Relativity for Accelerator Physicists

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Overview

- The principle of special relativity
- Lorentz transformation and its consequences
- 4-vectors: position, velocity, momentum, invariants. Derivation of $E=mc^2$
- Examples of the use of 4-vectors
- Inter-relation between $\beta$ and $\gamma$, momentum and energy
- An accelerator problem in relativity
Reading

- W. Rindler: Introduction to Special Relativity (OUP 1991)
- D. Lawden: An Introduction to Tensor Calculus and Relativity
- N.M.J. Woodhouse: Special Relativity (Springer 2002)
Historical background

- Groundwork by Lorentz in studies of electrodynamics, with crucial concepts contributed by Einstein to place the theory on a consistent basis.

- Maxwell’s equations (1863) attempted to explain electromagnetism and optics through wave theory
  - light propagates with speed \( c = 3 \times 10^8 \text{ m/s} \) in “ether” but with different speeds in other frames
  - the ether exists solely for the transport of e/m waves
  - Maxwell’s equations not invariant under Galilean transformations
  - To avoid setting e/m apart from classical mechanics, assume light has speed \( c \) only in frames where source is at rest
  - And the ether has a small interaction with matter and is carried along with astronomical objects
Nonsense! Contradicted by:

- Aberration of star light (small shift in apparent positions of distant stars)
- Fizeau’s 1859 experiments on velocity of light in liquids
- Michelson-Morley 1907 experiment to detect motion of the earth through ether
- Suggestion: perhaps material objects contract in the direction of their motion

\[ L(v) = L_0 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \]

This was the last gasp of ether advocates and the germ of Special Relativity led by Lorentz, Minkowski and Einstein.
Simultaneity

- Two clocks A and B are synchronised if light rays emitted at the same time from A and B meet at the mid-point of AB.

- Frame F' moving with respect to F. Events simultaneous in F cannot be simultaneous in F'.

- Simultaneity is **not** absolute but frame dependent.
The Principle of Special Relativity

- A frame in which particles under no forces move with constant velocity is “inertial”
- Consider relations between inertial frames where measuring apparatus (rulers, clocks) can be transferred from one to another.
- Behaviour of apparatus transferred from F to F' is independent of mode of transfer.
- Apparatus transferred from F to F', then from F' to F", agrees with apparatus transferred directly from F to F".

The Principle of Special Relativity states that all physical laws take equivalent forms in related inertial frames, so that we cannot distinguish between the frames.
The Lorentz Transformation

- Must be linear to agree with standard Galilean transformation in low velocity limit
- Preserves wave fronts of pulses of light,
  \[ P \equiv x'^2 + y'^2 + z'^2 - c^2 t'^2 = 0 \]
  whenever \[ Q \equiv x''^2 + y''^2 + z''^2 - c^2 t''^2 = 0 \]

- Solution is the **Lorentz transformation** from frame F \((t,x,y,z)\) to frame F'\((t',x',y',z')\) moving with velocity \(v\) along the x-axis:

\[
\begin{align*}
  t' &= \gamma \left( t - \frac{vx}{c^2} \right) \\
  x' &= \gamma (x - vt) \\
  y' &= y \\
  z' &= z
\end{align*}
\]

where \( \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \)
Outline of Derivation

Set \( t' = \alpha t + \beta x \)
\( x' = \gamma x + \delta t \)
\( y' = \varepsilon y \)
\( z' = \zeta z \)

Then \( P = kQ \)

\[ c^2 t'^2 - x'^2 - y'^2 - z'^2 = k \left( c^2 t^2 - x^2 - y^2 - z^2 \right) \]

\[ c^2 (\alpha t + \beta x)^2 - (\gamma x + \delta t)^2 - \varepsilon^2 y^2 - \zeta^2 z^2 = k \left( c^2 t^2 - x^2 - y^2 - z^2 \right) \]

Equate coefficients of \( x, y, z, t \).

Isotropy of space \( \Rightarrow k = k(\vec{v}) = k(|\vec{v}|) = \pm 1 \)

Apply some common sense (e.g. \( \varepsilon, \zeta, k = +1 \) and not -1)
Rod AB of length L' fixed in F' at $x'_A$, $x'_B$. What is its length measured in F?

Must measure positions of ends in F at the same time, so events in F are $(t, x_A)$ and $(t, x_B)$. From Lorentz:

$$x'_A = \gamma (x_A - vt) \quad x'_B = \gamma (x_B - vt)$$

$$L' = x'_B - x'_A = \gamma (x_B - x_A) = \gamma L > L$$

Moving objects appear contracted in the direction of the motion.
Consequences: time dilatation

- Clock in frame $F$ at point with coordinates $(x,y,z)$ at different times $t_A$ and $t_B$
- In frame $F'$ moving with speed $v$, Lorentz transformation gives
  \[
  t'_A = \gamma \left( t_A - \frac{vx}{c^2} \right) \quad t'_B = \gamma \left( t_B - \frac{vx}{c^2} \right)
  \]
- So:
  \[
  \Delta t' = t'_B - t'_A = \gamma (t_B - t_A) = \gamma \Delta t > \Delta t
  \]

Moving clocks appear to run slow
Schematic Representation of the Lorentz Transformation

Length contraction $L < L'$
Rod at rest in $F'$. Measurement in $F$ at fixed time $t$, along a line parallel to $x$-axis

Time dilatation: $\Delta t < \Delta t'$
Clock at rest in $F$. Time difference in $F'$ from line parallel to $x'$-axis
Example: High Speed Train

- All clocks synchronised.
- Observers A and B at entrance and exit of tunnel say the train is moving, has contracted and has length

\[
\frac{100}{\gamma} = 100 \times \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} = 100 \times \left(1 - \frac{3}{4}\right)^{\frac{1}{2}} = 50 \text{m}
\]

- But the tunnel is moving relative to the driver and guard on the train and they say the train is 100 m in length but the tunnel has contracted to 50 m
A Simple Problem

- F=frame of tunnel (and A,B)
- F'=frame of train (and driver D and guard G)

\[
\begin{align*}
x_A &= 0 \quad x_B = 100 \\
x_D' &= 0 \quad x_G' = 100
\end{align*}
\]

- What does B’s clock read when the guard goes into the tunnel?

\[
\begin{align*}
t' &= \gamma \left( t + \frac{vx}{c^2} \right) & t &= \gamma \left( t' - \frac{vx'}{c^2} \right) \\
x' &= \gamma (x + vt) & x &= \gamma (x' - vt')
\end{align*}
\]

Coincident events \((t_B, x_B)\) and \((t'_G, x'_G)\)

\[
x'_G = \gamma (x_B + vt_B) \quad \Rightarrow \quad 100 = 2(100 + vt_B)
\]

\[
\Rightarrow \quad t_B = -\frac{100}{c\sqrt{3}}
\]
Simple Problem (continued)

- What does the guard’s clock read as he goes in?

\[
x_B = \gamma(x_G' - vt_G') \quad \Rightarrow \quad 100 = 2(100 - vt_G')
\]

\[
\Rightarrow \quad t_G' = +\frac{100}{\sqrt{3}c}
\]

- Where is the guard when his clock reads 0?

Put \( t_G' = 0 \) in \( x = \gamma(x_G' - vt_G') \)

\[
\Rightarrow \quad x = 2x_G' = 200 \text{ m}
\]

So the guard is still 100m from the tunnel entrance when his clock reads zero.
Example: $\pi$-mesons

- Mesons are created in the upper atmosphere, 90km from earth. Their half life is $\tau=2 \mu s$, so they can travel at most $2 \times 10^{-6} c=600m$ before decaying. So how do more than 50% reach the earth’s surface undecayed?

- Mesons see distance contracted by $\gamma$, so 
  $$v\tau \approx \left(\frac{90}{\gamma}\right) km$$

- Earthlings say mesons’ clocks run slow so their half-life is $\gamma \tau$ and 
  $$v(\gamma \tau) \approx 90 \text{ km}$$

- Both give 
  $$\beta \gamma = \frac{90 \text{ km}}{c \tau} = 150, \quad \beta \approx 1, \quad \gamma \approx 150$$
Invariants

- An invariant is a quantity that has the same value in all inertial frames.
- Lorentz transformation is based on invariance of
  \[ c^2 t^2 - (x^2 + y^2 + z^2) = (ct)^2 - \bar{x}^2 \]
- Write in terms of the 4-position vector \( X = (ct, \bar{x}) \) as \( X \cdot X \)
  If \( X = (x_0, \bar{x}), \ Y = (y_0, \bar{y}) \), define the invariant product
  \( X \cdot Y = x_0 y_0 - \bar{x} \cdot \bar{y} \)

- Fundamental invariant (preservation of speed of light):
  \[
  c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 = c^2 \Delta t^2 \left( 1 - \frac{\Delta x^2 + \Delta y^2 + \Delta z^2}{c^2 \Delta t^2} \right)
  \]
  \[
  = c^2 \Delta t^2 \left( 1 - \frac{v^2}{c^2} \right) = c^2 \left( \frac{\Delta t}{\gamma} \right)^2
  \]

- \( \tau \) is the invariant proper time where \( \Delta \tau = \Delta t / \gamma \)
- \( \tau \) is time in the rest frame
4-Vectors

- The Lorentz transformation can be written in matrix form

\[
\begin{pmatrix}
t' \\
x' \\
y' \\
z'
\end{pmatrix} = \gamma \begin{pmatrix}
t - \frac{vx}{c^2} \\
x - vt \\
y \\
z
\end{pmatrix}
\]

\[
\begin{pmatrix}
c t' \\
x' \\
y' \\
z'
\end{pmatrix} = \begin{pmatrix}
\gamma & -\frac{\gamma v}{c} & 0 & 0 \\
-\frac{\gamma v}{c} & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
c t \\
x \\
y \\
z
\end{pmatrix}
\]

An object made up of 4 elements which transforms like X is called a 4-vector

(Analogous to the 3-vector of classical mechanics)

Position 4-vector \( X = (ct, \vec{x}) \)
4-Vectors in S.R. Mechanics

- **Velocity:**
  \[ V = \frac{dX}{d\tau} = \gamma \frac{dX}{dt} = \gamma \frac{d}{dt}(ct, \vec{x}) = \gamma (c, \vec{v}) \]

- **Note invariant**
  \[ V \cdot V = \gamma^2 (c^2 - \vec{v}^2) = c^2 \]

- **Momentum**
  \[ P = m_0 V = m_0 \gamma (c, \vec{v}) = (mc, \vec{p}) \]

\[ m = m_0 \gamma \quad \text{is relativistic mass} \]

\[ \vec{p} = m_0 \gamma \vec{v} = m\vec{v} \quad \text{is the 3-momentum} \]
Example of Transformation: Addition of Velocities

- A particle moves with velocity \( \vec{u} = (u_x, u_y, u_z) \) in frame F, so has 4-velocity \( V = \gamma_u (c, \vec{u}) \).
- Add velocity \( \vec{v} = (v, 0, 0) \) by transforming to frame F' to get new velocity \( \vec{w} \).
- Lorentz transformation gives \( (t \leftrightarrow \gamma, \, \vec{x} \leftrightarrow \gamma \vec{u}) \).

\[
\gamma_w = \gamma_v \left( \gamma_u + \frac{v \gamma_u u_x}{c^2} \right)
\]

\[
w_x = \frac{u_x + v}{1 + \frac{v u_x}{c^2}}
\]

\[
w_y = \frac{u_y}{\gamma_v \left( 1 + \frac{v u_x}{c^2} \right)}
\]

\[
w_z = \frac{u_z}{\gamma_v \left( 1 + \frac{v u_x}{c^2} \right)}
\]
Einstein’s relation

- Momentum invariant: \( P \cdot P = m_0^2 (V \cdot V) = m_0^2 c^2 \)

- Differentiate: \( P \cdot \frac{dP}{d\tau} = 0 \Rightarrow V \cdot \frac{dP}{d\tau} = 0 \)

- From Newton’s 2nd Law expect 4-Force given by:
  \[ F = \frac{dP}{d\tau} = \gamma \frac{dP}{dt} = \gamma \frac{d}{dt} (mc, \vec{p}) = \gamma \left( c \frac{dm}{dt}, \frac{d\vec{p}}{dt} \right) = \gamma \left( c \frac{dm}{dt}, \vec{f} \right) \]

- But \( V \cdot \frac{dP}{d\tau} = 0 \Rightarrow V \cdot F = 0 \)

- So \( \frac{d}{dt} (mc^2) - \vec{v} \cdot \vec{f} = 0 \)

Rate of doing work, \( \vec{v} \cdot \vec{f} = \) rate of change of kinetic energy

Therefore kinetic energy:
\[ T = mc^2 + \text{constant} = m_0 c^2 (\gamma - 1) \]

\( E = mc^2 \) is total energy
Basic quantities used in Accelerator calculations

Relative velocity \( \beta = \frac{\sqrt{\gamma}}{c} \)

Velocity \( v = \beta c \)

Momentum \( p = mv = m_0 \gamma \beta c \)

Kinetic energy \( T = (m - m_0)c^2 = m_0 c^2 (\gamma - 1) \)

\[ \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \beta^2\right)^{-\frac{1}{2}} \]

\[ \Rightarrow (\beta \gamma)^2 = \frac{\gamma^2 v^2}{c^2} = \gamma^2 - 1 \Rightarrow \beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2} \]
Velocity as a function of energy

\[ T = m_0 (\gamma - 1)c^2 \]

\[ \gamma = 1 + \frac{T}{m_0 c^2} \]

\[ \beta = \sqrt{1 - \frac{1}{\gamma^2}} \]

\[ p = m_0 c \beta \gamma \]

For small \( \frac{v}{c} \), \( \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{1}{2} \frac{v^2}{c^2} \)

so \( T = m_0 c^2 (\gamma - 1) \approx \frac{1}{2} m_0 v^2 \)
Relationships between small variations in parameters \( \Delta E, \Delta T, \Delta p, \Delta \beta, \Delta \gamma \)

\[
(\beta \gamma)^2 = \gamma^2 - 1
\]

\[
\Rightarrow \beta \gamma \Delta(\beta \gamma) = \gamma \Delta \gamma
\]

\[
\Rightarrow \beta \Delta(\beta \gamma) = \Delta \gamma \quad (1)
\]

\[
\frac{1}{\gamma^2} = 1 - \beta^2
\]

\[
\Rightarrow \frac{1}{\gamma^3} \Delta \gamma = \beta \Delta \beta \quad (2)
\]

\[
\frac{\Delta p}{p} = \frac{\Delta(m_0 \gamma \beta c)}{m_0 \gamma \beta c} = \frac{\Delta(\beta \gamma)}{\beta \gamma}
\]

\[
= \frac{1}{\beta^2} \frac{\Delta \gamma}{\gamma} = \frac{1}{\beta^2} \frac{\Delta E}{E}
\]

\[
= \gamma^2 \frac{\Delta \beta}{\beta}
\]

\[
= \frac{\gamma}{\gamma + 1} \frac{\Delta T}{T} \quad \text{(exercise)}
\]
<table>
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<tr>
<th>[ \frac{\Delta \beta}{\beta} ]</th>
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<td>[ \frac{\Delta \beta}{\beta} = \frac{\Delta \beta}{\beta} ]</td>
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<td>[ \frac{1}{\gamma(\gamma + 1)} \frac{\Delta T}{T} ]</td>
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<td>[ \frac{\Delta T}{T} ]</td>
<td>[ \frac{\gamma}{\gamma - 1} \frac{\Delta \gamma}{\gamma} ]</td>
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4-Momentum Conservation

- Equivalent expression for 4-momentum
  \[ P = m_0 \gamma(c, \vec{v}) = (mc, \vec{p}) = \left( \frac{E}{c}, \vec{p} \right) \]

- Invariant
  \[ m_0^2 c^2 = P \cdot P = \frac{E^2}{c^2} - \vec{p}^2 \]
  \[ \frac{E^2}{c^2} = m_0^2 c^2 + \vec{p}^2 \]

- Classical momentum conservation laws \( \rightarrow \) conservation of 4-momentum. Total 3-momentum and total energy are conserved.
  \[ \sum P_i = \text{constant} \]
  \[ \sum E_i \quad \text{and} \quad \sum \vec{p}_i \quad \text{constant} \]
Example of use of invariants

- Two particles have equal rest mass $m_0$.
  - Frame 1: one particle at rest, total energy is $E_1$.
  - Frame 2: centre of mass frame where velocities are equal and opposite, total energy is $E_2$.

Problem: Relate $E_1$ to $E_2$
Total energy $E_1$ (Fixed target experiment)

Total energy $E_2$ (Colliding beams expt)

\[ P_1 = \left( \frac{E_1 - m_0 c^2}{c}, \vec{p} \right) \]

\[ P_2 = \left( m_0 c, 0 \right) \]

\[ P_1 = \left( \frac{E_2}{2c}, \vec{p}' \right) \]

\[ P_2 = \left( \frac{E_2}{2c}, -\vec{p}' \right) \]

Invariant: \[ P_2 \cdot (P_1 + P_2) \]

\[ m_0 c \times \frac{E_1}{c} - 0 \times p = \frac{E_2}{2c} \times \frac{E_2}{c} + p' \times 0 \]

\[ \Rightarrow \quad 2m_0 c^2 E_1 = E_2^2 \]
Accelerator Problem

- In an accelerator, a proton $p_1$ with rest mass $m_0$ collides with an anti-proton $p_2$ (with the same rest mass), producing two particles $W_1$ and $W_2$ with equal mass $M_0 = 100m_0$
  - Expt 1: $p_1$ and $p_2$ have equal and opposite velocities in the lab frame. Find the minimum energy of $p_2$ in order for $W_1$ and $W_2$ to be produced.
  - Expt 2: in the rest frame of $p_1$, find the minimum energy $E'$ of $p_2$ in order for $W_1$ and $W_2$ to be produced.
Note: \[ \frac{E^2}{c^2} = \vec{p}^2 + m_0^2 c^2 \implies \text{same } m_0, \text{ same } p \text{ mean same } E. \]

Total 3-momentum is zero before collision and so is zero after impact.

4-momenta before collision:

\[ P_1 = \left( \frac{E}{c}, \vec{p} \right) \quad P_2 = \left( \frac{E}{c}, -\vec{p} \right) \]

4-momenta after collision:

\[ P'_1 = \left( \frac{E'}{c}, \vec{q} \right) \quad P'_2 = \left( \frac{E'}{c}, -\vec{q} \right) \]

Energy conservation \( \Rightarrow E = E' \rightarrow \) rest energy = \( M_0 c^2 = 100 \ m_0 c^2 \)
Before collision:

\[ P_1 = \left( m_0c, \vec{0} \right) \quad P_2 = \left( \frac{E'}{c}, \vec{p} \right) \]

Total energy is

\[ E_1 = E' + m_0c^2 \]

Use previous result \( 2m_0c^2 E_1 = E_2^2 \) to relate \( E_1 \) to total energy \( E_2 \) in C.O.M frame

\[ 2m_0c^2 E_1 = E_2^2 \]

\[ \Rightarrow \quad 2m_0c^2 \left( E' + m_0c^2 \right) = (2E)^2 > \left( 200 m_0c^2 \right)^2 \]

\[ \Rightarrow \quad E' > \left( 2 \times 10^4 - 1 \right) m_0c^2 \approx 20,000 m_0c^2 \]