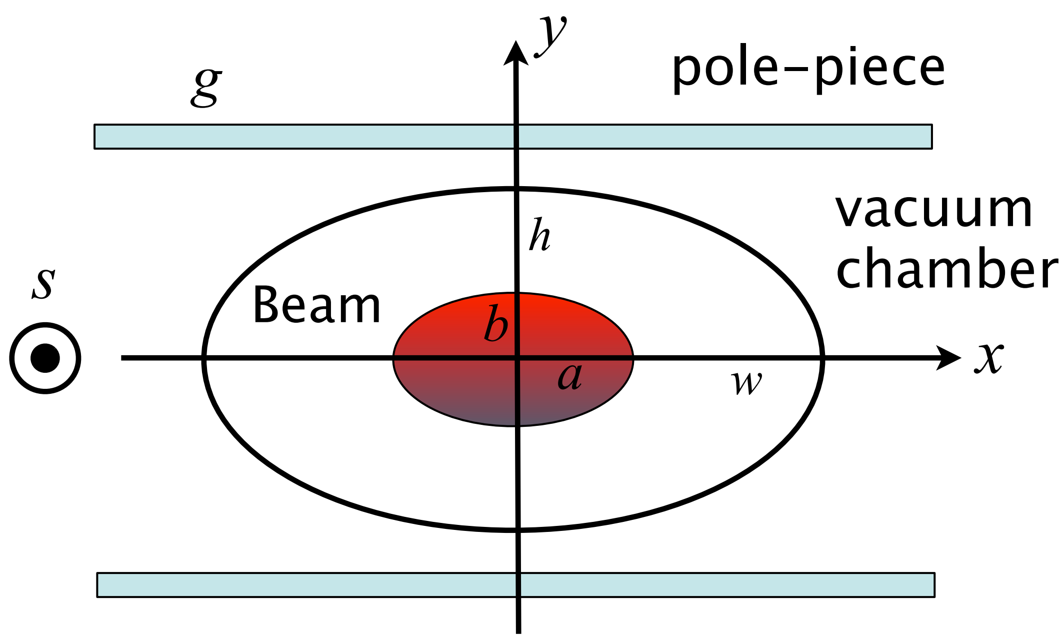
These give the coupled set of equations for beam particles and beam envelope:

$$\begin{aligned} x'' + k_x(s)x - \frac{2K}{a+b}\frac{x}{a} &= 0 \\ y'' + k_y(s)y - \frac{2K}{a+b}\frac{y}{b} &= 0 \\ a'' + k_x(s)a - \frac{\epsilon_x^2}{a^3} - \frac{2K}{a+b} &= 0 \\ b'' + k_y(s)b - \frac{\epsilon_y^2}{b^3} - \frac{2K}{a+b} &= 0 \end{aligned}$$

Numerical integration:
Codes (Agile, KVBL) can design and optimise a linear focusing channel or ring by varying field gradients etc, similar to MADX.

Note: phase advance (tune) is $\int \frac{1}{\hat{\beta}}\,\mathrm{d}s = \int \frac{\epsilon_x}{a^2}\,\mathrm{d}s \qquad \Longrightarrow \qquad Q \approx \frac{R}{\langle\beta\rangle} = \frac{R\epsilon_x}{\langle a^2\rangle}$

Laslett Coefficients



Uniform, elliptical beam in an elliptical vacuum chamber

$$\Delta Q_{y,\mathrm{inc}} = -\frac{N r_0 \langle \beta_y \rangle}{\pi \beta^2 \gamma} \left(\underbrace{\frac{\zeta_0^y}{b^2\gamma^2}}_{\text{Direct}} + \underbrace{\frac{\zeta_1^y}{h^2}}_{\text{electr. image}} + \underbrace{\beta^2 \frac{\zeta_2^y}{g^2}}_{\text{magnetic image}} \right)$$

$$\Delta Q_{y,\mathrm{coh}} = -\frac{N r_0 \langle \beta_y \rangle}{\pi \beta^2 \gamma} \left(\frac{\xi_1^y}{h^2} + \beta^2 \frac{\xi_2^y}{g^2} \right)$$

Similarly for ΔQ_x. In general $|\Delta Q_y| > |\Delta Q_x|$

$\xi_1, \xi_2, \zeta_0, \zeta_1, \zeta_2$ are the Laslett Coefficients (L.J. Laslett 1963)

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Laslett coefficients	**Circular** $(\mathbf{a} = \mathbf{b}, \mathbf{w} = \mathbf{h})$	**Elliptical** $(\mathbf{e.g.}\ \mathbf{w} = \mathbf{2h})$	**Parallel plates** $(\mathbf{h}/\mathbf{w} = \mathbf{0})$
ζ_0^x	$\frac{1}{2}$	$\frac{b^2}{a(a+b)}$	
ζ_0^y	$\frac{1}{2}$	$\frac{b}{a+b}$	
ζ_1^x	0	-0.172	-0.206
ζ_1^y	0	0.172	$0.206\ (= \pi^2/48)$
ξ_1^x	$\frac{1}{2}$	0.083	0
ξ_1^y	$\frac{1}{2}$	0.55	$0.617\ (= \pi^2/16)$
ζ_2^x	$-0.411\ (= -\pi^2/24)$	-0.411	-0.411
ζ_2^y	0.411	0.411	0.411
ζ_2^x	0	0	0
ζ_2^y	$0.617\ (= \pi^2/16)$	0.617	0.617

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Incoherent ΔQ: Practical Formulae

$$\Delta Q_y = -\frac{r_0}{\pi}\left(\frac{q^2}{A}\right)\frac{N}{\beta^2\gamma^3}\frac{F_y G_y}{B_f}\left\langle \frac{\beta_y}{b(a+b)} \right\rangle$$

$$\left\langle \frac{\beta_y}{b(a+b)} \right\rangle = \left\langle \frac{\beta_y}{b^2(1+\frac{a}{b})} \right\rangle \approx \frac{1}{\epsilon_y\left(1+\sqrt{\frac{\epsilon_x Q_y}{\epsilon_y Q_x}}\right)}$$

$$\Delta Q_{x,y} = -\frac{r_0}{\pi}\left(\frac{q^2}{A}\right)\frac{N}{\beta^2\gamma^3}\frac{F_{x,y} G_{x,y}}{B_f}\frac{1}{\epsilon_{x,y}\left(1+\sqrt{\frac{\epsilon_{y,x} Q_{x,y}}{\epsilon_{x,y} Q_{y,x}}}\right)}$$

$q/A =$ charge/mass ratio for ions, e.g $\frac{q}{A} = \frac{6}{16}$ for ${}_{16}O^{6+}$

$F_{x,y} =$ "Form factor" depending on images etc

$G_{x,y} =$ "Form factor" depending on particle distribution ($G = 1$ for uniform)

$B_f = \frac{\bar{\lambda}}{\hat{\lambda}} = \frac{\bar{I}}{\hat{I}}$, Bunching factor

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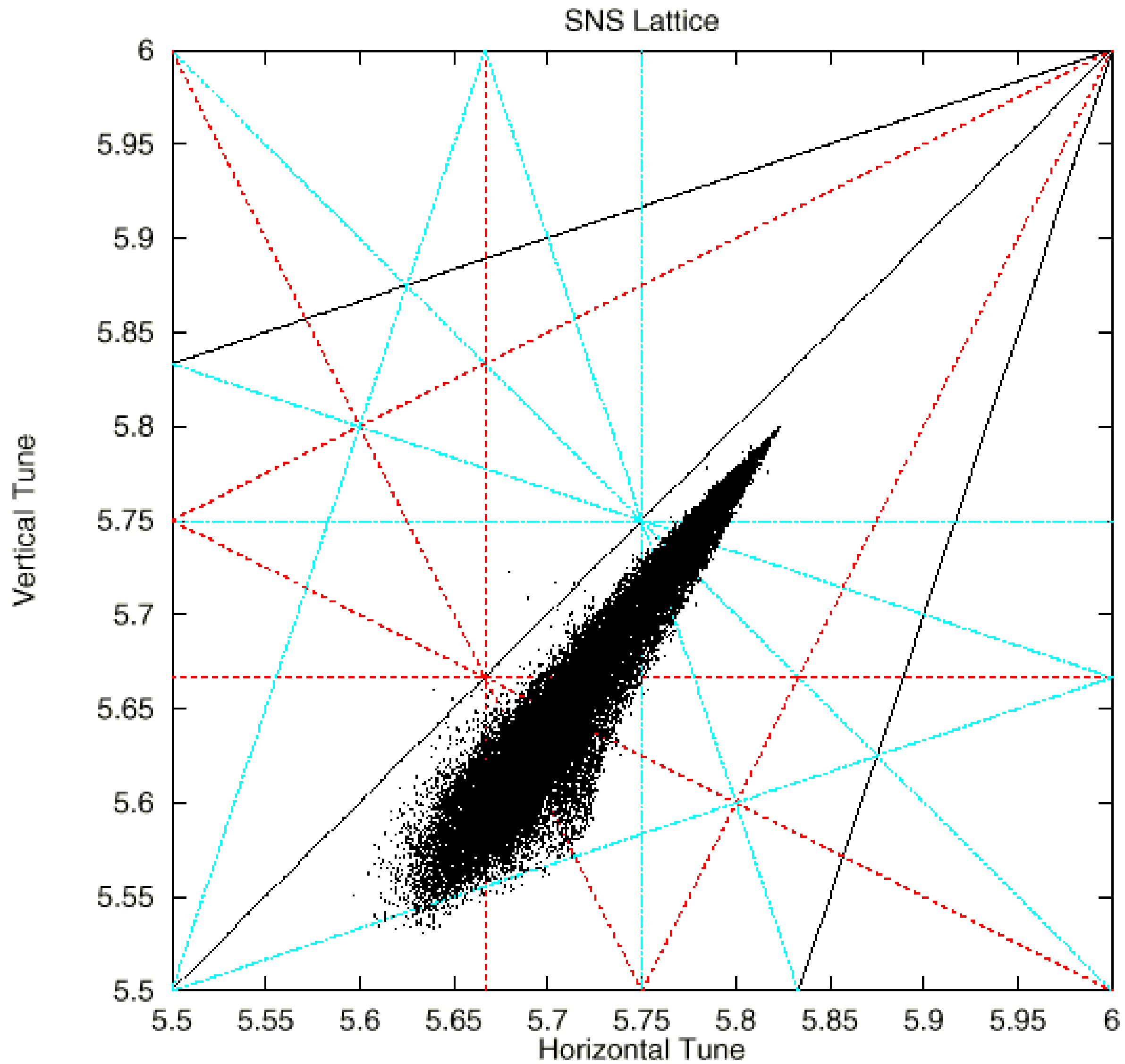
Q-Spread during Accumulation



Tune-spread “necktie” diagram during beam injection/accumulation in the SNS. Modelled using the ORBIT code with non-linear space-charge.

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