Relativity for Accelerator Physicists

Christopher R. Prior

Fellow and Tutor in Mathematics

Trinity College, Oxford

Accelerator Science and Technology Centre

Rutherford Appleton Laboratory, U.K.

Overview

- The principle of special relativity
- Lorentz transformation and its consequences
- 4-vectors: position, velocity, momentum, invariants. Derivation of $E=mc^2$
- Examples of the use of 4-vectors
- Inter-relation between β and γ, momentum and energy
- An accelerator problem in relativity

Reading

- W. Rindler: Introduction to Special Relativity (OUP 1991)
- D. Lawden: An Introduction to Tensor Calculus and Relativity
- N.M.J. Woodhouse: Special Relativity (Springer 2002)
- A.P. French: Special Relativity, MIT Introductory Physics Series (Nelson Thomes)

Historical background

- Groundwork by Lorentz in studies of electrodynamics, with crucial concepts contributed by Einstein to place the theory on a consistent basis.
- Maxwell's equations (1863) attempted to explain electromagnetism and optics through wave theory
  - light propagates with speed $c = 3\times10^8$ m/s in "ether" but with different speeds in other frames
  - the ether exists solely for the transport of e/m waves
  - Maxwell's equations not invariant under Galilean transformations
  - To avoid setting e/m apart from classical mechanics, assume light has speed *c* only in frames where source is at rest
  - And the ether has a small interaction with matter and is carried along with astronomical objects

Nonsense! Contradicted by:

- Aberration of star light (small shift in apparent positions of distant stars)
- Fizeau's 1859 experiments on velocity of light in liquids
- Michelson-Morley 1907 experiment to detect motion of the earth through ether
- Suggestion: perhaps material objects contract in the direction of their motion

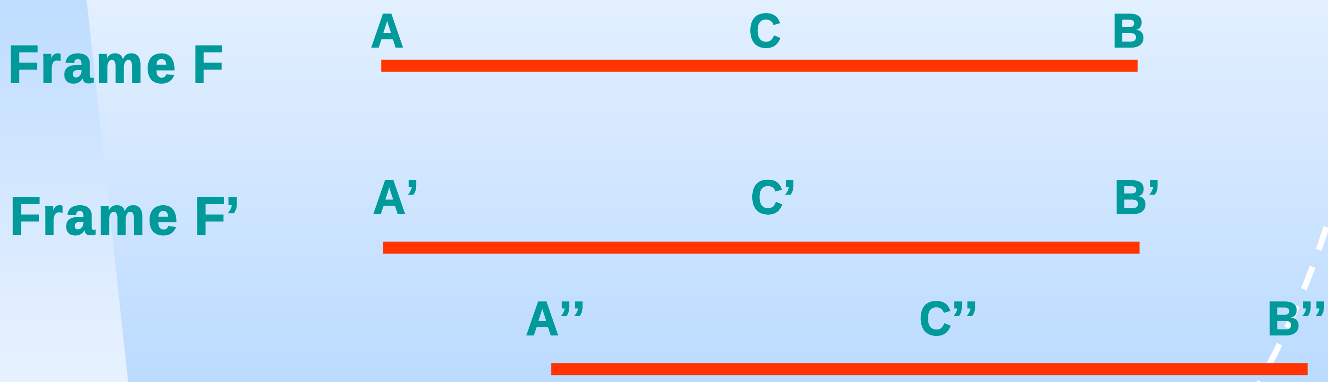
$$L(v) = L_0 \left(1 - \frac{v^2}{c^2}\right)^{1/2}$$

This was the last gasp of ether advocates and the germ of Special Relativity led by Lorentz, Minkowski and Einstein.

TRINITY COLLEGE • OXFORD

Simultaneity

- Two clocks A and B are synchronised if light rays emitted at the same time from A and B meet at the mid-point of AB



- Frame F' moving with respect to F. Events simultaneous in F cannot be simultaneous in F'.
- Simultaneity is not absolute but frame dependent.

TRINITY COLLEGE • OXFORD

The Principle of Special Relativity

- A frame in which particles under no forces move with constant velocity is "inertial"
- Consider relations between inertial frames where measuring apparatus (rulers, clocks) can be transferred from one to another.
- Behaviour of apparatus transferred from F to F' is independent of mode of transfer
- Apparatus transferred from F to F', then from F' to F'', agrees with apparatus transferred directly from F to F''.
- *The Principle of Special Relativity states that all physical laws take equivalent forms in related inertial frames, so that we cannot distinguish between the frames.*

TRINITY COLLEGE • OXFORD

The Lorentz Transformation

- Must be linear to agree with standard Galilean transformation in low velocity limit
- Preserves wave fronts of pulses of light,

i.e. $P \equiv x^2 + y^2 + z^2 - c^2t^2 = 0$

whenever $Q \equiv x'^2 + y'^2 + z'^2 - c^2t'^2 = 0$

- Solution is the **Lorentz transformation** from frame F (t,x,y,z) to frame F'(t',x',y',z') moving with velocity v along the x-axis:

$$\left.\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}\right\} \quad \text{where} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Outline of Derivation

Set $t' = \alpha t + \beta x$

$x' = \gamma x + \delta t$

$y' = \varepsilon y$

$z' = \varsigma z$

Then $P = kQ$

$\Leftrightarrow c^2 t'^2 - x'^2 - y'^2 - z'^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$

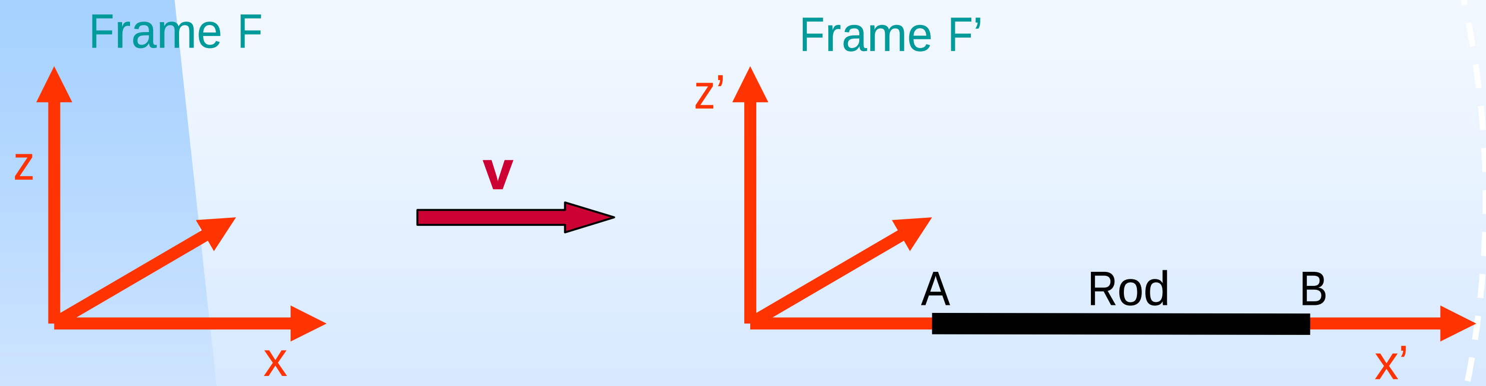
$\Rightarrow c^2(\alpha t + \beta x)^2 - (\gamma x + \delta t)^2 - \varepsilon^2 y^2 - \varsigma^2 z^2 = k\left(c^2 t^2 - x^2 - y^2 - z^2\right)$

Equate coefficients of x, y, z, t.

Isotropy of space $\Rightarrow k = k(\vec{v}) = k(|\vec{v}|) = \pm 1$

Apply some common sense (e.g. $\varepsilon, \varsigma, k = +1$ and not -1)

Consequences: length contraction



Rod AB of length L' fixed in F' at x'_A, x'_B. What is its length measured in F?

Must measure positions of ends in F at the same time, so events in F are (t,x_A) and (t,x_B). From Lorentz:

$$x'_A = \gamma\left(x_A - vt\right) \qquad x'_B = \gamma\left(x_B - vt\right)$$

$$L' = x'_B - x'_A = \gamma\left(x_B - x_A\right) = \gamma L > L$$

Moving objects appear contracted in the direction of the motion

Consequences: time dilatation

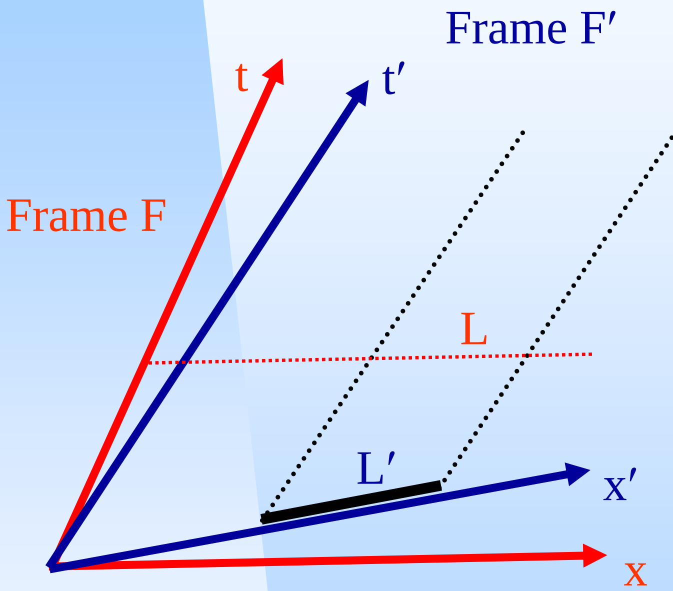
- Clock in frame F at point with coordinates (x,y,z) at different times t_A and t_B
- In frame F' moving with speed v, Lorentz transformation gives

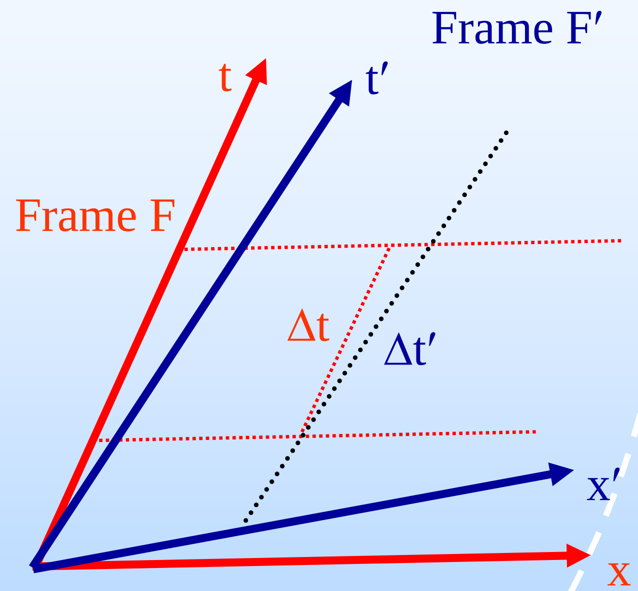
$$t'_A = \gamma\left(t_A - \frac{vx}{c^2}\right) \qquad t'_B = \gamma\left(t_B - \frac{vx}{c^2}\right)$$

- So: $$\Delta t' = t'_B - t'_A = \gamma\left(t_B - t_A\right) = \gamma\Delta t > \Delta t$$

Moving clocks appear to run slow

Schematic Representation of the Lorentz Transformation





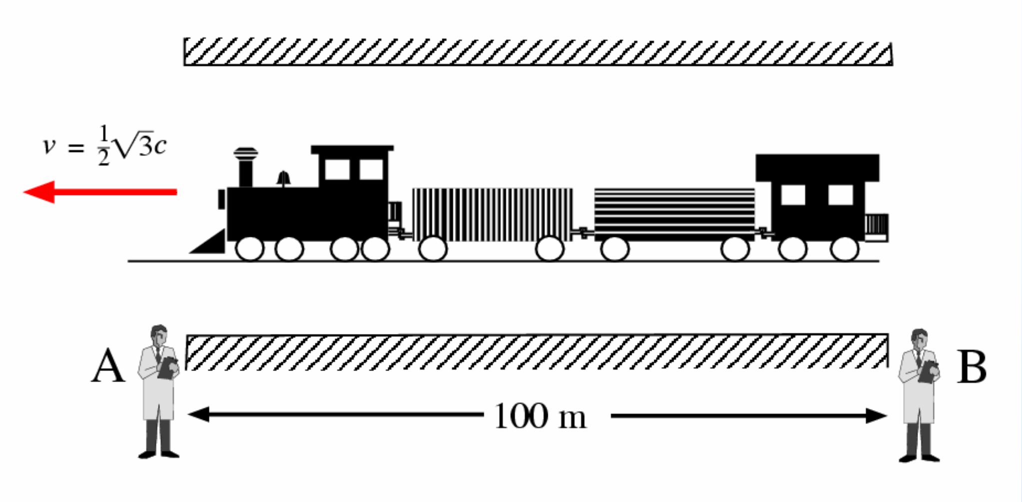
Length contraction $L<L'$

Rod at rest in F′. Measurement in F at fixed time t, along a line parallel to x-axis

Time dilatation: $\Delta t<\Delta t'$

Clock at rest in F. Time difference in F′ from line parallel to x′-axis

Example: High Speed Train



- All clocks synchronised.
- Observers A and B at entrance and exit of tunnel say the train is moving, has contracted and has length

$$\frac{100}{\gamma} = 100 \times \left(1 - \frac{v^2}{c^2}\right)^{1/2} = 100 \times \left(1 - \frac{3}{4}\right)^{1/2} = 50\text{m}$$

- But the tunnel is moving relative to the driver and guard on the train and they say the train is 100 m in length but the tunnel has contracted to 50 m

TRINITY
COLLEGE • OXFORD

A Simple Problem

- F=frame of tunnel (and A,B)
- F'=frame of train (and driver D and guard G)

$$x_A = 0 \quad x_B = 100$$
$$x'_D = 0 \quad x'_G = 100$$

- What does B's clock read when the guard goes into the tunnel?

$$t' = \gamma\left(t + \frac{vx}{c^2}\right) \qquad t = \gamma\left(t' - \frac{vx'}{c^2}\right)$$
$$x' = \gamma(x + vt) \qquad x = \gamma(x' - vt')$$

Coincident events (t_B, x_B) and (t'_G, x'_G)

$$x'_G = \gamma(x_B + vt_B) \quad \Rightarrow \quad 100 = 2(100 + vt_B)$$
$$\Rightarrow \quad t_B = -\frac{100}{c\sqrt{3}}$$

Simple Problem (continued)

- What does the guard's clock read as he goes in?

$$x_B = \gamma(x'_G - vt'_G) \quad \Rightarrow \quad 100 = 2(100 - vt'_G)$$
$$\Rightarrow \quad t'_G = +\frac{100}{\sqrt{3}c}$$

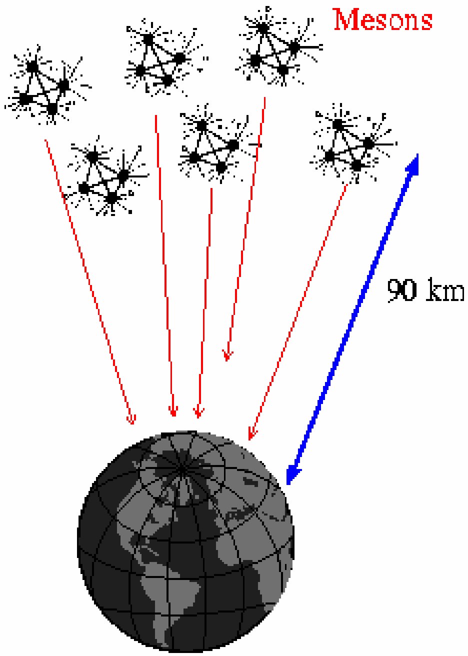
- Where is the guard when his clock reads 0?

Put $t'_G = 0$ in $x = \gamma(x'_G - vt'_G)$
$$\Rightarrow \quad x = 2x'_G = 200\,m$$

So the guard is still 100m from the tunnel entrance when his clock reads zero.

TRINITY COLLEGE • OXFORD

Example: π-mesons



Half-life = 2 x 10^{-6} sec

so required velocity = 90/2 x 10^{-6} = 4.5 x 10^{7} km/sec

= 150 c

- Mesons are created in the upper atmosphere, 90km from earth. Their half life is τ=2 μs, so they can travel at most 2 ×10^{-6}c=600m before decaying. So how do more than 50% reach the earth's surface undecayed?
- Mesons see distance contracted by γ, so

$$v\tau \approx \left(\frac{90}{\gamma}\right)\text{km}$$

- Earthlings say mesons' clocks run slow so their half-life is γτ and

$$v(\gamma\tau) \approx 90\,\text{km}$$

- Both give

$$\beta\gamma = \frac{90\text{ km}}{c\tau} = 150, \quad \beta \approx 1, \quad \gamma \approx 150$$

Invariants

- An invariant is a quantity that has the same value in all inertial frames.
- Lorentz transformation is based on invariance of

$$c^2t^2-(x^2+y^2+z^2)=(ct)^2-\vec{x}^2$$

- Write in terms of the 4-position vector $\mathrm{X}=(ct,\vec{x})$ as $\mathrm{X}\bullet\mathrm{X}$

If $\mathrm{X}=(x_0,\vec{x}),\ \mathrm{Y}=(y_0,\vec{y}),$ define the invariant product

$$\mathrm{X}\bullet\mathrm{Y}=x_0y_0-\vec{x}\cdot\vec{y}$$

- Fundamental invariant (preservation of speed of light):

$$c^2\Delta t^2-\Delta x^2-\Delta y^2-\Delta z^2=c^2\Delta t^2\left(1-\frac{\Delta x^2+\Delta y^2+\Delta z^2}{c^2\Delta t^2}\right)$$

$$=c^2\Delta t^2\left(1-\frac{v^2}{c^2}\right)\quad=c^2\left(\frac{\Delta t}{\gamma}\right)^2$$

- τ is the invariant proper time where $\Delta\tau=\Delta t/\gamma$
- τ is time in the rest frame

4-Vectors

- The Lorentz transformation can be written in matrix form

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

$$\Rightarrow \begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{\gamma v}{c} & 0 & 0 \\ -\frac{\gamma v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

An object made up of 4 elements which transforms like X is called a 4-vector

(analogous to the 3-vector of classical mechanics)

Position 4-vector $\mathrm{X} = (ct, \vec{x})$

4-Vectors in S.R. Mechanics

- Velocity: $$V = \frac{dX}{d\tau} = \gamma \frac{dX}{dt} = \gamma \frac{d}{dt}(ct, \vec{x}) = \gamma\,(c, \vec{v})$$
- Note invariant $$V \bullet V = \gamma^2 (c^2 - \vec{v}^2) = c^2$$
- Momentum $$P = m_0 V = m_0 \gamma (c, \vec{v}) = (mc, \vec{p})$$

$m = m_0 \gamma$ is relativistic mass

$\vec{p} = m_0 \gamma \vec{v} = m\vec{v}$ is the 3-momentum

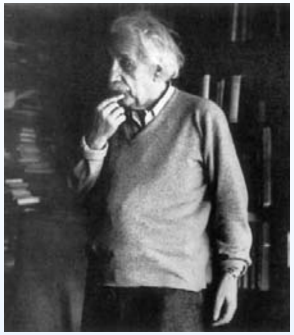
Example of Transformation: Addition of Velocities

- A particle moves with velocity $\vec{u} = (u_x, u_y, u_z)$ in frame F, so has 4-velocity $V = \gamma_u(c, \vec{u})$
- Add velocity $\vec{v} = (v, 0, 0)$ by transforming to frame F′ to get new velocity $\vec{w}$.
- Lorentz transformation gives $(t \leftrightarrow \gamma, \quad \vec{x} \leftrightarrow \gamma\vec{u})$

$$\begin{aligned} \gamma_w &= \gamma_v\left(\gamma_u + \frac{v\gamma_u u_x}{c^2}\right) \\ \gamma_w w_x &= \gamma_v(\gamma_u u_x + v\gamma_u) \\ \gamma_w w_y &= \gamma_u u_y \\ \gamma_w w_z &= \gamma_u u_z \end{aligned} \quad \Rightarrow \quad w_x = \frac{u_x + v}{1 + \frac{vu_x}{c^2}}$$

$$w_y = \frac{u_y}{\gamma_v\left(1 + \frac{vu_x}{c^2}\right)} \qquad w_z = \frac{u_z}{\gamma_v\left(1 + \frac{vu_x}{c^2}\right)}$$

Einstein's relation



- Momentum invariant $P \bullet P = m_0^2 \,(V \bullet V) = m_0^2 c^2$
- Differentiate $P \bullet \dfrac{dP}{d\tau} = 0 \quad \Rightarrow \quad V \bullet \dfrac{dP}{d\tau} = 0$
- From Newton's 2nd Law expect 4-Force given by

$$F = \frac{dP}{d\tau} = \gamma \frac{dP}{dt} = \gamma \frac{d}{dt}(mc, \vec{p}) = \gamma \left(c\frac{dm}{dt}, \frac{d\vec{p}}{dt} \right) = \gamma \left(c\frac{dm}{dt}, \vec{f} \right)$$

- But $V \bullet \dfrac{dP}{d\tau} = 0 \quad \Rightarrow \quad V \bullet F = 0$
- So $\dfrac{d}{dt}\left(mc^2\right) - \vec{v} \cdot \vec{f} = 0$

Rate of doing work, $\vec{v} \cdot \vec{f}$ = rate of change of kinetic energy

Therefore kinetic energy

$$T = mc^2 + \text{constant} = m_0 c^2 (\gamma - 1)$$

$E=mc^2$ is total energy

Basic quantities used in Accelerator calculations

Relative velocity $\beta = \mathrm{v}/\mathrm{c}$

Velocity $v = \beta\mathrm{c}$

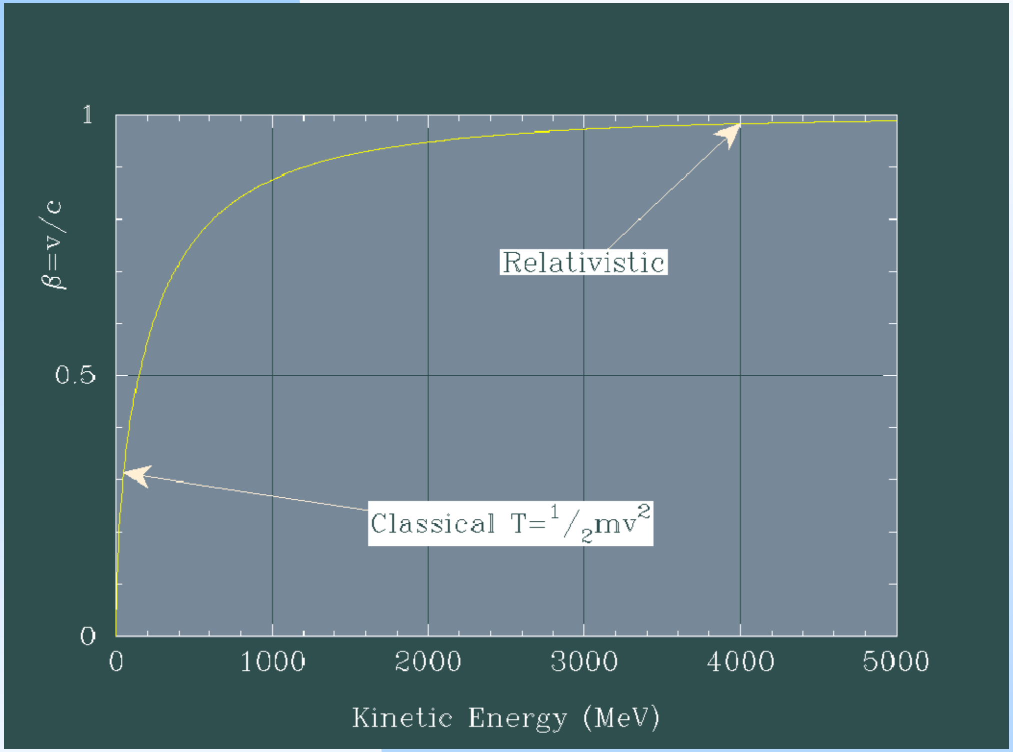
Momentum $p = mv = m_0\gamma\beta c$

Kinetic energy $T = (m\text{-}m_0)c^2 = m_0c^2(\gamma - 1)$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 - \beta^2\right)^{-\frac{1}{2}}$$

$$\Rightarrow \quad (\beta\gamma)^2 = \frac{\gamma^2 v^2}{c^2} = \gamma^2 - 1 \quad \Rightarrow \quad \beta^2 = \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

Velocity as a function of energy



$$T = m_0(\gamma - 1)c^2$$

$$\gamma = 1 + T/m_0c^2$$

$$\beta = \sqrt{1 - 1/\gamma^2}$$

$$p = m_o c\beta\gamma$$

For small v/c, $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{1}{2}\frac{v^2}{c^2}$

so $T = m_0c^2(\gamma - 1) \approx \frac{1}{2}m_0v^2$

Relationships between small variations in parameters ΔE, ΔT, Δp, Δβ, Δγ

$$(\beta\gamma)^2 = \gamma^2 - 1$$

$$\Rightarrow \quad \beta\gamma\,\Delta(\beta\gamma) = \gamma\,\Delta\gamma$$

$$\Rightarrow \quad \beta\,\Delta(\beta\gamma) = \Delta\gamma \qquad (1)$$

$$\frac{1}{\gamma^2} = 1 - \beta^2$$

$$\Rightarrow \quad \frac{1}{\gamma^3}\Delta\gamma = \beta\,\Delta\beta \qquad (2)$$

$$\frac{\Delta p}{p} = \frac{\Delta(m_0\gamma\beta c)}{m_0\gamma\beta c} = \frac{\Delta(\beta\gamma)}{\beta\gamma}$$

$$= \frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma} = \frac{1}{\beta^2}\frac{\Delta E}{E}$$

$$= \gamma^2\frac{\Delta\beta}{\beta}$$

$$= \frac{\gamma}{\gamma+1}\frac{\Delta T}{T} \qquad \text{(exercise)}$$

	$\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\Delta E}{E} = \frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\beta}{\beta} =$	$\frac{\Delta\beta}{\beta}$	$\frac{1}{\gamma^2}\frac{\Delta p}{p}$ $\frac{\Delta p}{p} - \frac{\Delta\gamma}{\gamma}$	$\frac{1}{\gamma(\gamma+1)}\frac{\Delta T}{T}$	$\frac{1}{\beta^2\gamma^2}\frac{\Delta\gamma}{\gamma}$ $\frac{1}{\gamma^2-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta p}{p} =$	$\gamma^2\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p}$	$\frac{\gamma}{\gamma+1}\frac{\Delta T}{T}$	$\frac{1}{\beta^2}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta T}{T} =$	$\gamma(\gamma+1)\frac{\Delta\beta}{\beta}$	$\left(1+\frac{1}{\gamma}\right)\frac{\Delta p}{p}$	$\frac{\Delta T}{T}$	$\frac{\gamma}{\gamma-1}\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta E}{E} =$	$(\beta\gamma)^2\frac{\Delta\beta}{\beta}$	$\beta^2\frac{\Delta p}{p}$	$\left(1-\frac{1}{\gamma}\right)\frac{\Delta T}{T}$	$\frac{\Delta\gamma}{\gamma}$
$\frac{\Delta\gamma}{\gamma} =$	$(\gamma^2-1)\frac{\Delta\beta}{\beta}$	$\frac{\Delta p}{p} - \frac{\Delta\beta}{\beta}$		

4-Momentum Conservation

- Equivalent expression for 4-momentum $$P = m_0\gamma(c,\vec{v}) = (mc,\vec{p}) = \left(E/c,\vec{p}\right)$$

- Invariant $$m_0^2c^2 = P \bullet P = \frac{E^2}{c^2} - \vec{p}^{\,2}$$

$$E^2/c^2 = m_0^2c^2 + \vec{p}^{\,2}$$

- Classical momentum conservation laws → conservation of 4-momentum. Total 3-momentum and total energy are conserved.

$$\sum_{\text{particles}, i} P_i = \text{constant}$$

$$\Rightarrow \sum_{\text{particles}, i} E_i \text{ and } \sum_{\text{particles}, i} \vec{p}_i \text{ constant}$$

TRINITY COLLEGE • OXFORD

Example of use of invariants

- Two particles have equal rest mass m_0.
  - Frame 1: one particle at rest, total energy is E_1.
  - Frame 2: centre of mass frame where velocities are equal and opposite, total energy is E_2.

Problem: Relate E_1 to E_2

Total energy E_1

(Fixed target experiment)

$$P_1 = \left(\frac{E_1 - m_0 c^2}{c}, \vec{p} \right) \qquad P_2 = \left(m_0 c, 0 \right)$$

Total energy E_2

(Colliding beams expt)

$$P_1 = \left(\frac{E_2}{2c}, \vec{p}' \right) \qquad P_2 = \left(\frac{E_2}{2c}, -\vec{p}' \right)$$

$$\text{Invariant:} \quad P_2 \bullet \left(P_1 + P_2 \right)$$

$$m_0 c \times \frac{E_1}{c} - 0 \times p = \frac{E_2}{2c} \times \frac{E_2}{c} + p' \times 0$$

$$\Rightarrow \qquad 2 m_0 c^2 E_1 = E_2^2$$

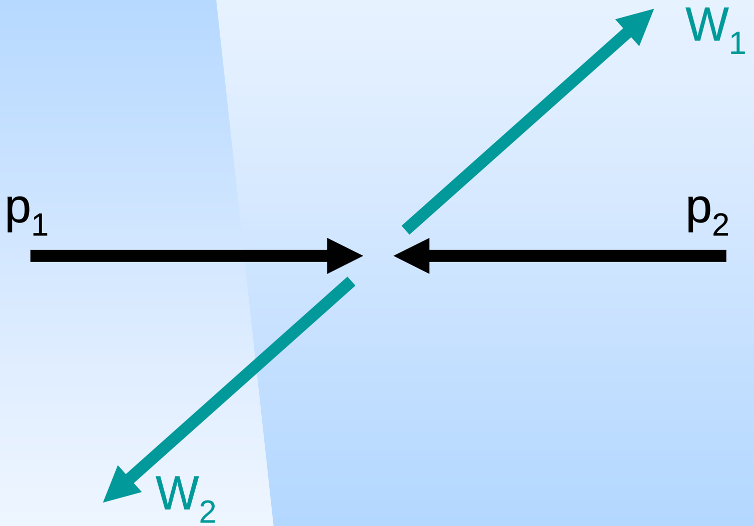
Accelerator Problem

- In an accelerator, a proton p_1 with rest mass m_0 collides with an anti-proton p_2 (with the same rest mass), producing two particles W_1 and W_2 with equal mass $M_0=100m_0$
  - Expt 1: p_1 and p_2 have equal and opposite velocities in the lab frame. Find the minimum energy of p_2 in order for W_1 and W_2 to be produced.
  - Expt 2: in the rest frame of p_1, find the minimum energy E' of p_2 in order for W_1 and W_2 to be produced

TRINITY COLLEGE • OXFORD

Note: $E^2/c^2 = \vec{p}^{\,2} + m_0^2 c^2$ $\Rightarrow$ same m_0, same p mean same E.

Experiment 1



Total 3-momentum is zero before collision and so is zero after impact

4-momenta before collision:

$$P_1 = \left(E/c, \vec{p}\right) \quad P_2 = \left(E/c, -\vec{p}\right)$$

4-momenta after collision:

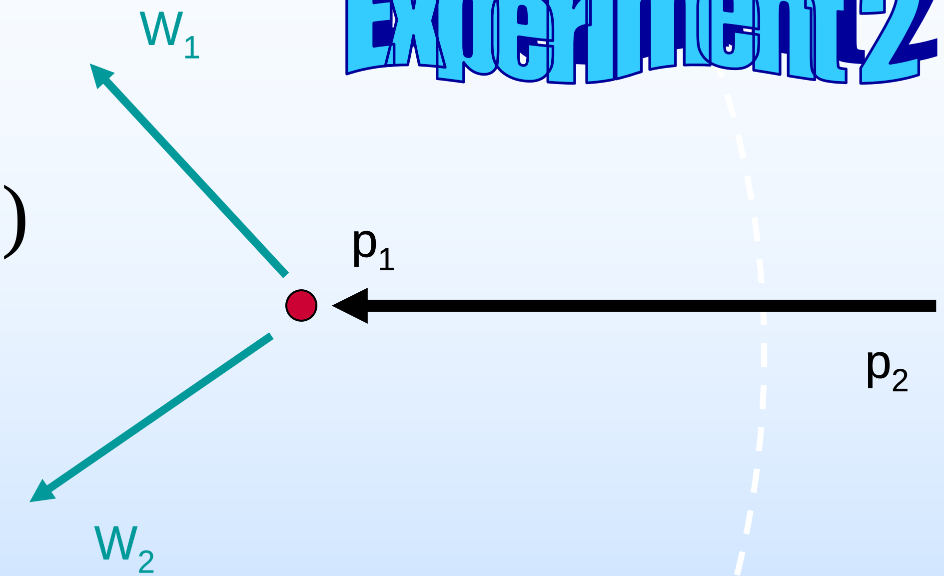
$$P_1 = \left(E'/c, \vec{q}\right) \quad P_2 = \left(E'/c, -\vec{q}\right)$$

Energy conservation $\Rightarrow$ *E=E′* > rest energy = M_0c^2 = 100 m_0c^2

Experiment 2

Before collision:

$$P_1 = \left(m_0 c, \vec{0}\right) \quad P_2 = \left(E'/c, \vec{p}\right)$$



Total energy is

$$E_1 = E' + m_0 c^2$$

Use previous result $2m_0c^2 E_1 = E_2^2$ to relate E_1 to total energy E_2 in C.O.M frame

$$2m_0 c^2 E_1 = E_2^2$$

$$\Rightarrow \quad 2m_0 c^2\left(E' + m_0 c^2\right) = (2E)^2 > \left(200\, m_0 c^2\right)^2$$

$$\Rightarrow \quad E' > \left(2\times 10^4 - 1\right) m_0 c^2 \approx 20{,}000\, m_0 c^2$$