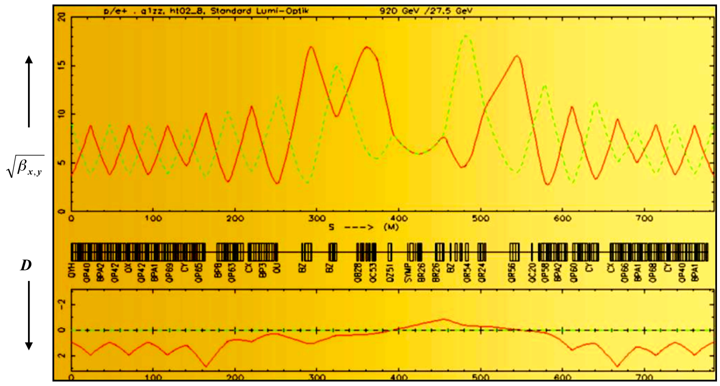
Lattice Design in Particle Accelerators

Bernhard Holzer

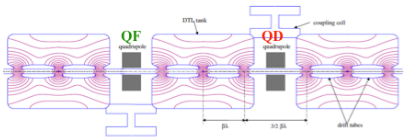


... in the context of space charge dominated beams

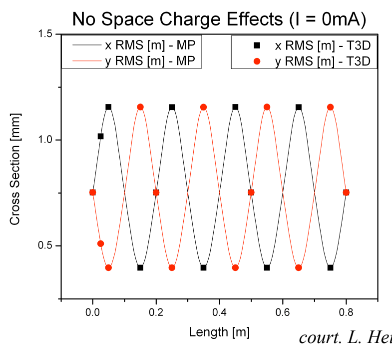
Lattice Cells and Space Charge: a practical example

Space Charge effect in a ring: Tune Shift $\Delta Q \propto \frac{I_p}{\varepsilon_{x,y}\,\beta^2\gamma^3}$

Space Charge effect in a Linac: distortion of the focusing effect



CERN Linac 4 design



court. L. Hein [2]

Kin. Energy ~ 3MeV , m_0 ~ 938 MeV

$$k_{s.c.} = 3*10^{-4}\ {}^{1}\!/\!{}_{m^2}$$

$$k_q = 5.2*10^{-1}\ {}^{1}\!/\!{}_{m^2}$$

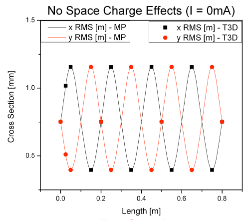
intergrating aling the linac we get: $\frac{\int k_{s.c.}dl}{\int k_q dl} \approx 75\%$

the space charge introduces an enourmous optical error that has to be compensated.

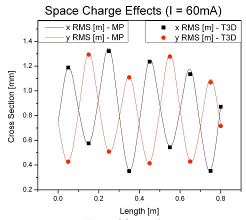
effect of quadrupole errors on the optics

tune shift
$$\Delta Q = \int_{s0}^{s0+l} \frac{\Delta k(s)\beta(s)ds}{4\pi}$$

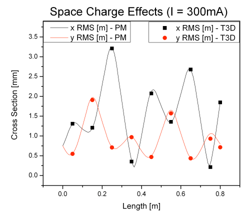
beta beat
$$\Delta\beta(s_0) = \frac{\beta_0}{2\sin 2\pi Q} \int_{s1}^{s1+l} \beta(s_1)\Delta K \cos\left(2\left|\psi_{s1} - \psi_{s0}\right| - 2\pi Q\right) ds$$



I = 0 mA



I = 60 mA



I = 300 mA

How to get the optical solution

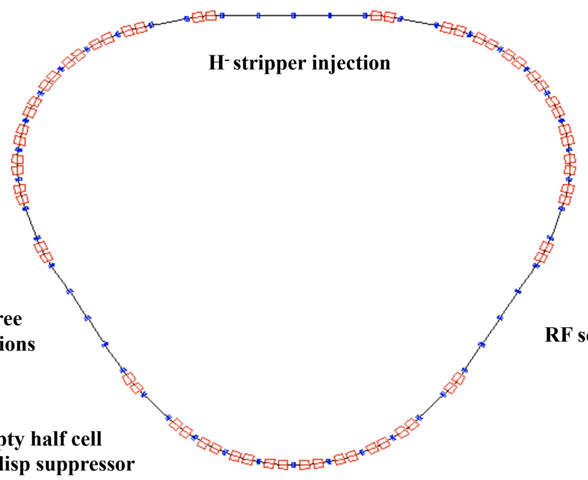
1. **Matching the focusing structure in order to obtain the ‚old‘ solution (for local spots only)**
2. **Find a new *periodical* solution, which includes the Space Charge Effects (Linac4)**
3. **Optimise the Lattice while minimising the losses (Linac4 TL)**

Lattice Design:

example: RCS (rapid cycling synchrotron)

regular arc structure

H^- stripper injection



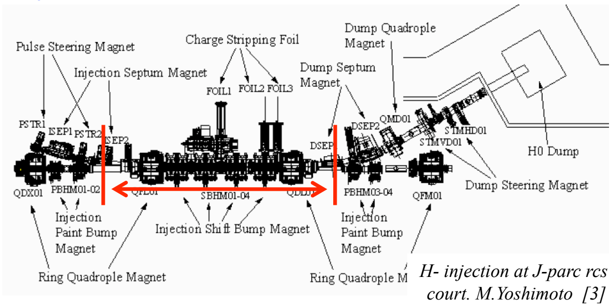
dispersion free straight sections

RF section

empty half cell -> disp suppressor

Lattice Design:

H⁻ Injection The Low Beta Insertion



H- injection at J-parc rcs
court. M.Yoshimoto [3]

Question to the audience:
what will happen to the beam parameters α, β, γ if we stop focusing for a while …?

$$\begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{s2} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2\\ -m_{11}m_{21} & m_{12}m_{21}+m_{22}m_{11} & -m_{12}m_{22}\\ m_{12}^2 & -2m_{22}m_{21} & m_{22}^2\end{pmatrix} * \begin{pmatrix}\beta\\ \alpha\\ \gamma\end{pmatrix}_{s1}$$

$$M = \begin{pmatrix} m_{11} & m_{12}\\ m_{21} & m_{22}\end{pmatrix} = \begin{pmatrix} 1 & l\\ 0 & 1\end{pmatrix}$$

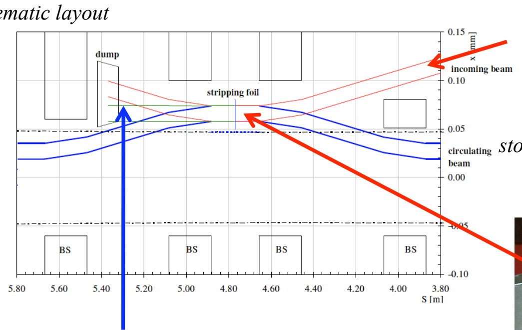
transfer matrix for a drift:

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$
$$\alpha(s) = \alpha_0 - \gamma_0 s$$
$$\gamma(s) = \gamma_0$$

„0“ refers to the position of the last lattice element
„s“ refers to the position in the drift

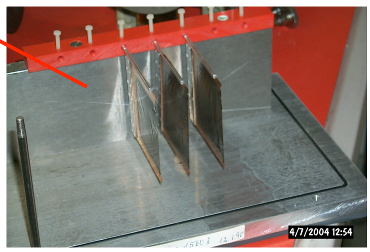
H⁻ Injection requirements: the Foil

schematic layout



H⁻ beam

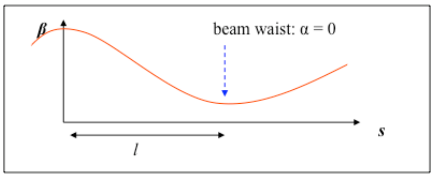
stored proton beam



1% H0, 0.1% H-, -> Absorber

Lattice Design: ***free space,***
dispersion free, symmetric waist,
α = 0, D = 0
local orbit bumps: **injection bump, phase space paint bump**

and let's start at the symmetry point:
in the center of a drift
$\alpha = 0$

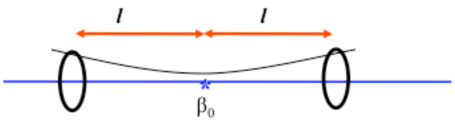


$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2$$

as $\quad \alpha_0 = 0, \quad \rightarrow \quad \gamma_0 = \frac{1+\alpha_0^2}{\beta_0} = \frac{1}{\beta_0}$

and we get for the β function in the neighborhood of the symmetry point

$$\boxed{\beta(\ell) = \beta_0 + \frac{\ell^2}{\beta_0}}$$



Nota bene:
1.) this is very bad !!!
2.) this is a direct consequence of the conservation of phase space density (... in our words: ε = const) … and there is no way out.
3.) Thank you, Mr. Liouville !!!

Find the β at the center of the drift that leads to the lowest maximum β at the end:

$$\frac{d\hat{\beta}}{d\beta_0} = 1 - \frac{\ell^2}{\beta_0^2} = 0 \qquad \boxed{\rightarrow \beta_0 = \ell \qquad \rightarrow \hat{\beta} = 2\beta_0}$$

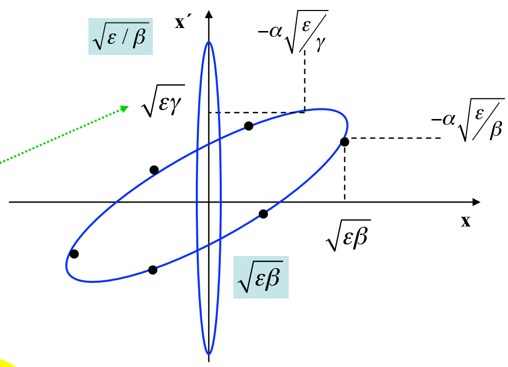
Low-β Insertions:... and how it looks in phase space

A mini-β insertion is always a kind of special symmetric drift space.
→greetings from Liouville

$$\alpha^* = 0$$

$$\gamma^* = \frac{1+\alpha^2}{\beta} = \frac{1}{\beta^*}$$

$$\sigma'^* = \sqrt{\frac{\varepsilon}{\beta^*}}$$



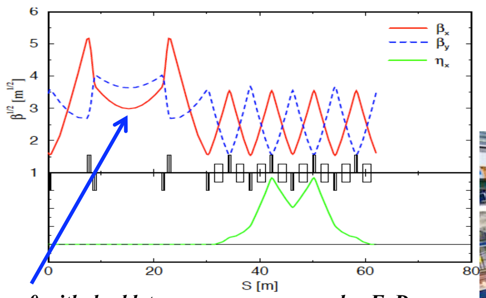
$$\beta^* = \frac{\sigma^*}{\sigma'^*}$$

at a symmetry point β is just the ratio of beam dimension and beam divergence.

The α = 0 Insertion:

How to create a low β insertion:

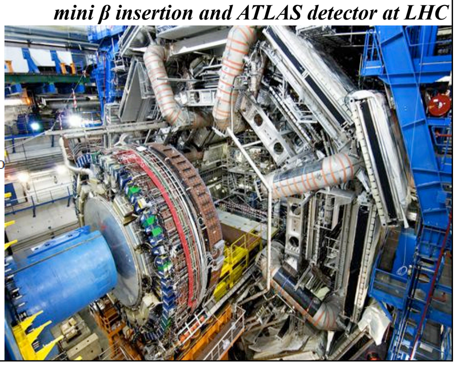
* ***symmetric drift space** (length adequate for the experiment)*
* ***quadrupole doublet (or triplet)** on each side (as close as possible)*
* ***additional quadrupole lenses to match twiss parameters** to the periodic cell in the arc*



α = 0 with doublets and free space for injection hardware

regular FoDo

mini β insertion and ATLAS detector at LHC



parameters to be optimised & matched to the periodic solution:

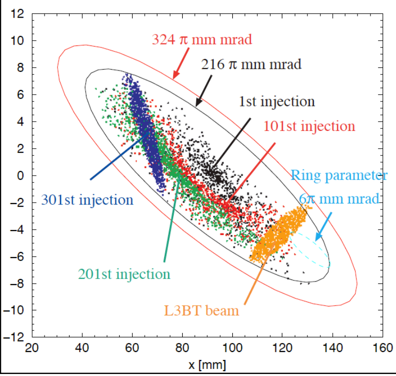
$\alpha_x, \alpha_y, D_x, D_x'$ $\quad \beta_x, \beta_y, Q_x, Q_y$

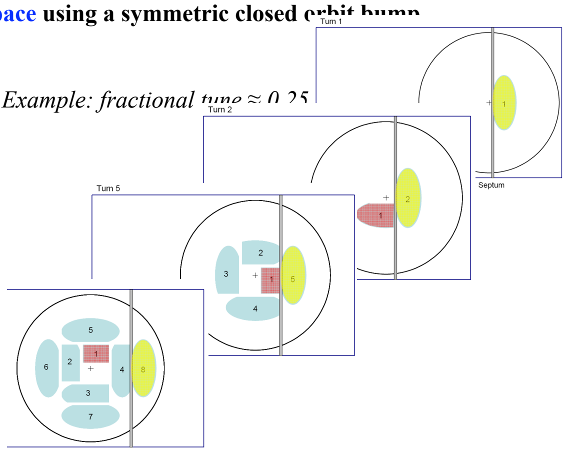
Lattice Design: **Multiturn Injection**

... and how it looks in phase space

Idea: Inject a beam of moderate intensity,
accumulate via H^- stripping injection
scan the transverse phase space using a symmetric closed orbit bump

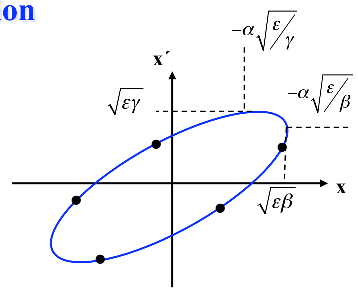
Example: fractional tune ≈ 0.25





J-Parc: Footprints of the circulating proton bunch *just a remark:* **Liouville during Acceleration**

$$\varepsilon = \gamma(s)\,x^2(s) + 2\alpha(s)x(s)x'(s) + \beta(s)\,x'^2(s)$$



Beam Emittance corresponds to the area covered in the x, x´ Phase Space Ellipse

Liouville: Area in phase space is constant.

But so sorry ... $\varepsilon \neq$ const !

Classical Mechanics:

phase space = diagram of the two canonical variables
position & momentum
x $\quad$ p_x

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad ; \quad L = T - V = kin.\,Energy - pot.\,Energy$$

According to Hamiltonian mechanics:
phase space diagram relates the variables q and p

q = position = x
p = momentum = γmv = mc$\gamma\beta_x$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \quad ; \quad \beta_x = \frac{\dot{x}}{c}$$

Liouvilles Theorem: $\int p\,dq = const$

for convenience (i.e. because we are lazy bones) we use in accelerator theory:

$$x' = \frac{dx}{ds} = \frac{dx}{dt}\frac{dt}{ds} = \frac{\beta_x}{\beta} \qquad \text{where } \beta_x = v_x / c$$

$$\int p\,dq = mc\int \gamma\beta_x dx$$

$$\int p\,dq = mc\gamma\beta \underbrace{\int x'dx}_{\varepsilon}$$

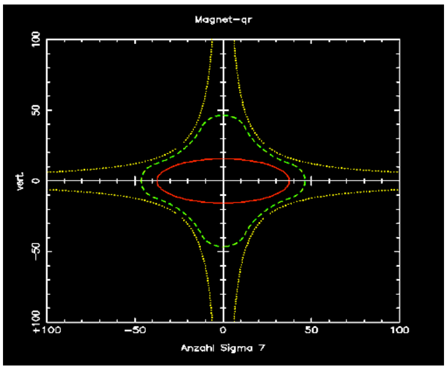
$$\Rightarrow \quad \varepsilon = \int x'dx \propto \frac{1}{\beta\gamma}$$

the beam emittance shrinks during acceleration $\varepsilon \sim 1/\gamma$

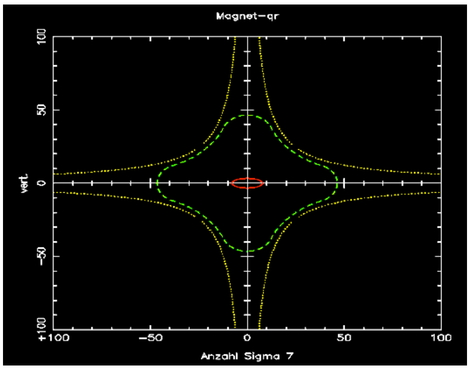
Example: HERA proton ring

injection energy: 40 GeV $\gamma = 43$
flat top energy: 920 GeV $\gamma = 980$

emittance ε (40GeV) $= 1.2 * 10^{-7}$
ε (920GeV) $= 5.1 * 10^{-9}$



7 σ beam envelope at E = 40 GeV



… and at E = 920 GeV

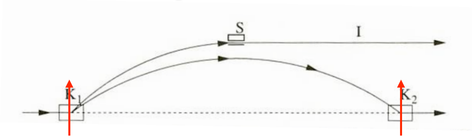
Lattice Design: Multiturn Injection

Deliberate Orbit Displacement: Bumps

create a configuration of orbit kicks to shift the closed orbit at a location in the ring …
… without changing the orbit anywhere else.

needed for injection, extraction, steering two beams in a collider with resp. to each other …

1.) the trivial bump: 2 coils



$$\Delta\phi = \pi$$

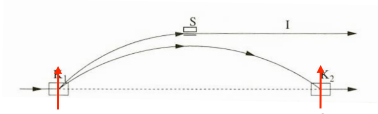
transformation matrix expressed in Twiss form: $\begin{pmatrix} x \\ x' \end{pmatrix}_2 = M * \begin{pmatrix} x \\ x' \end{pmatrix}_1$

(… see ground school)

$$M = \begin{pmatrix} \sqrt{\frac{\beta_2}{\beta_1}}(\cos(\psi_2-\psi_1)+\alpha_1\sin(\psi_2-\psi_1)) & \sqrt{\beta_1\beta_2}\sin(\psi_2-\psi_1) \\ \frac{(\alpha_1-\alpha_2)\cos(\psi_2-\psi_1)-(1+\alpha_1\alpha_2)\sin(\psi_2-\psi_1)}{\sqrt{\beta_1\beta_2}} & \sqrt{\frac{\beta_1}{\beta_2}}(\cos(\psi_2-\psi_1)-\alpha_2\sin(\psi_2-\psi_1)) \end{pmatrix}$$

for $\Delta\psi = 180°$ we get

$$\begin{pmatrix} x \\ x' \end{pmatrix}_2 = \begin{pmatrix} -\sqrt{\frac{\beta_2}{\beta_1}} & 0 \\ \frac{\alpha_2-\alpha_1}{\sqrt{\beta_1\beta_2}} & -\sqrt{\frac{\beta_1}{\beta_2}} \end{pmatrix} * \begin{pmatrix} 0 \\ \delta \end{pmatrix}_1$$



x = 0,
x´= δ$_1$

δ$_2$

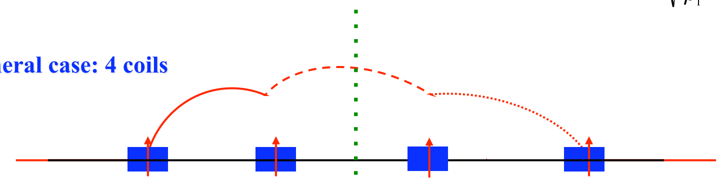
$$x_2 = 0$$

$$x_2' = -\sqrt{\frac{\beta_1}{\beta_2}} * \delta_1$$

the kick,of the second coil has to compensate the angle of the orbit at position s$_2$

$$\delta_2 = -x_2' = \sqrt{\frac{\beta_2}{\beta_1}} * \delta_1$$

2.) the general case: 4 coils



δ$_1$ δ$_2$ sp δ$_3$ δ$_4$

transformation from pos. 1 → 3: **transformation from pos. 2 → 3:**

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{sp} = M_{1->sp} * \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} + M_{2->sp} * \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix}, \qquad \begin{pmatrix} x \\ x' \end{pmatrix}_{sp} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} * \begin{pmatrix} 0 \\ \delta_1 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} * \begin{pmatrix} 0 \\ \delta_2 \end{pmatrix}$$

Symmetric Orbit Displacement: 4-Bumps

$$x_{sp} = a_{12} * \delta_1 + b_{12} * \delta_2$$
$$x'_{sp} = a_{22} * \delta_1 + b_{22} * \delta_2$$

$$\longrightarrow \quad \delta_1 = \frac{b_{22}x_{sp} - b_{12}x'_{sp}}{a_{12}b_{22} - a_{22}b_{12}}, \qquad \delta_2 = \frac{a_{12}x'_{sp} - a_{22}x_{sp}}{a_{12}b_{22} - a_{22}b_{12}}$$

using again

$$M_{1\to sp} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_{sp}}{\beta_1}}(\cos(\psi_{sp}-\psi_1) + \alpha_1 \sin(\psi_{sp}-\psi_1)) & \sqrt{\beta_1\beta_{sp}} \sin(\psi_{sp}-\psi_1) \\ \frac{(\alpha_1-\alpha_{sp})\cos(\psi_{sp}-\psi_1) - (1+\alpha_1\alpha_{sp})\sin(\psi_{sp}-\psi_1)}{\sqrt{\beta_1\beta_{sp}}} & \sqrt{\frac{\beta_1}{\beta_{sp}}}(\cos(\psi_{sp}-\psi_1) - \alpha_{sp}\sin(\psi_{sp}-\psi_1)) \end{pmatrix}$$

$$M_{2\to sp} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta_{sp}}{\beta_2}}(\cos(\psi_{sp}-\psi_2) + \alpha_2 \sin(\psi_{sp}-\psi_2)) & \sqrt{\beta_2\beta_{sp}} \sin(\psi_{sp}-\psi_2) \\ \frac{(\alpha_2-\alpha_{sp})\cos(\psi_{sp}-\psi_2) - (1+\alpha_2\alpha_{sp})\sin(\psi_{sp}-\psi_2)}{\sqrt{\beta_2\beta_{sp}}} & \sqrt{\frac{\beta_2}{\beta_{sp}}}(\cos(\psi_{sp}-\psi_2) - \alpha_{sp}\sin(\psi_{sp}-\psi_2)) \end{pmatrix}$$

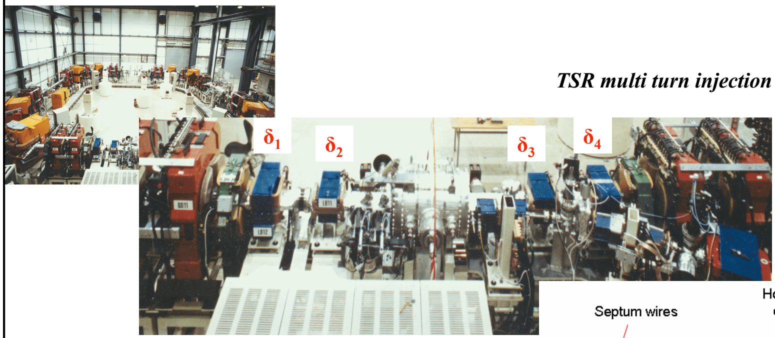
knowing that α_{sp}=0 ... and requiring that x'_{sp} =0 we get conditions for the kick strengths of the dipoles:

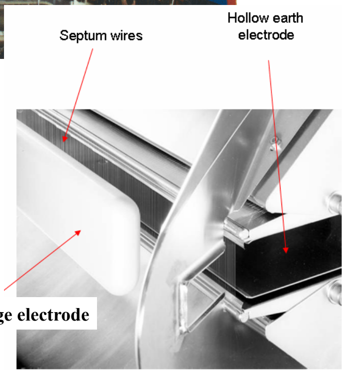
$$\delta_1 = \frac{1}{\sqrt{\beta_1\beta_{sp}}} * \frac{\cos(\psi_{sp}-\psi_2)}{\sin(\psi_2-\psi_1)} * x_{sp} \qquad \delta_2 = \frac{-1}{\sqrt{\beta_2\beta_{sp}}} * \frac{\cos(\psi_{sp}-\psi_1)}{\sin(\psi_2-\psi_1)} * x_{sp}$$

and for the kick strengths δ_3 and δ_4 in analog manner.

Symmetric Orbit Displacement: 4-Bumps

TSR multi turn injection bump

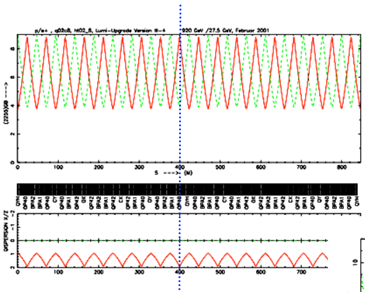




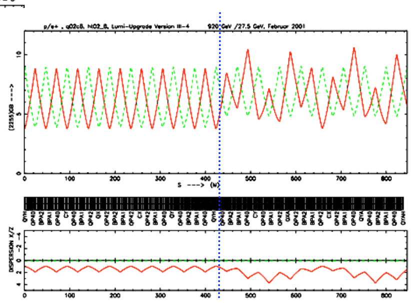
special requirements: multi turn injection
of heavy ions
smallest distance between stored & injected beam required
bump amplitude is reduced turn by turn for phase space painting

Linac4 / PSB: painting bump   = 100 µs
„dc"- bump = 5 ms

Transferline / Injection
optics conditions



Transferline: matched beam optics.
twiss parameters at start correspond to periodic Twiss



Transferline: un-matched beam optics
at half the way:

twiss parameters at start correspond to periodic Twiss
quadstrengths reduced by 20 % for second part

Transferline / Injection
optics conditions

- SPS to LHC transfer line TI 8 – beta functions

Twis parameters at start and end of the transfer line are fixed

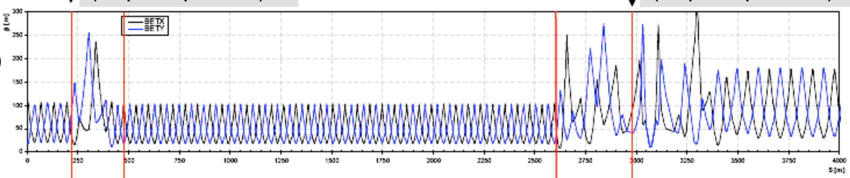
Extraction

$(\alpha_x,\beta_x,\alpha_y,\beta_y,D,D')^{ext}$

Injection

$(\alpha_x,\beta_x,\alpha_y,\beta_y, D,D')^{inj}$

β(m)



FoDo arc

SPS

FoDo transferline

LHC

$(\alpha_x(s),\beta_x(s),\alpha_y(s),\beta_y(s), D(s),D'(s))^{trans}$

Regular FODO lattice

Initial matching section

Final matching section

at least 6 individually powered quadrupoles needed

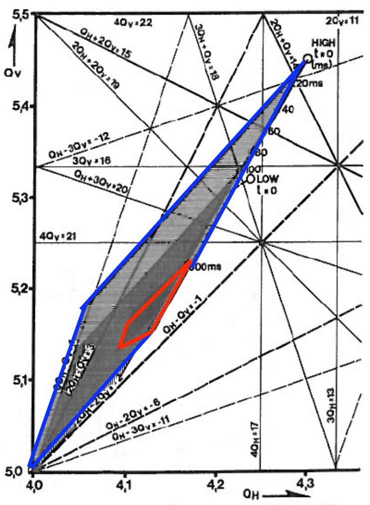
Lattice Design: Space charge effects in rings
the "neck-tie problem"

space charge tune shift: $\Delta Q_{sc} \propto \frac{N_p}{\beta^2\gamma^3}$

large range in working diagram required at low energy shrinking fast as acceleration starts

this is where the "R" comes from in RCS

→ **install trim quadrupoles to adjust the working point during acceleration**



court. K.H. Schindl [4]

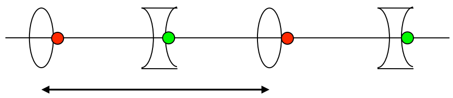
effect of quadrupole changes on the optics:

it changes the tune $\Delta Q = \frac{1}{4\pi}\int \Delta k * \beta ds$

but it affects also the optics $\frac{\Delta\beta}{\beta} = \frac{1}{2\sin 2\pi Q}\int \beta_t \Delta k \cos 2(\phi_t - \phi_s - \pi Q)dt$

Space charge effects in rings:

trim quadrupoles to reduce the "neck-tie problem"



use several (i.e. more than one) trim quadrupoles in the lattice ...
and put them ... 90^0 apart

$$\frac{\Delta\beta}{\beta} = \frac{1}{2\sin 2\pi Q}\int \{\beta_t \Delta k \cos 2(\phi_t - \phi_s - \pi Q) + \beta_t \Delta k \cos 2(\phi_t + 90° - \phi_s - \pi Q)\} dt$$

$$\frac{\Delta\beta}{\beta} = \frac{1}{2\sin 2\pi Q}\int \{\beta_t \Delta k \cos 2(\phi_t - \phi_s - \pi Q) + \beta_t \Delta k \cos 2(\phi_t - \phi_s - \pi Q + 90°)\} dt$$

$$\frac{\Delta\beta}{\beta} \approx \frac{\beta_t \Delta k l}{2\sin 2\pi Q}\{\cos 2(\phi_t - \phi_s - \pi Q) + \cos 2(\phi_t - \phi_s - \pi Q) * \underbrace{\cos(2*90°)}_{-1} - \sin 2(\phi_t - \phi_s - \pi Q) * \underbrace{\sin(2*90°)}_{0}\}$$

$$\boxed{\frac{\Delta\beta}{\beta} \approx 0}$$

we can install fast quadrupoles to adjust the tune on the flight without any effect on the optics !

Lattice Design: Dispersion, Trajectories for Δp / p ≠ 0

$$x'' + x\left(\frac{1}{\rho^2} - k\right) = \frac{\Delta p}{p} \cdot \frac{1}{\rho}$$

general solution:

$$x(s) = x_h(s) + x_i(s)$$

$$\begin{cases} x_h''(s) + K(s) \cdot x_h(s) = 0 \\ x_i''(s) + K(s) \cdot x_i(s) = \dfrac{1}{\rho} \cdot \dfrac{\Delta p}{p} \end{cases}$$

Normalise with respect to Δp/p:

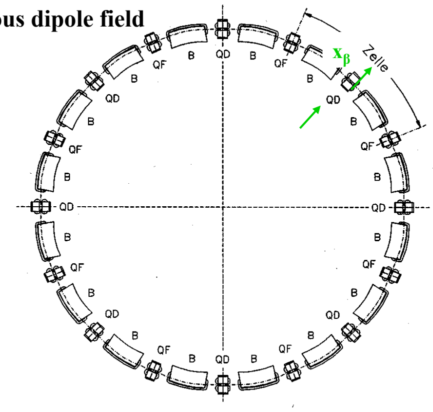
$$D(s) = \frac{x_i(s)}{\Delta p/p}$$

Dispersion function D(s)

* **is that special orbit, an ideal particle would have for Δp/p = 1**
* **the orbit of any particle is the sum of the well known x_β and the dispersion**
* **as D(s) is just another orbit it will be subject to the focusing properties of the lattice**

Dispersion

Example: homogeneous dipole field



oit for $\Delta p/p > 0$

$$: D(s)\cdot\frac{\Delta p}{p}$$

Matrix formalism:

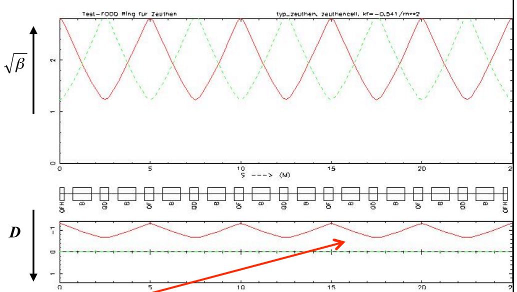
$$x(s) = x_\beta(s) + D(s)\cdot\frac{\Delta p}{p}$$

$$x(s) = m_{11}(s) * x_0 + m_{12}(s) * x_0^{'} + D(s)\frac{\Delta p}{p}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} * \begin{pmatrix} x \\ x' \end{pmatrix}_{s0} + \frac{\Delta p}{p}\begin{pmatrix} D \\ D' \end{pmatrix}$$

or expressed as 3x3 matrix

$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_s = \begin{pmatrix} m_{11} & m_{12} & D \\ m_{21} & m_{22} & D' \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_0$$



Example: ***longitudinal phase space painting***

$$x_\beta = 1...2\,mm$$
$$D(s) \approx 1...2\,m$$
$$\Delta p/p \approx 1.2\,MeV/160\,MeV = 0.4\%$$

$$\Delta x_{\Delta p/p} = 5.6\,mm$$

off energy beam still has to follow the same trajectory at the injection point
→ Dispersion must vanish at the collision point

Calculate D, D´: ... takes a couple of sunny Sunday evenings !

$$D(s) = m_{12}(s)\int_{s0}^{s1}\frac{1}{\rho}m_{11}(\tilde{s})d\tilde{s} - m_{11}(s)\int_{s0}^{s1}\frac{1}{\rho}m_{12}(\tilde{s})d\tilde{s}$$

Example: Drift

$$M_{Drift} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \qquad D(s) = m_{12}(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} m_{11}(\tilde{s})d\tilde{s}}_{=0} - m_{11}(s)\underbrace{\int_{s0}^{s1} \frac{1}{\rho} m_{12}(\tilde{s})d\tilde{s}}_{=0}$$

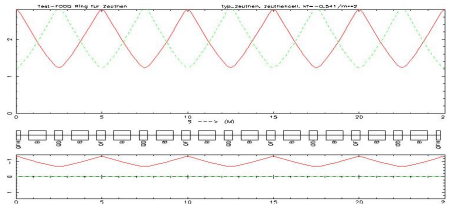
Example: Dipole

$$M_{foc} = \begin{pmatrix} \cos(\sqrt{|K|}s) & \frac{1}{\sqrt{|K|}}\sin(\sqrt{|K|}s) \\ -\sqrt{|K|}\sin(\sqrt{|K|}s) & \cos(\sqrt{|K|}s) \end{pmatrix}_0$$

$$K = \frac{1}{\rho^2} - \cancel{k}$$

$$s = l_B$$

$$M_{Dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix} \rightarrow \quad D(s) = \rho\cdot(1-\cos\frac{l}{\rho}), \quad D'(s) = \sin\frac{l}{\rho}$$



periodic dispersion in a FoDo cell including dipoles

$$\hat{D} = \frac{\ell^2}{\rho} * \frac{(1+\frac{1}{2}\sin\frac{\mu}{2})}{\sin^2\frac{\mu}{2}} \qquad \check{D} = \frac{\ell^2}{\rho} * \frac{(1-\frac{1}{2}\sin\frac{\mu}{2})}{\sin^2\frac{\mu}{2}}$$

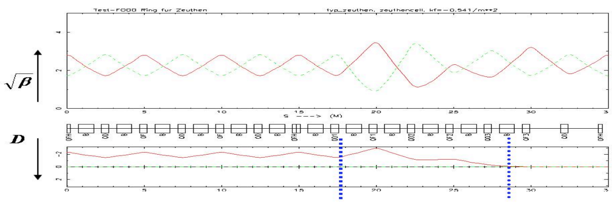
Lattice Design: Dispersion Suppressor Schemes

a.) The straight forward one: ***use additional quadrupole lenses to match the optical parameters ... including the D(s), D´(s) terms***

* Dispersion suppressed by 2 quadrupole lenses,

* β and α restored to the values of the periodic solution by 4 additional quadrupoles

$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \quad \begin{matrix} D(s), \quad D'(s) \\ \beta_x(s), \alpha_x(s) \\ \beta_y(s), \alpha_y(s) \end{matrix}$$



6 additional quadrupole lenses required

periodic FoDo structure — *matching section including 6 additional quadrupoles* — *dispersion free section, regular FoDo without dipoles*

! easy,

! flexible: it works for any phase advance per cell

! does not change the geometry of the storage ring,

! can be used to match between different lattice structures (i.e. phase advances)

! additional power supplies needed (→ expensive)

! requires stronger quadrupoles

! due to higher β values: more aperture required

b.) The clever one: ***missing bend suppressors***

… turn it the other way round:

Start with D = D' = 0 and create dispersion – using dipoles - in such a way, that it fits exactly the conditions at the centre of the first regular quadrupoles:

$$D(s) = S(s)\int_{s0}^{s1} \frac{1}{\rho} C(\tilde{s})\,d\tilde{s} - C(s)\int_{s0}^{s1} \frac{1}{\rho} S(\tilde{s})\,d\tilde{s} \longrightarrow D(s) = \hat{D}, \quad D'(s) = 0$$

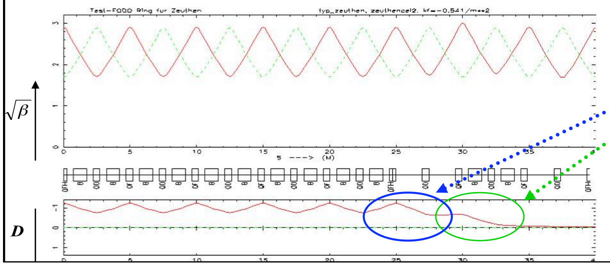
conditions for the (missing) dipole field scheme:

$$\frac{2m+n}{2}\Phi_C = (2k+1)\frac{\pi}{2}$$

$$\sin\frac{n\Phi_C}{2} = \frac{1}{2}\,,\ k = 0,2\ \dots\ or$$

$$\sin\frac{n\Phi_C}{2} = \frac{-1}{2}\,,\ k = 1,3\ \dots$$

m = number of cells without dipoles followed by n regular arc cells.



Example:

phase advance in the arc Φ_C = 60°
number of suppr. cells m = 1
number of regular cells n = 1

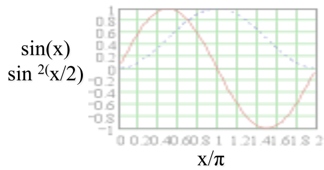
c.) The Half Bend Dispersion Suppressor

condition for vanishing dispersion:

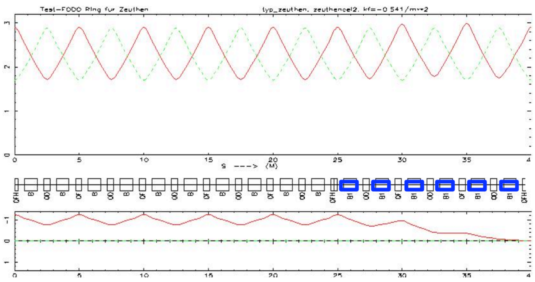
$$2*\delta_{supr}*\sin^2(\frac{n\Phi_c}{2}) = \delta_{arc}$$

so if we require $\delta_{supr} = \frac{1}{2}*\delta_{arc}$

we get $\sin^2(\frac{n\Phi_c}{2}) = 1$



or, which is equivalent $\sin(n\Phi_c) = 0 \qquad n\Phi_c = k*\pi, \qquad k = 1,3,\dots$



in the n suppressor cells the phase advance has to accumulate to a odd multiple of π

strength of suppressor dipoles is half as strong as that of arc dipoles, δ_{suppr} = 1/2 δ_{arc}

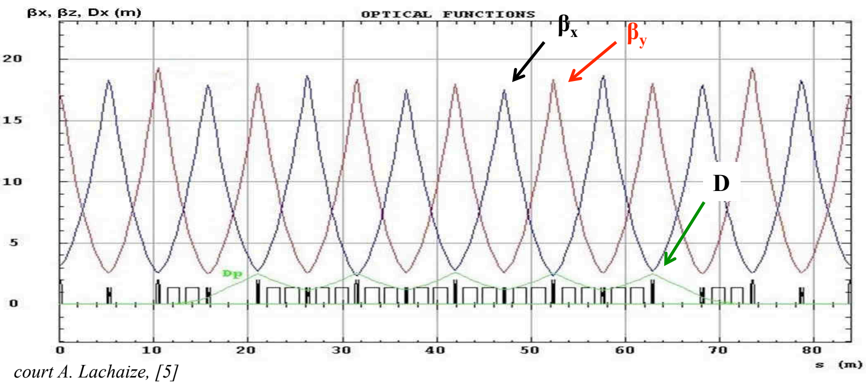
Example: phase advance in the arc Φ_C = 60°
number of suppr. cells n = 3

d.) ... and a very compact compromise

*** leave out the dipoles in the last half cell of the arc**
*** adjust (i.e. match) the arc quadrupoles with the constraint D = D' = 0 at the end of the structure**

-> no independent quads needed
-> Dispersion vanishes indeed
-> beta functions are modified over the complete ring, $\Phi_c = 101^0$

-> but very compact and cheep.



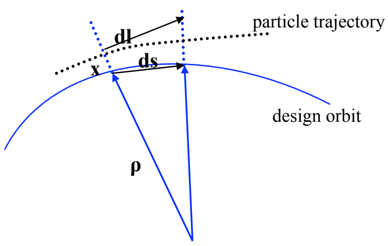
court A. Lachaize, [5]

Lattice Design: Momentum Compaction Factor & γ_{tr}

particle with a displacement x to the design orbit
→ path length dl ...

$$\frac{dl}{ds} = \frac{\rho + x}{\rho}$$

$$\rightarrow \; dl = \left(1 + \frac{x}{\rho(s)}\right) ds$$



circumference of an off-energy closed orbit

$$l_{\Delta E} = \oint dl = \oint \left(1 + \frac{x_{\Delta E}}{\rho(s)}\right) ds$$

remember:

$$x_{\Delta E}(s) = D(s)\frac{\Delta p}{p}$$

$$\delta l_{\Delta E} = \frac{\Delta p}{p} \oint \left(\frac{D(s)}{\rho(s)}\right) ds$$

*** The lengthening of the orbit for off-momentum particles is given by the dispersion function and the bending radius.**

Definition: $\frac{\delta l_\varepsilon}{L} = \alpha_p \frac{\Delta p}{p}$ $\rightarrow \alpha_p = \frac{1}{L}\oint\left(\frac{D(s)}{\rho(s)}\right)ds$

For first estimates assume: $\frac{1}{\rho} = const.$

$$\int_{dipoles} D(s)\,ds \approx l_{\Sigma(dipoles)} \cdot \langle D \rangle_{dipole}$$

$$\alpha_p = \frac{1}{L}\, l_{\Sigma(dipoles)} \cdot \langle D \rangle \frac{1}{\rho} = \frac{1}{L}\, 2\pi\rho \cdot \langle D \rangle \frac{1}{\rho} \quad \rightarrow \quad \alpha_p \approx \frac{2\pi}{L}\langle D \rangle \approx \frac{\langle D \rangle}{R}$$

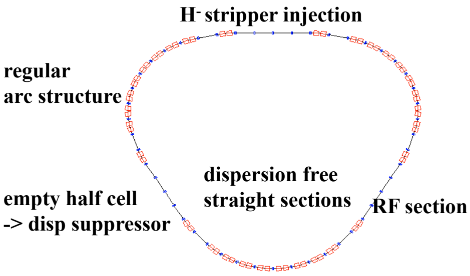
... and now the key point:

$\gamma_{tr}^2 = \frac{1}{\alpha_p}$

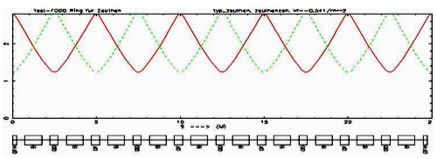
small dispersion $\longrightarrow$ **small** α_p $\longrightarrow$ **large** γ_{tr}

Resume: an example

RCS (rapid cycling synchrotron)



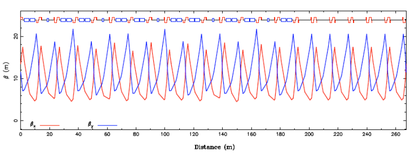
Resume:lattices



FoDo

small gradients, smooth optical functions, very flexible & robust

no long straight sections



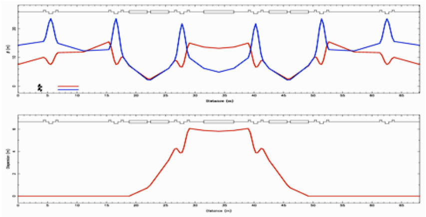
Doublet structure

made for longer straight sections

higher gradients, larger betas

court. G.H. Rees, [6]

Resume: Lattice structures

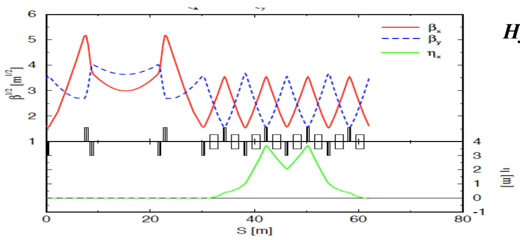


court. G.H. Rees, [6]

Triplet structure

long straights with / without dispersion

unregular betas, higher gradients



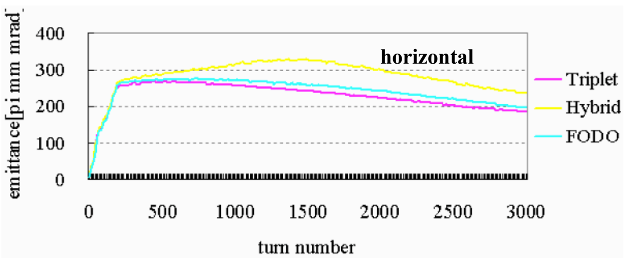
court. J. Wei, [7]

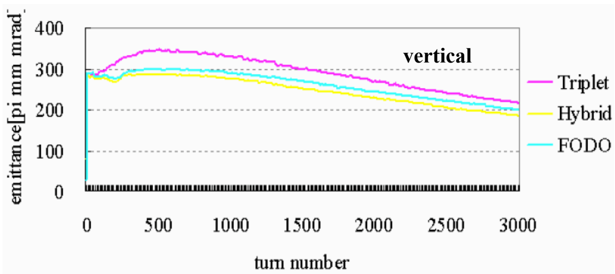
Hybrid: FoDo & Doublet

long straights with / without dispersion

smooth & regular betas, low gradients

Resume: Lattice structures and emittance preservation





court. S. Xu, [9]

Special Acknowledgments
and recommendations for further reading

A. Lombardi, — *[1]"Beam dynamics in Linac 4 at CERN"*

L. Hein, — *[2]"Space charge effects in FoDo structures", priv. com.*

M. Yoshimoto et al, — *[3]"Present status of injection and extraction system of 3 GeV RCS at J-PARC"*

K.H. Schindl — *[4] "Space Charge", CAS report, CERN 2006-02*

A. Lachaize et al. — *[5] "Design of low energy rings"*

G.H. Rees, — *[6] "options for a 50Hz,10MW, short pulse spallation source", "lattices for 8 and 30 GeV proton drivers"*

J. Wei et al, — *[7] "FODO/Doublet Lattice for the SNS accumulator ring"*

C. Prior, — *[8] "The synchrotron option for a multi-megawatt proton driver"*

S. Xu, S. Wang, S. Fang: — *[9] "Study of space charge effects for RCS/ CSNS"*

Appendix I: Dispersion: Solution of the Inhomogenious Equation of Motion

the dispersion function is given by

$$D(s) = S(s)*\int \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C(s)*\int \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}$$

proof:

$$D'(s) = S'(s)*\int \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} + S(s)*\frac{C(\tilde{s})}{\rho(\tilde{s})} - C'(s)*\int \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s} - C(s)\frac{S(\tilde{s})}{\rho(\tilde{s})}$$

$$D'(s) = S'(s)*\int \frac{C}{\rho} d\tilde{s} - C'(s)*\int \frac{S}{\rho} d\tilde{s}$$

$$D''(s) = S''(s)*\int \frac{C}{\rho} d\tilde{s} + S'\frac{C}{\rho} - C''(s)*\int \frac{S}{\rho} d\tilde{s} - C'\frac{S}{\rho}$$

$$D''(s) = S''(s)*\int \frac{C}{\rho} d\tilde{s} - C''(s)* + \frac{1}{\rho})\underbrace{(CS' - SC')}_{= \det(M) = 1}$$

$$D''(s) = S''(s)*\int \frac{C}{\rho} d\tilde{s} - C''(s)*\int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

now the principal trajectories S and C fulfill the homogeneous equation

$$S''(s) = -K*S \quad , \qquad C''(s) = -K*C$$

and so we get:

$$D''(s) = -K * S(s) * \int \frac{C}{\rho} d\tilde{s} + K * C(s) * \int \frac{S}{\rho} d\tilde{s} + \frac{1}{\rho}$$

$$D''(s) = -K * D(s) + \frac{1}{\rho}$$

$$D''(s) + K * D(s) = \frac{1}{\rho}$$

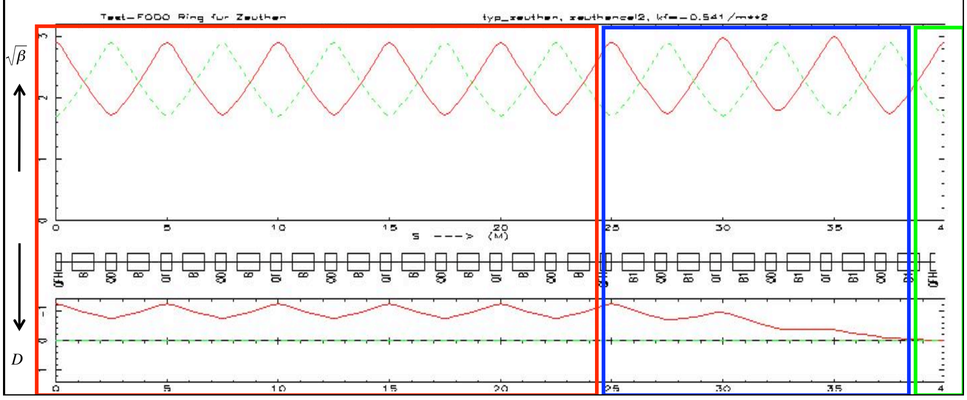
qed.

Appendix II: Dispersion Suppressors

... the calculation of the half bend scheme in full detail (for purists only)

1.) the lattice is split into 3 parts: ***(Gallia divisa est in partes tres)***

* periodic solution of the arc	periodic β, periodic dispersion D
* section of the dispersion suppressor	periodic β, dispersion vanishes
* FoDo cells without dispersion	periodic β, D = D′ = 0



2.) calculate the dispersion D in the periodic part of the lattice

transfer matrix of a periodic cell:

$$M_{0\to S} = \begin{pmatrix} \sqrt{\frac{\beta_S}{\beta_0}}(\cos\phi + \alpha_0 \sin\phi) & \sqrt{\beta_S\beta_0}\sin\phi \\ \frac{(\alpha_0 - \alpha_S)\cos\phi - (1+\alpha_0\alpha_S)\sin\phi}{\sqrt{\beta_S\beta_0}} & \sqrt{\frac{\beta_S}{\beta_0}}(\cos\phi - \alpha_S \sin\phi) \end{pmatrix}$$

for the transformation from one symmetriy point to the next (i.e. one cell) we have:
Φ_C = phase advance of the cell, $\alpha = 0$ at a symmetry point. The index "*c*" refers to the periodic solution of one cell.

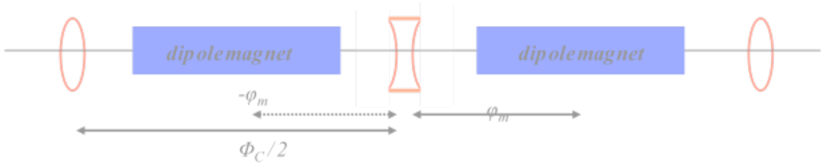
$$M_{Cell} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\Phi_C & \beta_C \sin\Phi_C & D(l) \\ \frac{-1}{\beta_C}\sin\Phi_C & \cos\Phi_C & D'(l) \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix elements D and D' are given by the C and S elements in the usual way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}$$

$$D'(l) = S'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C'(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}$$

here the values C(*l*) and S(*l*) refer to the symmetry point of the cell (middle of the quadrupole) and the integral is to be taken over the dipole magnet where $\rho \neq 0$. For ρ = const the integral over C(s) and S(s) is approximated by the values in the middle of the dipole magnet.



Transformation of C(s) from the symmetry point to the center of the dipole:

$$C_m = \sqrt{\frac{\beta_m}{\beta_C}}\cos\Delta\Phi = \sqrt{\frac{\beta_m}{\beta_C}}\cos(\frac{\Phi_C}{2} \pm \varphi_m) \qquad S_m = \beta_m\beta_C \sin(\frac{\Phi_C}{2} \pm \varphi_m)$$

where β_C is the periodic β function at the beginning and end of the cell, β_m its value at the middle of the dipole and φ_m the phase advance from the quadrupole lens to the dipole center.

Now we can solve the intergal for D and D':

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}$$

$$D(l) = \beta_C \sin\Phi_C * \frac{L}{\rho} * \sqrt{\frac{\beta_m}{\beta_C}} * \cos(\frac{\Phi_C}{2} \pm \varphi_m) - \cos\Phi_C * \frac{L}{\rho}\sqrt{\beta_m\beta_C} * \sin(\frac{\Phi_C}{2} \pm \varphi_m)$$

$$D(l) = \delta\sqrt{\beta_m \beta_C}\left\{\sin\Phi_C\left[\cos(\frac{\Phi_C}{2}+\varphi_m)+\cos(\frac{\Phi_C}{2}-\varphi_m)\right]-\right.$$

$$\left.-\cos\Phi_C\left[\sin(\frac{\Phi_C}{2}+\varphi_m)+\sin(\frac{\Phi_C}{2}-\varphi_m)\right]\right\}$$

I have put $\delta = L/\rho$ for the strength of the dipole

remember the relations $\cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$D(l) = \delta\sqrt{\beta_m \beta_C}\left\{\sin\Phi_C * 2\cos\frac{\Phi_C}{2} * \cos\varphi_m - \cos\Phi_C * 2\sin\frac{\Phi_C}{2} * \cos\varphi_m\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m\left\{\sin\Phi_C * \cos\frac{\Phi_C}{2} * - \cos\Phi_C * \sin\frac{\Phi_C}{2}\right\}$$

remember: $\sin 2x = 2\sin x * \cos x$

$\cos 2x = \cos^2 x - \sin^2 x$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m\left\{2\sin\frac{\Phi_C}{2} * \cos^2\frac{\Phi_C}{2} - (\cos^2\frac{\Phi_C}{2} - \sin^2\frac{\Phi_C}{2}) * \sin\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}\left\{2\cos^2\frac{\Phi_C}{2} - \cos^2\frac{\Phi_C}{2} + \sin^2\frac{\Phi_C}{2}\right\}$$

$$D(l) = 2\delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * \sin\frac{\Phi_C}{2}$$

in full analogy one derives the expression for D‘:

$$\boldsymbol{D'(l)} = 2\delta\sqrt{\boldsymbol{\beta_m / \beta_c}} * \cos\boldsymbol{\varphi_m} * \cos\frac{\boldsymbol{\Phi_c}}{2}$$

As we refer the expression for D and D‘ to a periodic struture, namely a FoDo cell we require periodicity conditons:

$$\begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix} = M_C * \begin{pmatrix} D_C \\ D'_C \\ 1 \end{pmatrix}$$

and by symmetry: $D'_C = 0$

With these boundary conditions the Dispersion in the FoDo is determined:

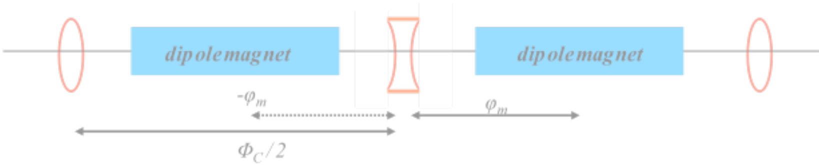
$$D_C * \cos\Phi_C + \delta\sqrt{\beta_m \beta_C} * \cos\varphi_m * 2\sin\frac{\Phi_C}{2} = D_C$$

(A1)
$$D_C = \delta\sqrt{\beta_m\beta_C} * \cos\varphi_m / \sin\frac{\Phi_C}{2}$$

This is the value of the periodic dispersion in the cell evaluated at the position of the dipole magnets.

3.) Calculate the dispersion in the suppressor part:

We will now move to the second part of the dispersion suppressor: The section where ... starting from D=D‘=0 the dispesion is generated ... or turning it around where the Dispersion of the arc is reduced to zero.
The goal will be to generate the dispersion in this section in a way that the values of the periodic cell that have been calculated above are obtained.



The relation for D, generated in a cell still holds in the same way:

$$D(l) = S(l) * \int_0^l \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(l) * \int_0^l \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

as the dispersion is generated in a number of *n* cells the matrix for these *n* cells is

$$M_n = M_C^n = \begin{pmatrix} \cos n\Phi_C & \beta_C \sin n\Phi_C & D_n \\ \frac{-1}{\beta_C}\sin n\Phi_C & \cos n\Phi_C & D'_n \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_n = \beta_C \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m) * \sqrt{\frac{\beta_m}{\beta_C}} - \\ - \cos n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \sqrt{\beta_m\beta_C} * \sin(i\Phi_C - \frac{1}{2}\Phi_C \pm \varphi_m)$$

$$D_n = \sqrt{\beta_m\beta_C} * \sin n\Phi_C * \delta_{\sup r} * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2} \pm \varphi_m) - \sqrt{\beta_m\beta_C} * \delta_{\sup r} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2} \pm \varphi_m)$$

remember: $\sin x + \sin y = 2\sin\frac{x+y}{2} * \cos\frac{x-y}{2}$ $\qquad \cos x + \cos y = 2\cos\frac{x+y}{2} * \cos\frac{x-y}{2}$

$$D_n = \delta_{\sup r} * \sqrt{\beta_m\beta_C} * \sin n\Phi_C * \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m - \\ -\delta_{\sup r} * \sqrt{\beta_m\beta_C} * \cos n\Phi_C \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * 2\cos\varphi_m$$

$$D_n = 2\delta_{supr} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sum_{i=1}^{n} \cos((2i-1)\frac{\Phi_C}{2}) * \sin n\Phi_C - \sum_{i=1}^{n} \sin((2i-1)\frac{\Phi_C}{2}) * \cos n\Phi_C \right\}$$

$$D_n = 2\delta_{supr} * \sqrt{\beta_m \beta_C} * \cos\varphi_m \left\{ \sin n\Phi_C \left\{ \frac{\sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} - \cos n\Phi_C * \left\{ \frac{\sin\frac{n\Phi_C}{2} * \sin\frac{n\Phi_C}{2}}{\sin\frac{\Phi_C}{2}} \right\} \right\}$$

$$D_n = \frac{2\delta_{supr} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin n\Phi_C * \sin\frac{n\Phi_C}{2} * \cos\frac{n\Phi_C}{2} - \cos n\Phi_C * \sin^2\frac{n\Phi_C}{2} \right\}$$

set for more convenience $x = n\Phi_C/2$

$$D_n = \frac{2\delta_{supr} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ \sin 2x * \sin x * \cos x - \cos 2x * \sin^2 x \right\}$$

$$D_n = \frac{2\delta_{supr} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} \left\{ 2\sin x \cos x * \cos x \sin x - (\cos^2 x - \sin^2 x)\sin^2 x \right\}$$

(A2)

$$D_n = \frac{2\delta_{supr} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2}$$

and in similar calculations:

$$D'_n = \frac{2\delta_{supr} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin n\Phi_C$$

This expression gives the dispersion generated in a certain number of *n* cells as a function of the dipole kick δ in these cells.
At the end of the dispersion generating section the value obtained for D(s) and D‘(s) has to be equal to the value of the periodic solution:

→equating *(A1)* and *(A2)* gives the conditions for the matching of the periodic dispersion in the arc to the values D = D‘= 0 afte the suppressor.

$$D_n = \frac{2\delta_{supr} * \sqrt{\beta_m \beta_C} * \cos\varphi_m}{\sin\frac{\Phi_C}{2}} * \sin^2\frac{n\Phi_C}{2} = \delta_{arc}\sqrt{\beta_m \beta_C} * \frac{\cos\varphi_m}{\sin\frac{\Phi_C}{2}}$$

$$\left.\begin{array}{l}\rightarrow 2\delta_{\sup r}\sin^2(\frac{n\Phi_C}{2}) = \delta_{arc} \\ \rightarrow \sin(n\Phi_C) = 0\end{array}\right\} \delta_{\sup r} = \frac{1}{2}\delta_{arc}$$

and at the same time the phase advance in the arc cell has to obey the relation:

$$n\Phi_C = k*\pi,\ k = 1,3,\ ...$$