Multi-Particle Effects: Space Charge

Karlheinz SCHINDL - CERN/AB

Direct space charge (Self fields)
- Fields and forces
- Defocusing effect of space charge
- Incoherent tune shift in a synchrotron

Image fields
- Image effect on incoherent tune shift
- Coherent tune shift
- “Laslett” coefficients

Bunched beams
- Effect of longitudinal motion
- Space-charge limited synchrotrons
- How to remove the space-charge limit

P.J. Bryant, Betatron frequency shifts due to self and image fields, CAS Aarhus 1986, CERN 87-10, p. 62
Space Charge Force

Two Particles

- Coulomb repulsion
- Magnetic attraction

Many Particles

- Force in beam centre = 0
- Force larger near beam edge

- Force in beam centre = 0
- Force larger near beam edge
**Direct Space Charge - Fields**

- $\eta$...charge density in Cb/m$^3$
- $\lambda$...constant line charge $\pi a^2 \eta$
- $I$...constant current $\lambda \beta c = \pi a^2 \eta \beta c$
- $a$...beam radius

**Electric**

- $\vec{E} = E_r$
- $\text{div} \vec{E} = \frac{\eta}{\varepsilon_0}$

**Magnetic**

- $\vec{B} = B_\phi$
- $\text{curl} \vec{B} = \mu_0 \vec{J}$

Current density ($\beta c \eta$)

\[ \iiint \text{div} \vec{E} \, dV = \iint \vec{E} \, dS \quad \oint B \, r \, d\phi = \iint \text{curl} \vec{B} \, dA \]

Apply these integrals over

- cylinder radius $r$
- length $l$
- cross section radius $r$

\[ r^2 \pi l \frac{\eta}{\varepsilon_0} = E_r 2r\pi l \]

\[ B_\phi 2\pi r = \mu_0 r^2 \pi \beta c \eta \]

\[ E_r = \frac{I}{2\pi \varepsilon_0 \beta c} \frac{r}{a^2} \]

\[ B_\phi = \frac{I}{2\pi \varepsilon_0 c^2} \frac{r}{a^2} \]
**Force on a Test Particle Inside the Beam**

\[ F = e (E + v \times B) \]
\[ F_r = e (E_r - v_s B_\phi) \]
\[ F_r = \frac{eI}{2 \pi \epsilon_0 c} \left(1 - \beta^2\right) \frac{r}{a^2} = \frac{eI}{2 \pi \epsilon_0 c \gamma^2} \frac{1}{a^2} \frac{r}{\gamma^2} \]

- **Space charge force**
- **Circular beam**
- **Uniform charge density**
- \( F_x, F_y \) linear in \( x, y \)
- **Force** \( \to 0 \) for \( \gamma \gg 1 \) (\( \beta \approx 1 \))
- **Defocusing lens** in either plane
Space Charge in a Transport Line

In a transport line, the focusing by quadrupoles is counteracted by space charge, making focusing weaker.

\[ x'' + K(s)x = 0 \]

\[ x'' + (K(s) + K_{SC}(s))x = 0 \]

Transport line with quadrupoles

Transport line with quadrupoles and space charge

\[ x'' = \frac{d^2x}{ds^2} = \frac{1}{\beta^2 c^2} \frac{d^2x}{dt^2} = \frac{1}{\beta^2 c^2} \frac{F_x}{m_0 \gamma} = \frac{2r_0 I}{ea^2 \beta^3 \gamma^3 c} x \quad \text{where} \quad r_0 = \frac{e^2}{4 \pi \varepsilon_0 m_0 c^2} \]

\[ x'' + \left( K(s) - \frac{2r_0 I}{ea^2 \beta^3 \gamma^3 c} \right) x = 0 \]
Incoherent Tune Shift in a Synchrotron

- Beam not bunched (so no acceleration)
- Uniform density in the circular x-y cross section (not very realistic)

\[ x'' + (K(s) + K_{SC}(s))x = 0 \Rightarrow Q_{x0} \text{ (external)} + \Delta Q_x \text{ (space charge)} \]

For small "gradient errors" \( k_x \)

\[ \Delta Q_x = \frac{1}{4\pi} \int_0^{2R\pi} k_x(s)\beta_x(s)ds = \frac{1}{4\pi} \int_0^{2R\pi} K_{SC}(s)\beta_x(s)ds \]

\[ \Delta Q_x = -\frac{1}{4\pi} \int_0^{2R\pi} \frac{2r_0I}{e\beta^3\gamma^3c} \frac{\beta_x(s)}{a^2} ds = -\frac{r_0RI}{e\beta^3\gamma^3c} \left( \frac{\beta_x(s)}{a^2(s)} \right) = -\frac{r_0RI}{e\beta^3\gamma^3cE_x} \]

\[ \Delta Q_{x,y} = -\frac{r_0N}{2\pi E_{x,y} \beta^2\gamma^3} \]

using \( I = (Ne\beta c)/(2R\pi) \) with

\( N \ldots \text{number of particles in ring} \)

\( E_{x,y} \ldots \text{emittance containing 100% of particles} \)

- "Direct" space charge, unbunched beam in a synchrotron
- Vanishes for \( \gamma \gg 1 \)
- Important for low-energy machines
- Independent of machine size \( 2\pi R \) for a given \( N \)
Incoherent Tune Shift: Image Effects

Image (line) charges created by two parallel conducting plates, distance 2h

Electric field around a line charge

\[ E_y = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{d} \]

Image Field \( E_{iy} \) generated by the n-th pair of line charges

\[ E_{iy} = \frac{(-1)^{n+1} \lambda}{2\pi\varepsilon_0} \left( \frac{1}{2nh-y} - \frac{1}{2nh+y} \right) = \frac{(-1)^{n+1} \lambda}{4\pi\varepsilon_0} \frac{y}{n^2h^2} \]

\[ E_{iy} = \frac{\lambda}{2\pi\varepsilon_0} \left( \frac{1}{2h-y} - \frac{1}{2h+y} \right) \]

\[ E_{iy} = \frac{\lambda}{2\pi\varepsilon_0} \left( \frac{1}{4h+y} - \frac{1}{4h-y} \right) \]
Image Effect of Parallel Conducting Plates ctd.

$$E_{iy} = \sum_{n=1}^{\infty} E_{iny} = \frac{\lambda}{4\pi\varepsilon_0 h^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} y = \frac{\lambda}{4\pi\varepsilon_0 h^2} \frac{\pi^2}{12} y$$

Vertical image field $E_{iy}$:
- vanishes at $y=0$
- linear in $y$
- vertical defocusing
- large if vacuum chamber small (small $h$)

$$\text{div} \bar{E}_i = 0 = \frac{\partial E_{ix}}{\partial x} + \frac{\partial E_{iy}}{\partial y} \Rightarrow E_{ix} = -\frac{\lambda}{4\pi\varepsilon_0 h^2} \frac{\pi^2}{12} x$$

because between the conducting plates no image charges

$$F_{ix} = -\frac{e\lambda}{\pi\varepsilon_0 h^2} \frac{\pi^2}{48} x$$

From these image forces $F_{ix}$ and $F_{iy}$ \(\Rightarrow K_{SC} \Rightarrow \Delta Q_{x,y}\)

Vertical image field $E_{iy}$:
- vanishes at $y=0$
- linear in $y$
- vertical defocusing
- large if vacuum chamber small (small $h$)

$$\Delta Q_x = -\frac{2r_0 IR \langle \beta_x \rangle}{ec\beta^3 \gamma} \left( \frac{1}{2\langle a^2 \rangle \gamma^2} - \frac{\pi^2}{48h^2} \right)$$

**tune shift**
- **direct**
- **image**

$$\Delta Q_y = -\frac{2r_0 IR \langle \beta_y \rangle}{ec\beta^3 \gamma} \left( \frac{1}{2\langle b^2 \rangle \gamma^2} + \frac{\pi^2}{48h^2} \right)$$

- Image effects do not vanish for large $\gamma$, thus **not negligible** for electron machines
- **Electrical** image effects normally focusing in horizontal, defocusing in vertical plane
- Image effects also due to ferromagnetic boundary (e.g. synchrotron magnets)
Incoherent and Coherent Motion

Incoherent motion

- Test particle in a beam whose centre of mass does not move
- The beam environment does not "see" any motion
- Each particle features its individual amplitude and phase

Coherent motion

- The centre of mass moves doing betatron oscillation as a whole
- The beam environment (e.g. a position monitor "sees" the "coherent motion")
- On top of the coherent motion, each particle has still its individual one
Coherent Tune Shift, Round Beam Pipe

\[ E_{ix}(\bar{x}) = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{b - \bar{x}} \approx \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{b \rho^2} = \frac{\lambda}{2\pi\varepsilon_0} \frac{1}{\rho^2} \bar{x} \]

\[ F_{ix}(\bar{x}) = \frac{e\lambda}{2\pi\varepsilon_0} \frac{1}{\rho^2} \bar{x} \]

\[ \Delta Q_{x,y,coh} = -\frac{r_0 R \langle \beta_{x,y} \rangle I}{ec\beta^3 \gamma \rho^2} = -\frac{r_0 \langle \beta_{x,y} \rangle N}{2\pi\beta^2 \gamma \rho^2} \]

- Coherent tune shift, round pipe
- Coherent tune shift, round pipe
- same in vertical plane (y) due to symmetry
- force linear in \( \bar{x} \)
- force positive hence defocusing in both planes
- negative (defocusing) both planes
- only weak dependence on \( \gamma \)
- \( \Delta Q_{coh} \) always negative
The “Laslett”* Coefficients

\[
\Delta Q_{y,\text{inc}} = -\frac{N_{r_0} \langle \beta \rangle}{\beta^2 \gamma \pi} \left( \frac{\varepsilon_0^y}{b^2} + \frac{\varepsilon_1^y}{h^2} + \beta^2 \frac{\varepsilon_2^y}{g^2} \right)
\]

\[
\Delta Q_{y,\text{coh}} = -\frac{N_{r_0} \langle \beta \rangle}{\beta^2 \gamma \pi} \left( \frac{\xi_1^y}{h^2} + \beta^2 \frac{\xi_2^y}{g^2} \right)
\]

<table>
<thead>
<tr>
<th>Laslett coefficients</th>
<th>Circular ((a = b, w = h))</th>
<th>Elliptical ((e.g. w = 2h))</th>
<th>Parallel plates ((h/w = 0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\varepsilon_0^x)</td>
<td>1/2</td>
<td>-0.172</td>
<td>-0.206</td>
</tr>
<tr>
<td>(\varepsilon_0^y)</td>
<td>1/2</td>
<td>0.172</td>
<td>0.206</td>
</tr>
<tr>
<td>(\varepsilon_0^z)</td>
<td>0</td>
<td>1/2</td>
<td>0</td>
</tr>
<tr>
<td>(\xi_0^x)</td>
<td>1/2</td>
<td>0.083</td>
<td>0</td>
</tr>
<tr>
<td>(\xi_0^y)</td>
<td>1/2</td>
<td>0.55</td>
<td>0.617(\pi^2/16)</td>
</tr>
<tr>
<td>(\varepsilon_1^x)</td>
<td>-0.411(-(\pi^2/24))</td>
<td>-0.411</td>
<td>-0.411</td>
</tr>
<tr>
<td>(\varepsilon_1^y)</td>
<td>0.411((\pi^2/24))</td>
<td>0.411</td>
<td>0.411</td>
</tr>
<tr>
<td>(\xi_1^x)</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(\xi_1^y)</td>
<td>0.617((\pi^2/16))</td>
<td>0.617</td>
<td>0.617</td>
</tr>
</tbody>
</table>

Uniform, elliptical beam in an elliptical beam pipe. Similar formulae for \(\Delta Q_x\).

In general, \(\Delta Q_y > \Delta Q_x\)

*L.J. Laslett, 1963

*\(\pi^2/48\)
Bunched Beam in a Synchrotron

What's different with bunched beams?

- **Q-shift** much larger in bunch centre than in tails
- **Q-shift** changes periodically with $\omega_s$
- peak **Q-shift** much larger than for unbunched beam with same $N$ (number of particles in the ring)
- **Q-shift** $\Rightarrow$ **Q-spread** over the bunch
Incoherent $\Delta Q$: A Practical Formula

\[
\Delta Q_y = -\frac{r_0}{\pi} \left( \frac{q^2}{A} \right) \frac{N}{\beta^2 \gamma^3} \frac{G_y}{B_f} \left( \frac{\beta_y}{b(a+b)} \right) \left( \frac{\beta_y}{b(1+a)} \right) \approx \frac{1}{E_y \left( 1 + \sqrt{\frac{E_y Q_y}{E_x Q_x}} \right)}
\]

\[
\Delta Q_{x,y} = -\frac{r_0}{\pi} \left( \frac{q^2}{A} \right) \frac{N}{\beta^2 \gamma^3} \frac{F_{x,y} G_{x,y}}{B_f} \left( \frac{E_y}{E_{x,y}} \right) \frac{1}{1 + \sqrt{\frac{E_{x,y} Q_{x,y}}{E_x Q_x}}} \]

$q/A$...... charge/mass number of ions ($1$ for protons, e.g. $6/16$ for $^{16}\text{O}^{6+}$)

$F_{x,y}$...... “Form factor” derived from Laslett’s image coefficients $\varepsilon_1^x, \varepsilon_1^y, \varepsilon_2^x, \varepsilon_2^y$ ($F \approx 1$ if dominated by direct space charge)

$G_{x,y}$...... Form factor depending on particle distribution in $x,y$. In general, $1 < G \leq 2$

- Uniform $G=1$ ($E_{x,y}$ 100% emittance)
- Gaussian $G=2$ ($E_{x,y}$ 95% emittance)

$B_f$...... “Bunching Factor”: average/peak line density $B_f = \frac{\lambda}{\bar{\lambda}} = \frac{\bar{I}}{I}$
A Space-Charge Limited Accelerator

CERN PS Booster Synchrotron
N = $10^{13}$ protons
$E_x^* = 80 \, \mu \text{rad m} \, [4 \, \beta \gamma \sigma_x^2/\beta_x] \, \text{hor. emittance}$
$E_y^* = 27 \, \mu \text{rad m} \, \text{vertical emittance}$
$B_r = 0.58$
$F_{x,y} = 1$
$G_x/G_y = 1.3/1.5$

- Direct space charge tune spread ~0.55 at injection, covering 2nd and 3rd order stop-bands
- "necktie"-shaped tune spread shrinks rapidly due to the $1/\beta^2 \gamma^3$ dependence
- Enables the working point to be moved rapidly to an area clear of strong stop-bands
How to Remove the Space-Charge Limit?

**Problem:** A large proton synchrotron is limited in $N$ because $\Delta Q_y$ reaches 0.3 ... 0.5 when filling the (vertical) acceptance.

**Solution:** Increase $N$ by raising the injection energy and thus $\beta^2 \gamma^3$ while keeping to the same $\Delta Q$. Ways to do this:

- **Direct space charge**
  
  $$\Delta Q_y \approx \frac{N}{E_y \beta^2 \gamma^3} \frac{\hat{I}}{I}$$

- Make a longer (higher-energy) **Linac** (by adding tanks as has been done in Fermilab)

- Add a small "Booster" **synchrotron** of radius $r = R/n$ with $n$ the number of batches (BNL) or rings (CERN)

<table>
<thead>
<tr>
<th></th>
<th>Linac (MeV)</th>
<th>Booster (GeV)</th>
<th>$n=R/r$</th>
<th>Potential gain in $N$</th>
<th>Achieved</th>
</tr>
</thead>
<tbody>
<tr>
<td>CERN PS</td>
<td>50</td>
<td>1</td>
<td>4(rings)</td>
<td>59</td>
<td>~15</td>
</tr>
<tr>
<td>BNL AGS</td>
<td>200</td>
<td>1.5</td>
<td>4(batches)</td>
<td>26</td>
<td>~8</td>
</tr>
</tbody>
</table>

**Potential gain in $N$:**

- FNAL
  
  $\beta^2 \gamma^3$
  
<table>
<thead>
<tr>
<th>$\beta^2 \gamma^3$</th>
<th>Linac</th>
<th>Booster</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.57</td>
<td>200 MeV</td>
<td>400 MeV</td>
</tr>
<tr>
<td>1.48</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

--- Potential gain 2.6 ---
Lecture Summary

"Direct" space charge generated by the self-field of the beam
- acts on incoherent motion but has no effect on coherent (dipolar) motion
- proportional to beam intensity
- defocusing in both transverse planes
- scales with $1/\gamma^3 \Rightarrow$ barely noticeable in high-energy hadron and low-energy lepton machines

Image effects due to mirror charges induced in the vacuum envelope
- proportional to beam intensity
- scales with $1/\gamma \Rightarrow$ not negligible for high-$\gamma$ beams and machines
- give rise to a further change in the incoherent motion, but focusing in one plane, defocusing in the other plane
- modify the transverse coherent motion (coherent Q-change)

Bunched beams: Space-charge defocusing depends on the particle’s position in the bunch leading to a Q-spread (rather than a shift)
- Direct space charge is a hard limit on intensity/emittance ratio
- can be overcome by a higher-energy injector
High Intensity Proton Beam in a FODO Line

Transverse phase planes
rad vs. m

IN  0 mA  OUT

Transverse envelopes
horizontal vertical
mm vs. m

100 mA

50 MeV

100 mA

Courtesy of Alessandra Lombardi/ CERN, 8/04