LONGITUDINAL DYNAMICS

by

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Introductory Level Course on Accelerator Physics
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summary

- Radio-Frequency Acceleration and Synchronism
- Properties of Radio-Frequency cavities
- Energy Gain
- Principle of Phase Stability and Consequences
- Synchronous linear accelerator
- Energy-Phase oscillations
- The capture phenomenon
- The Synchrotron
- RF cavities for Synchrotron
- Dispersion Effects in a Synchrotron
- Energy-Phase Equations in a Synchrotron
- Phase space motions

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And CERN Accelerator Schools (CAS) Proceedings

Main Characteristics of an Accelerator

ACCELERATION is the main job of an accelerator.

- The accelerator provides **kinetic energy** to charged particles, hence increasing their **momentum**.
- In order to do so, it is necessary to have an electric field $\vec{E}$, preferably along the direction of the initial momentum.

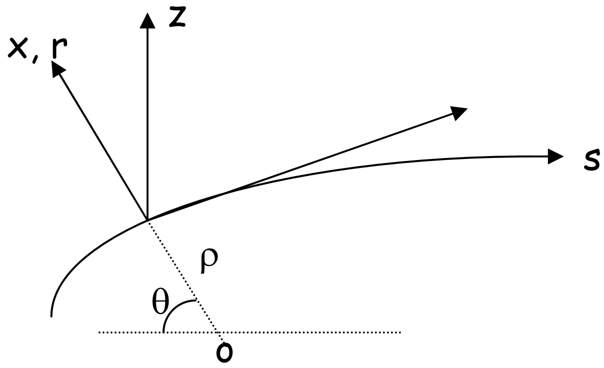
$$\boxed{\frac{dp}{dt} = eE}$$

BENDING is generated by a magnetic field perpendicular to the plane of the particle trajectory. The bending radius ρ obeys to the relation :

$$\boxed{\frac{p}{e} = B\rho}$$

FOCUSING is a second way of using a magnetic field, in which the bending effect is used to bring the particles trajectory closer to the axis, hence to increase the beam density.

Acceleration & Curvature



Within the assumption:

$$\vec{E} \rightarrow E_\theta$$

$$\vec{B} \rightarrow B_z$$

the Newton-Lorentz force:

$$\frac{d\vec{p}}{dt} = e\vec{E} + e\vec{v} \times \vec{B}$$

becomes:

$$\frac{d(mv_\theta)}{dt}\vec{u}_\theta - m\frac{v_\theta^2}{\rho}\vec{u}_r = eE_\theta\vec{u}_\theta - ev_\theta B_z\vec{u}_r$$

leading to:

$$\frac{dp_\theta}{dt} = eE_\theta$$

$$\frac{p_\theta}{e} = B_z\rho$$

Energy Gain

In relativistic dynamics, energy and momentum satisfy the relation:

$$E^2 = E_0^2 + p^2c^2 \qquad \left(E = E_0 + W\right)$$

Hence:

$$dE = vdp$$

The rate of energy gain per unit length of acceleration (along z) is then:

$$\frac{dE}{dz} = v\frac{dp}{dz} = \frac{dp}{dt} = eE_z$$

and the kinetic energy gained from the field along the z path is:

$$dW = dE = eE_z dz \qquad \Rightarrow \qquad W = e\int E_z dz = eV$$

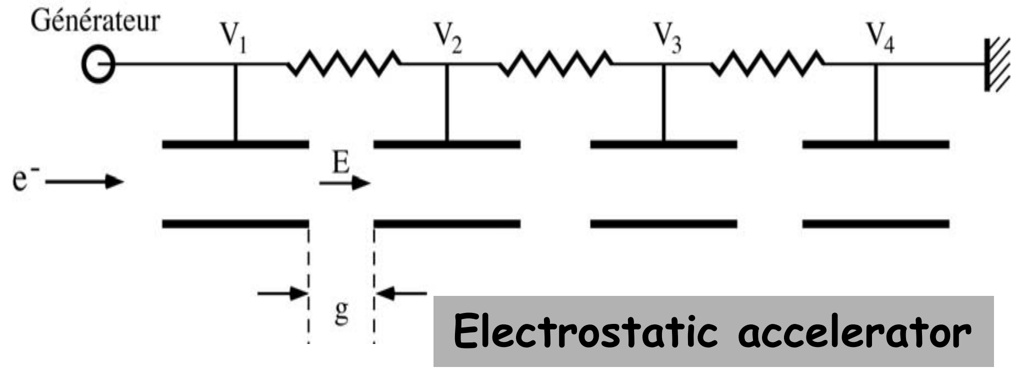
where V is just a potential

Methods of Acceleration

1_ Electrostatic Field

Energy gain : $W=n.e(V_2-V_1)$

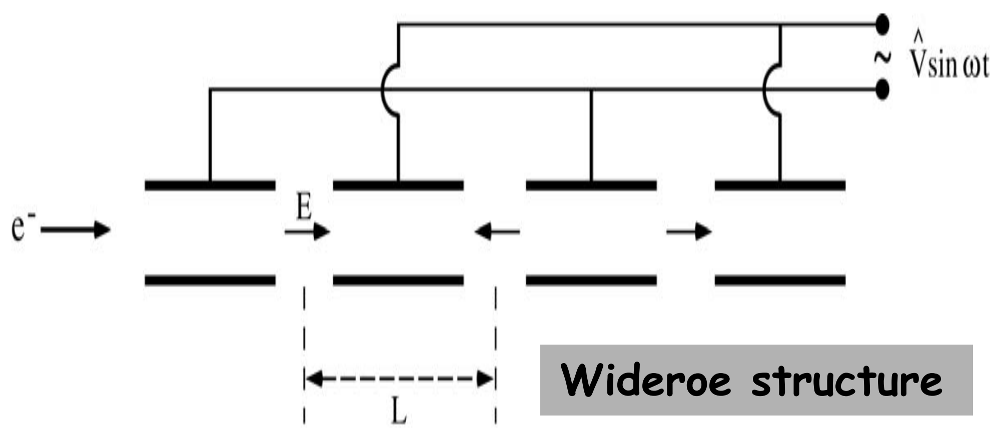
limitation : $V_{generator} = \Sigma\ V_i$



2_ Radio-frequency Field

Synchronism : ➡ L=vT/2

v=particle velocity **T= RF period**



Methods of Acceleration (2)

3_ Acceleration by induction

From MAXWELL EQUATIONS :

The electric field is derived from a scalar potential ϕ and a vector potential ***A***
The time variation of the magnetic field **H** generates an electric field **E**

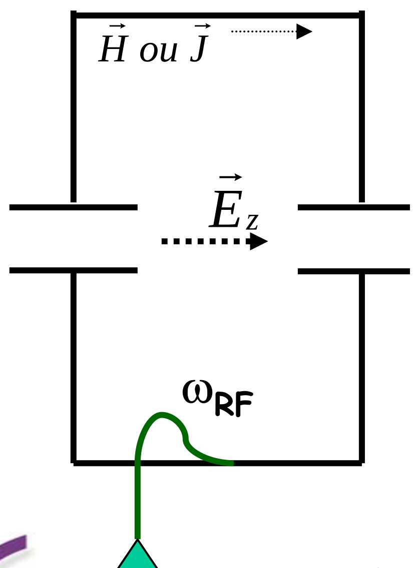
$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \mu \vec{H} = \vec{\nabla} \times \vec{A}$$

The advantage of Resonant Cavities

- Considering RF acceleration, it is obvious that when particles get high velocities the drift spaces get longer and one loses on the efficiency. The solution consists of using a higher operating frequency.

- The power lost by radiation, due to circulating currents on the electrodes, is proportional to the RF frequency. The solution consists of enclosing the system in a cavity which resonant frequency matches the RF generator frequency.

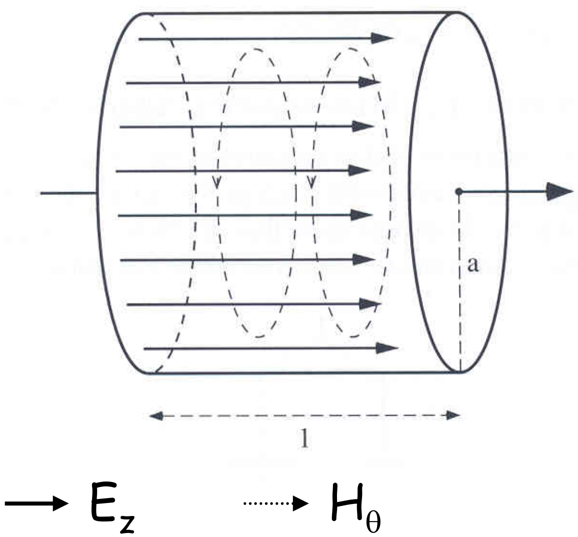


-Each such cavity can be independently powered from the RF generator.

- The electromagnetic power is now constrained in the resonant volume.

- Note however that joule losses will occur in the cavity walls (unless made of superconducting materials)

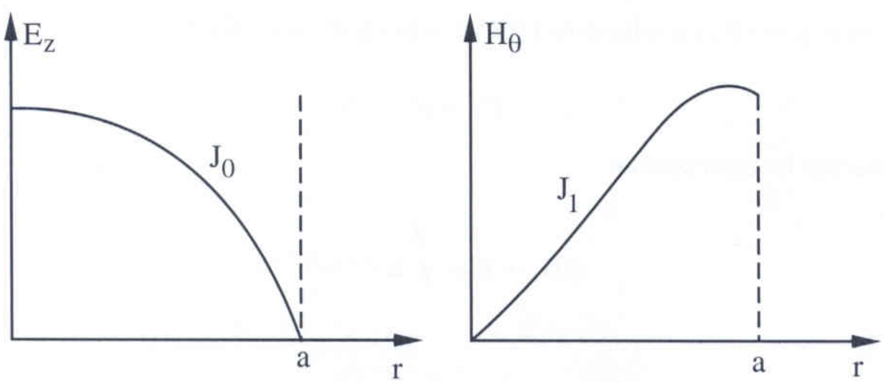
The Pill Box Cavity



From Maxwell's equations one can derive the wave equations :

$$\nabla^2 A - \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = 0 \qquad (A = E \text{ ou } H)$$

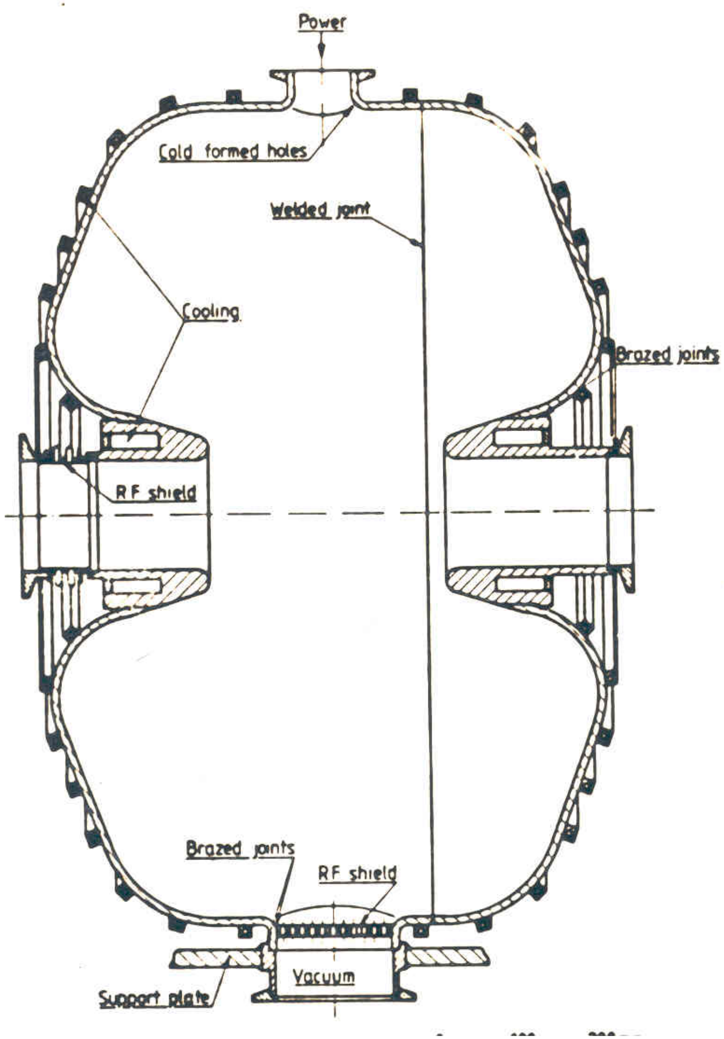
Solutions for E and H are oscillating modes, at discrete frequencies, of types TM ou TE. For l<2a the most simple mode, TM_{010}, has the lowest frequency ,and has only two field components:



$$\left.\begin{array}{l} E_z = J_0(kr) \\ H_\theta = -\dfrac{j}{Z_0} J_1(kr) \end{array}\right\} e^{j\omega t}$$

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \quad \lambda = 2{,}62\, a \quad Z_0 = 377\Omega$$

The Pill Box Cavity (2)



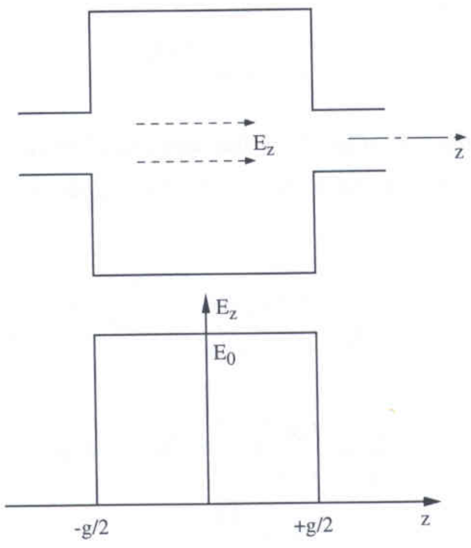
The design of a pill-box cavity can be sophisticated in order to improve its performances:

-A nose cone can be introduced in order to concentrate the electric field around the axis,

-Round shaping of the corners allows a better distribution of the magnetic field on the surface and a reduction of the Joule losses. It also prevent from multipactoring effects.

A good cavity is a cavity which efficiently transforms the RF power into accelerating voltage.

Transit Time Factor



Oscillating field at frequency ω and which amplitude is assumed to be constant all along the gap:

$$E_z = E_0 \cos\omega t = \frac{V}{g} \cos\omega t$$

Consider a particle passing through the middle of the gap at time t=0 :

$$z = vt$$

The total energy gain is: $$\Delta W = \frac{eV}{g} \int_{-g/2}^{g/2} \cos\omega \frac{z}{v} dz$$

$$\Delta W = eV \frac{\sin\theta/2}{\theta/2} = eVT$$

$$\theta = \frac{\omega g}{v} \quad \textit{transit angle}$$

$$T \quad \textit{transit time factor}$$

(0 < T < 1)

Transit Time Factor (2)

Consider the most general case and make use of complex notations:

$$\Delta E = e\Re_e \int_0^g E_z(z) e^{j\omega t} dz \qquad \omega t = \omega \frac{z}{v} - \psi_p$$

Ψ_p is the phase of the particle entering the gap with respect to the RF.

$$\Delta E = e\Re_e \left[e^{-j\psi_p} \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right]$$

$$\Delta E = e\Re_e \left[e^{-j\psi_p} e^{j\psi_i} \left| \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right| \right]$$

Introducing: $\phi = \psi_p - \psi_i$

$$\Delta E = e \left| \int_0^g E_z(z) e^{j\omega \frac{z}{v}} dz \right| \cos\phi$$

and considering the phase which yields the maximum energy gain:

$$T = \frac{\left| \int_0^g E_z(z) e^{j\omega t} dz \right|}{\int_0^g E_z(z) dz}$$

Important Parameters of Accelerating Cavities

Shunt Impedance

Relationship between gap voltage and wall losses.

$$P_d = \frac{V^2}{R}$$

Quality Factor

$$Q = \frac{\omega W_s}{P_d}$$

Relationship between stored energy in the volume and dissipated power on the walls.

$$\frac{R}{Q} = \frac{V^2}{\omega W_s}$$

Filling Time

$$P_d = -\frac{dW_s}{dt} = \frac{\omega}{Q} W_s$$

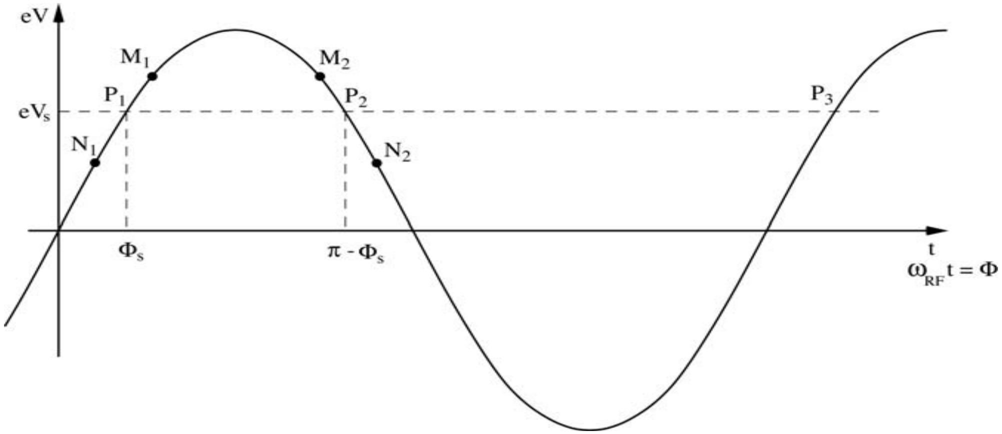
Exponential decay of the stored energy due to losses.

$$\tau = \frac{Q}{\omega}$$

Principle of Phase Stability

Let's consider a succession of accelerating gaps, operating in the 2π mode, for which the synchronism condition is fulfilled for a phase Φ_s .

For a 2π mode, the electric field is the same in all gaps at any given time.



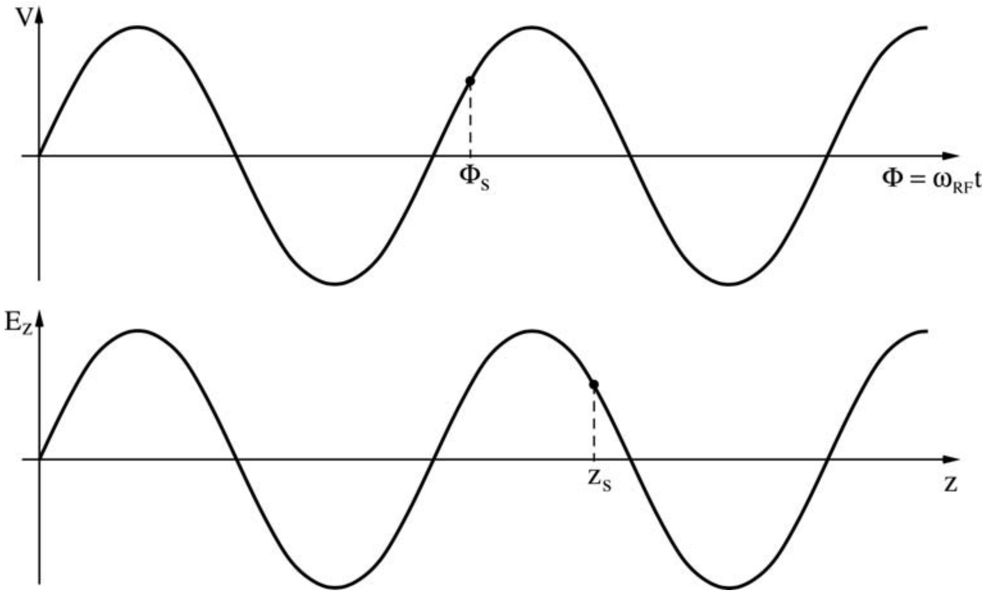
$$eV_S = e\hat{V}\sin\Phi_S$$ **is the energy gain in one gap for the particle to reach the next gap with the same RF phase: P_1 ,P_2, …… are fixed points.**

If an increase in energy is transferred into an increase in velocity, M_1 & N_1 will move towards P_1(stable), while M_2 & N_2 will go away from P_2 (unstable).

A Consequence of Phase Stability

Transverse Instability

Longitudinal phase stability means : $\dfrac{\partial V}{\partial t} > 0 \Rightarrow \dfrac{\partial E_z}{\partial z} < 0$

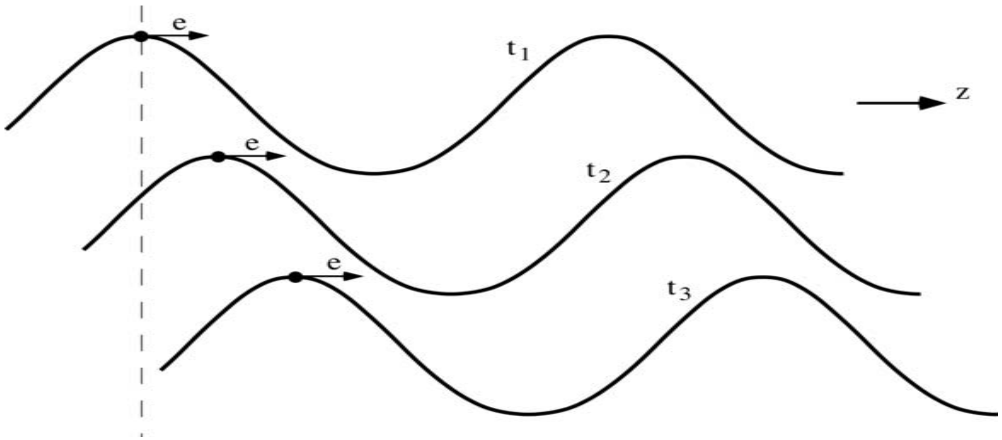


defocusing RF force

The divergence of the field is zero according to Maxwell : $\nabla.\vec{E} = 0 \Rightarrow \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_z}{\partial z} = 0 \Rightarrow \dfrac{\partial E_x}{\partial x} > 0$

External focusing (solenoid, quadrupole) is then necessary

The Traveling Wave Case



$$E_z = E_0 \cos(\omega_{RF} t - kz)$$

$$k = \frac{\omega_{RF}}{v_\varphi}$$

$$z = v(t - t_0)$$

The particle travels along with the wave, and k represents the wave propagation factor.

$$v_\varphi = phase\, velocity$$

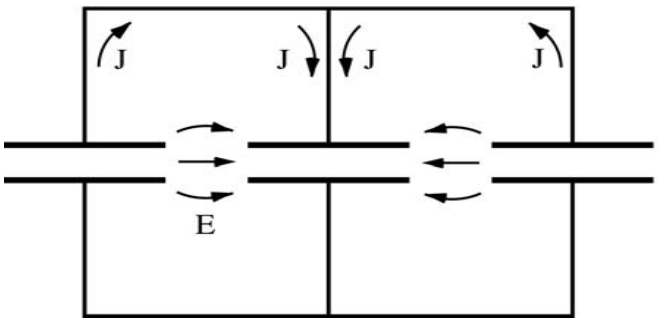
$$v = particle\, velocity$$

$$E_z = E_0 \cos\left(\omega_{RF} t - \omega_{RF} \frac{v}{v_\varphi} t - \phi_0\right)$$

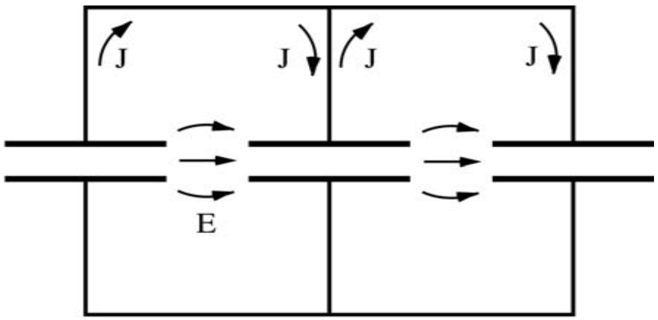
If synchronism satisfied: $v = v_\varphi \quad and \quad E_z = E_0 \cos\phi_0$

where ϕ_0 is the RF phase seen by the particle.

Multi-gaps Accelerating Structures:
A- Low Kinetic Energy Linac (protons,ions)

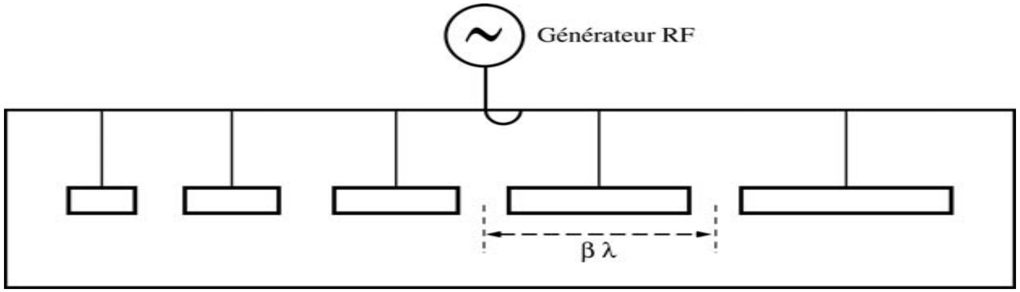


Mode π L= vT/2



Mode 2π L= vT = $\beta\lambda$

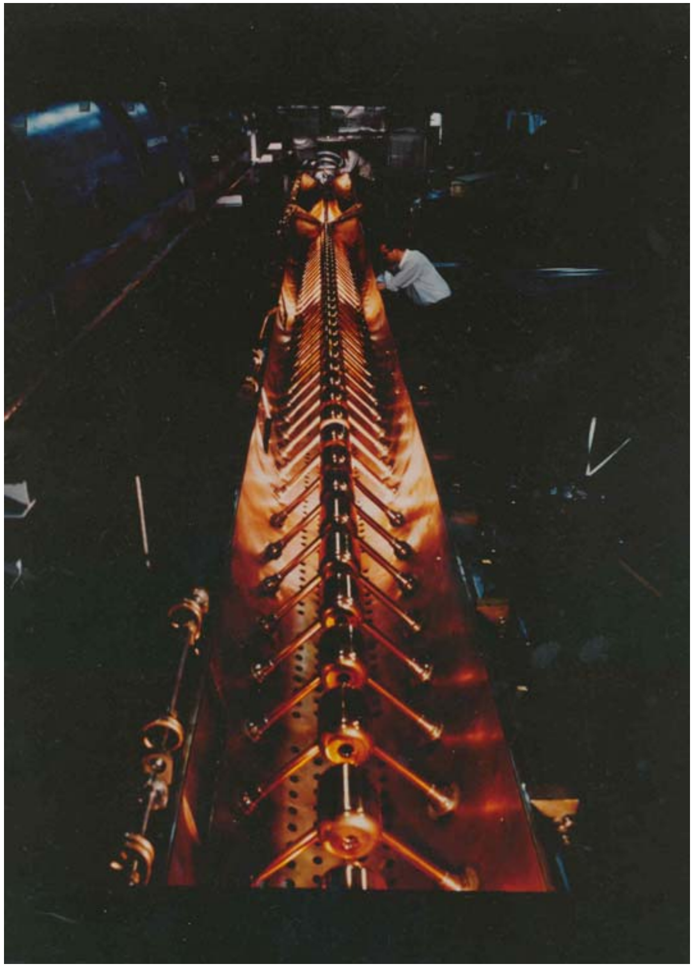
In « WIDEROE » structure radiated power $\propto \omega$ CV



ALVAREZ structure

In order to reduce the radiated power the gap is enclosed in a resonant volume at the operating frequency. A common wall can be suppressed if no circulating current in it for the chosen mode.

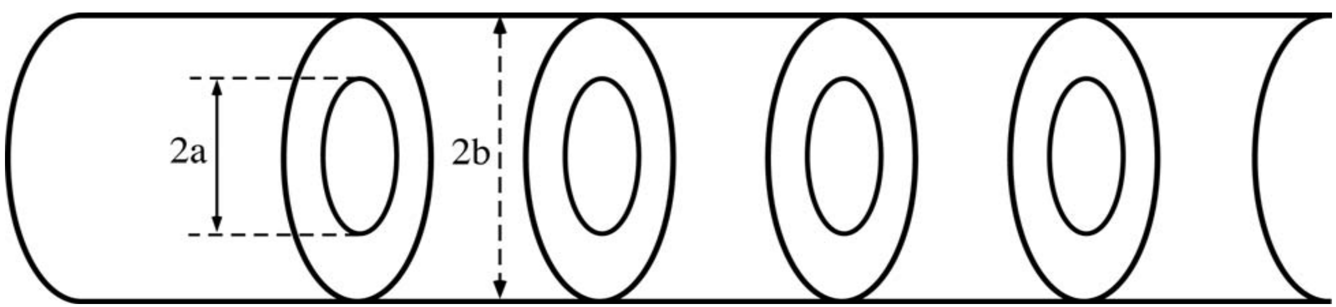
CERN Proton Linac



Multi-gaps Accelerating Structures:
B- High Energy Electron Linac

- When particles gets ultra-relativistic (v~c) the drift tubes become very long unless the operating frequency is increased. Late 40's the development of radar led to high power transmitters (klystrons) at very high frequencies (3 GHz).

- Next came the idea of suppressing the drift tubes using traveling waves. However to get a continuous acceleration the phase velocity of the wave needs to be adjusted to the particle velocity.



solution: slow wave guide with irises ⟶ iris loaded structure

Iris Loaded Structure for Electron Linac

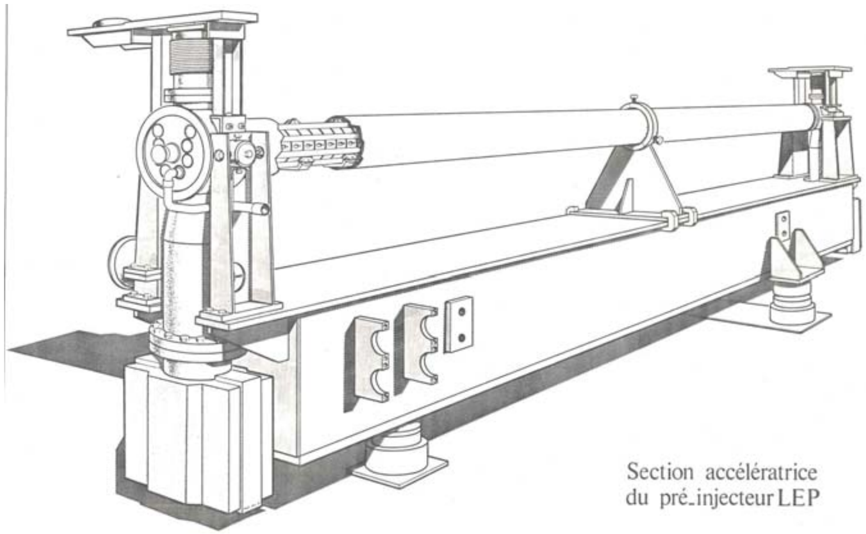
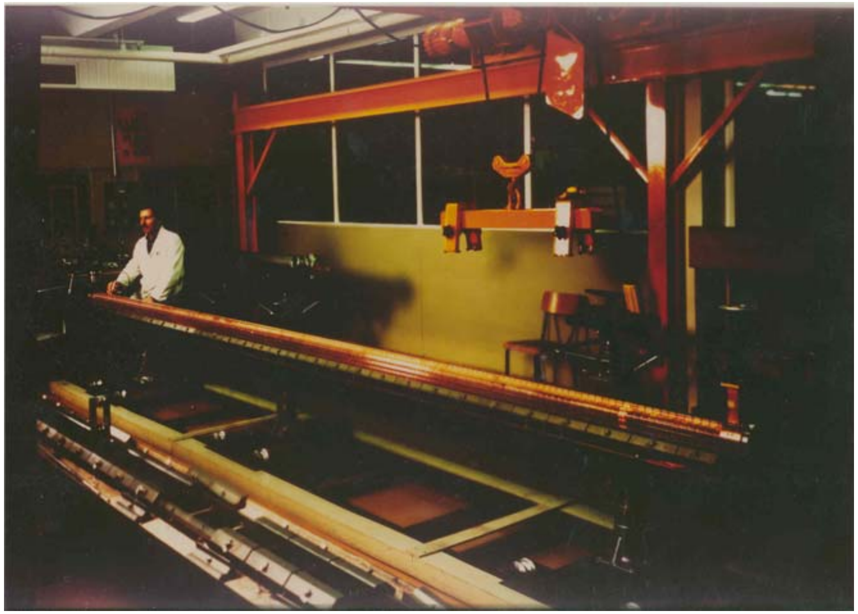


Photo of a CGR-MeV structure



4.5 m long copper structure, equipped with matched input and output couplers. Cells are low temperature brazed and a stainless steel envelope ensures proper vacuum.

Energy-phase Equations

- Rate of energy gain for the synchronous particle:

$$\frac{dE_s}{dz} = \frac{dp_s}{dt} = eE_0 \sin\phi_s$$

- Rate of energy gain for a non-synchronous particle, expressed in reduced variables, $w = W - W_s = E - E_s$ and $\varphi = \phi - \phi_s$:

$$\frac{dw}{dz} = eE_0\left[\sin(\phi_s + \varphi) - \sin\phi_s\right] \approx eE_0 \cos\phi_s . \varphi \quad (small\ \varphi)$$

- Rate of change of the phase with respect to the synchronous one:

$$\frac{d\varphi}{dz} = \omega_{RF}\left(\frac{dt}{dz} - \left(\frac{dt}{dz}\right)_s\right) = \omega_{RF}\left(\frac{1}{v} - \frac{1}{v_s}\right) \cong -\frac{\omega_{RF}}{v_s^2}(v - v_s)$$

Since:

$$v - v_s = c(\beta - \beta_s) \cong \frac{c}{2\beta_s}\left(\beta^2 - \beta_s^2\right) \cong \frac{w}{m_0 v_s \gamma_s^3}$$

Energy-phase Oscillations

one gets:

$$\frac{d\varphi}{dz} = -\frac{\omega_{RF}}{m_0 v_s^3 \gamma_s^3} w$$

Combining the two first order equations into a second order one:

$$\frac{d^2\varphi}{dz^2} + \Omega_s^2 \varphi = 0$$

with

$$\Omega_s^2 = \frac{eE_0 \omega_{RF} \cos\phi_s}{m_0 v_s^3 \gamma_s^3}$$

Stable harmonic oscillations imply:

$$\Omega_s^2 > 0 \quad and\ real$$

hence: $\cos\phi_s > 0$

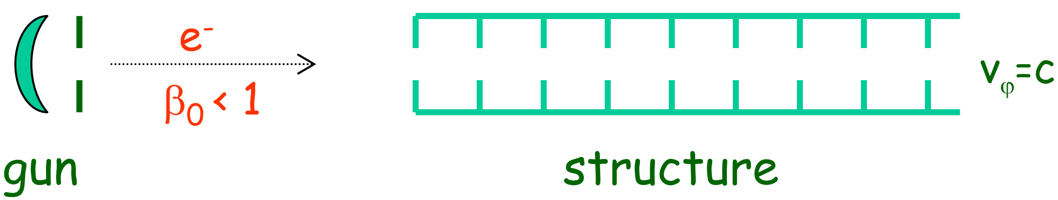
And since acceleration also means: $\sin\phi_s > 0$

One finally gets the results: $0 < \phi_s < \dfrac{\pi}{2}$

The Capture Problem

- Previous results show that at ultra-relativistic energies (γ >> 1) the longitudinal motion is frozen. Since this is rapidly the case for electrons, all traveling wave structures can be made identical (phase velocity=c).

- Hence the question is: can we capture low kinetic electrons energies (γ < 1), as they come out from a gun, using an iris loaded structure matched to c ?



$$E_z = E_0 \sin \phi(t)$$

The electron entering the structure, with velocity v < c, is not synchronous with the wave. The path difference, after a time dt, between the wave and the particle is:

$$dz = (c - v)dt$$

Since: $\phi = \omega_{RF} t - kz \quad with\ propagation\ factor\ k = \dfrac{\omega_{RF}}{v_\varphi} = \dfrac{\omega_{RF}}{c}$

one gets: $dz = \dfrac{c}{\omega_{RF}} d\phi = \dfrac{\lambda_g}{2\pi} d\phi$ and $\dfrac{d\phi}{dt} = \dfrac{2\pi}{\lambda_g} c(1 - \beta)$

The Capture Problem (2)

From Newton-Lorentz:

$$\frac{d}{dt}(mv) = m_0 c \frac{d}{dt}(\beta\gamma) = m_0 c \frac{d}{dt}\left(\frac{\beta}{\left(1-\beta^2\right)^{\frac{1}{2}}}\right) = eE_0 \sin\phi$$

Introducing a suitable variable: $\beta = \cos\alpha$

the equation becomes: $$\frac{d\alpha}{dt} = -\frac{eE_0}{m_0 c}\sin\phi \sin^2\alpha$$

Using $$\frac{d\phi}{dt} = \frac{d\phi}{d\alpha}\frac{d\alpha}{dt} \longrightarrow -\sin\phi\, d\phi = \frac{2\pi m_0 c^2}{\lambda_g e E_0}\frac{1-\cos\alpha}{\sin^2\alpha}d\alpha$$

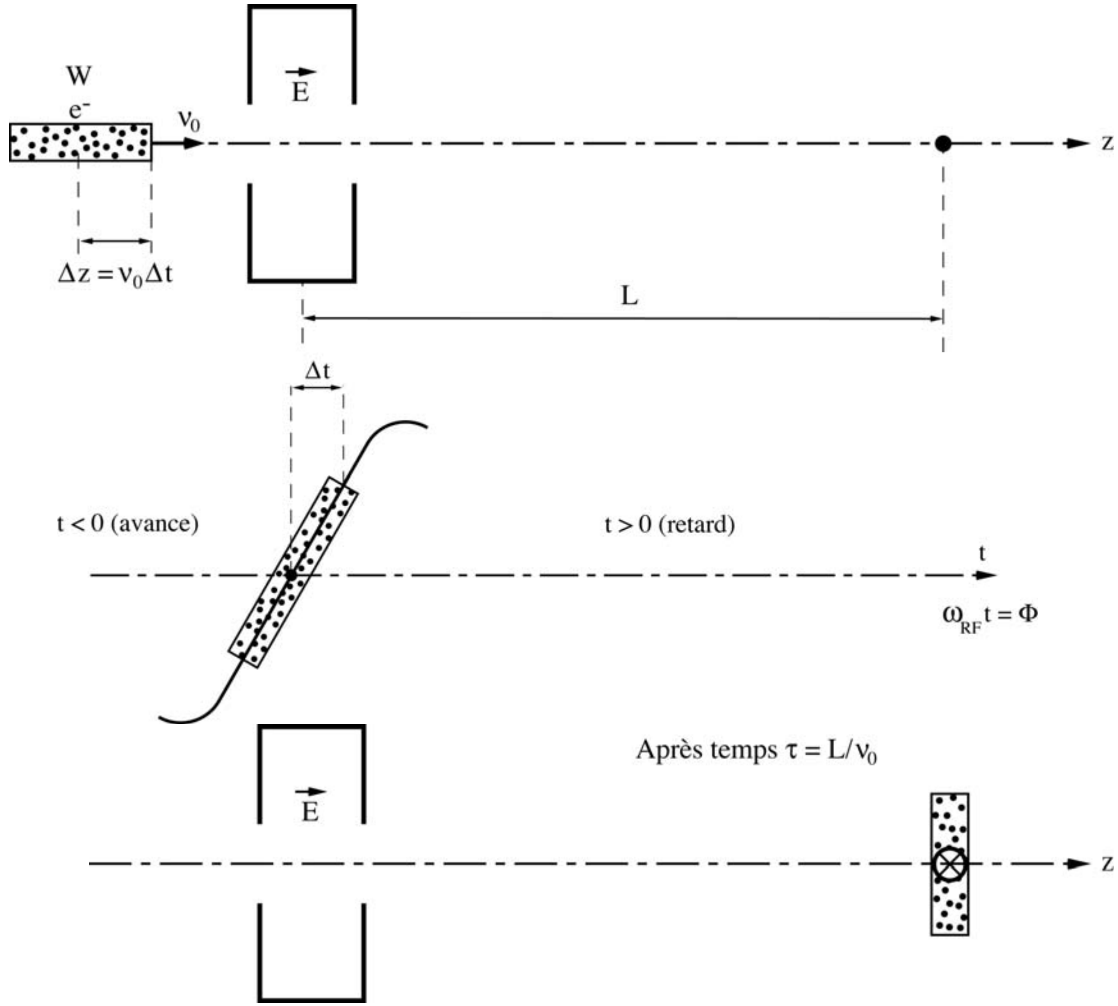
Integrating from t_0 to t $\longrightarrow$ $$\cos\phi_0 - \cos\phi = \frac{2\pi m_0 c^2}{e\lambda_g E_0}\left(\frac{1-\beta_0}{1+\beta_0}\right)^{\frac{1}{2}} \leq 2$$

(from $\beta=\beta_0$ to $\beta=1$)

Capture condition $\longrightarrow$ $$E_0 \geq \frac{\pi m_0 c^2}{e\lambda_g}\left(\frac{1-\beta_0}{1+\beta_0}\right)^{\frac{1}{2}}$$

Improved Capture With Pre-buncher

A long bunch coming from the gun enters an RF cavity; the reference particle is the one which has no velocity change. The others get accelerated or decelerated. After a distance L bunch gets shorter while energies are spread: bunching effect. This short bunch can now be captured more efficiently by a TW structure (v_φ=c).



Improved Capture With Pre-buncher (2)

The bunching effect is a space modulation that results from a velocity modulation and is similar to the phase stability phenomenon. Let's look at particles in the vicinity of the reference one and use a classical approach.

Energy gain as a function of cavity crossing time:

$$\Delta W = \Delta\left(\frac{1}{2} m_0 v^2\right) = m_0 v_0 \Delta v = eV_0 \sin\phi \approx eV_0 \phi$$

$$\Delta v = \frac{eV_0 \phi}{m_0 v_0}$$

Perfect linear bunching will occur after a time delay τ, corresponding to a distance L, when the path difference is compensated between a particle and the reference one:

$$\Delta v.\tau = \Delta z = v_0 t = v_0 \frac{\phi}{\omega_{RF}}$$

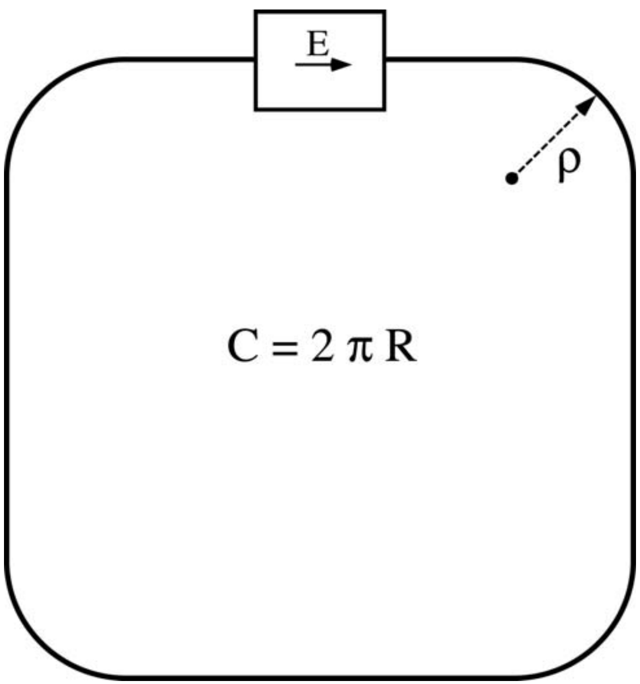
(assuming the reference particle enters the cavity at time t=0)

Since L = vτ one gets:

$$L = \frac{2 v_0 W}{eV_0 \omega_{RF}}$$

The Synchrotron

The synchrotron is a synchronous accelerator since there is a synchronous RF phase for which the energy gain fits the increase of the magnetic field at each turn. That implies the following operating conditions:



$e\hat{V}\sin\Phi \longrightarrow$ Energy gain per turn

$\Phi = \Phi_s = cte \longrightarrow$ Synchronous particle

$\omega_{RF} = h\omega_r \longrightarrow$ RF synchronism

$\rho = cte \quad R = cte \longrightarrow$ Constant orbit

$B\rho = P/e \Rightarrow B \longrightarrow$ Variable magnetic field

If v = c, ω_r hence ω_{RF} remain constant (ultra-relativistic e^-)

The Synchrotron (2)

Energy ramping is simply obtained by varying the B field:

$$p = eB\rho \quad \Rightarrow \quad \frac{dp}{dt} = e\rho\, B' \quad \Rightarrow (\Delta p)_{turn} = e\rho\, B' T_r = \frac{2\pi\, e\rho\, RB'}{v}$$

Since: $$E^2 = E_0^2 + p^2 c^2 \quad \Rightarrow \quad \Delta E = v\Delta p$$

$$(\Delta E)_{turn} = (\Delta W)_s = 2\pi e\rho RB' = e\hat{V}\sin\phi_s$$

- The number of stable synchronous particles is equal to the harmonic number h. They are equally spaced along the circumference.
- Each synchronous particle satifies the relation p=eBρ. They have the nominal energy and follow the nominal trajectory.

The Synchrotron (3)

During the energy ramping, the RF frequency increases to follow the increase of the revolution frequency :

$$\omega_r = \frac{\omega_{RF}}{h} = \omega(B, R_s)$$

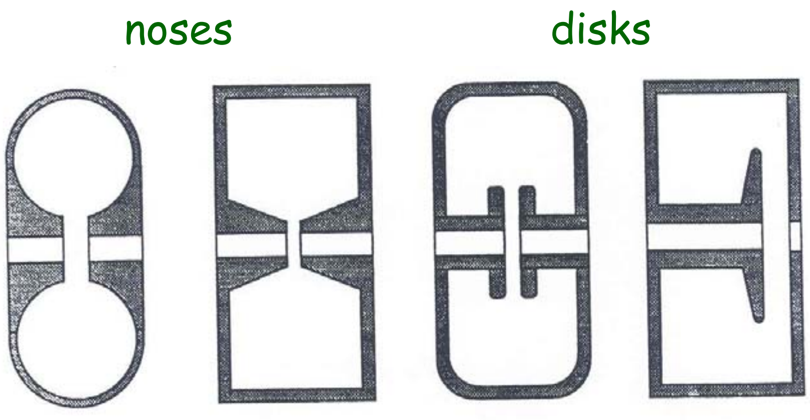
hence : $$\frac{f_{RF}(t)}{h} = \frac{v(t)}{2\pi R_s} = \frac{1}{2\pi}\frac{e}{m} < B(t) > \quad \Rightarrow \quad \frac{f_{RF}(t)}{h} = \frac{1}{2\pi}\frac{ec^2}{E_s(t)}\frac{r}{R_s}B(t)$$

Since $E^2 = m_0^2 c^4 + p^2 c^2$, the RF frequency must follow the variation of the B field with the law : $$\frac{f_{RF}(t)}{h} = \frac{c}{2\pi R_s}\left\{\frac{B(t)^2}{(m_0 c^2 / ecr)^2 + B(t)^2}\right\}^{1/2}$$ which asymptotically tends towards $f_r \to \frac{c}{2\pi R}$ when B becomes large compare to $(m_0c^2 / 2\pi r)$ which corresponds to $v \longrightarrow c$ ($pc \gg m_0c^2$). In practice the B field can follow the law:

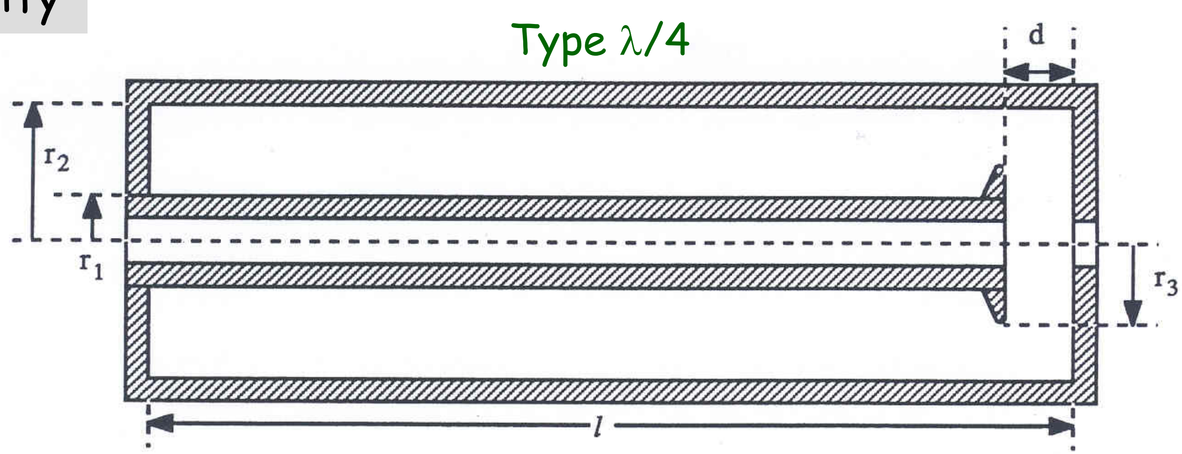
$$B(t) = \frac{B}{2}(1 - \cos\omega t) = B\sin^2\frac{\omega}{2}t$$

Single Gap Types Cavities

Pill-box variants

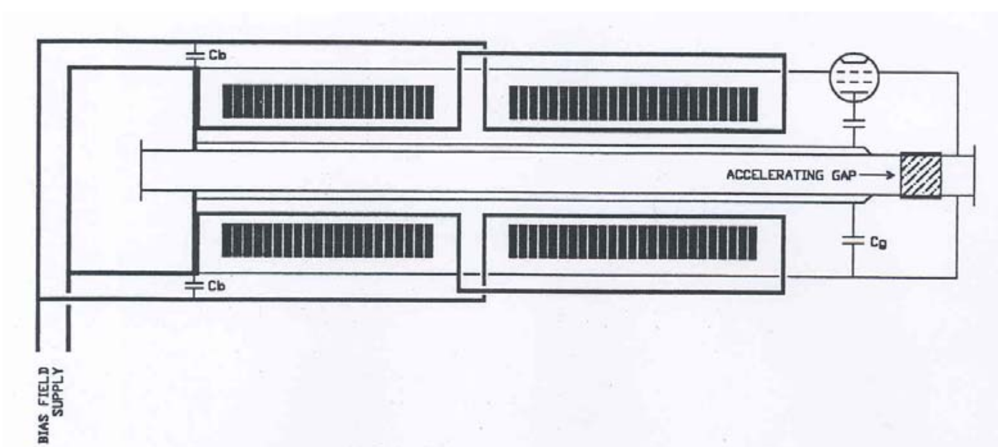


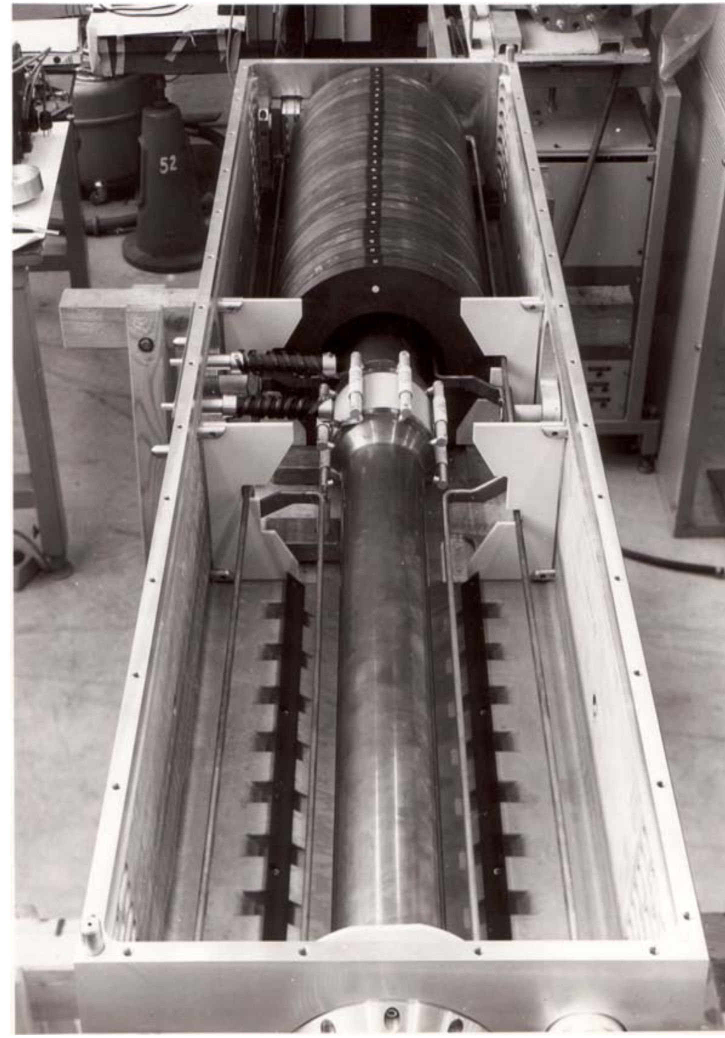
Coaxial cavity



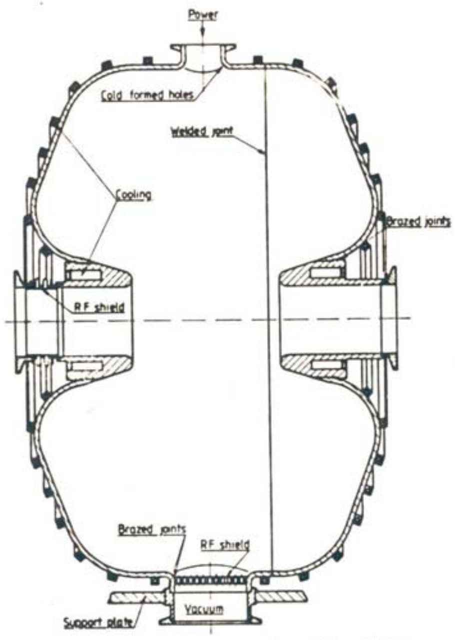
Ferrite Loaded Cavities

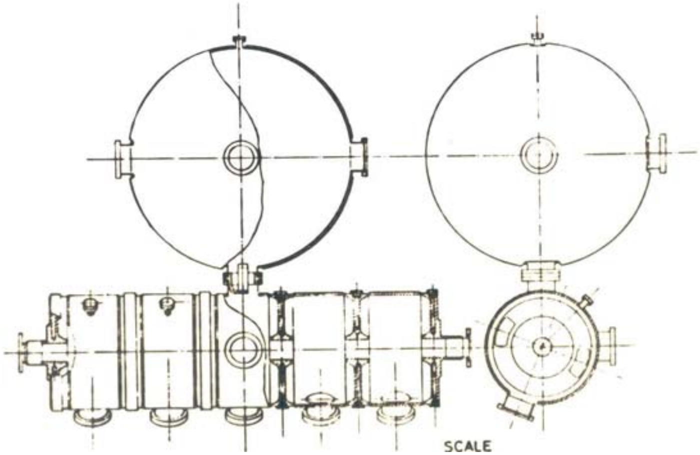
- Ferrite toroids are placed around the beam tube which allow to reach lower frequencies at reasonable size.
- Polarizing the ferrites will change the resonant frequency, hence satisfying energy ramping in protons and ions synchrotrons.

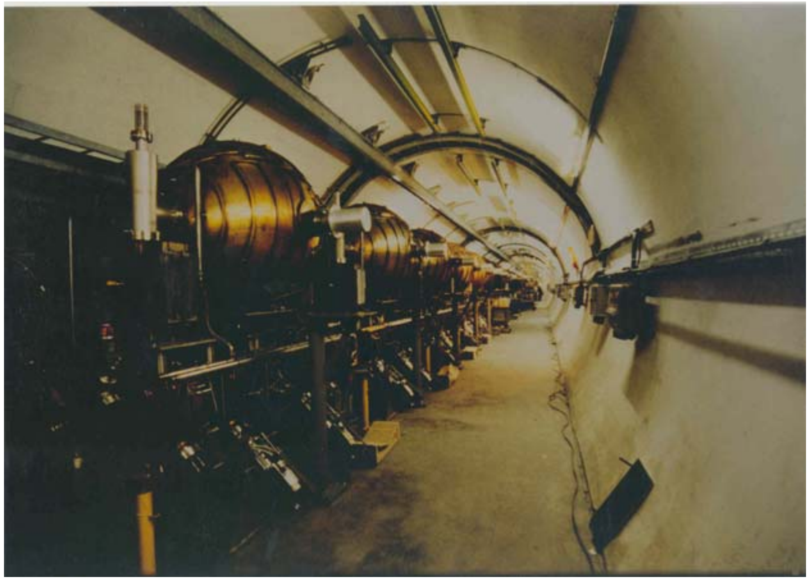


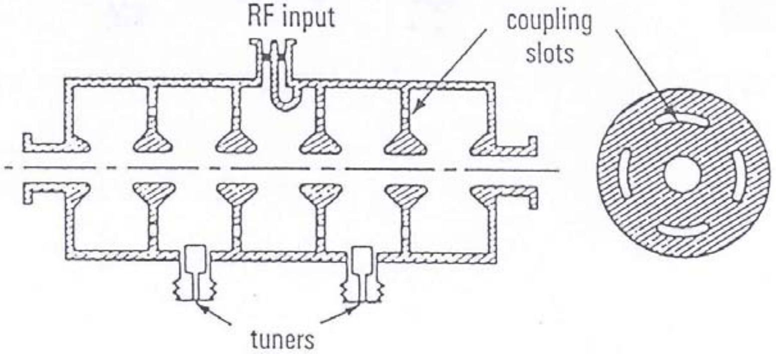


High Q cavities for e^- Synchrotrons

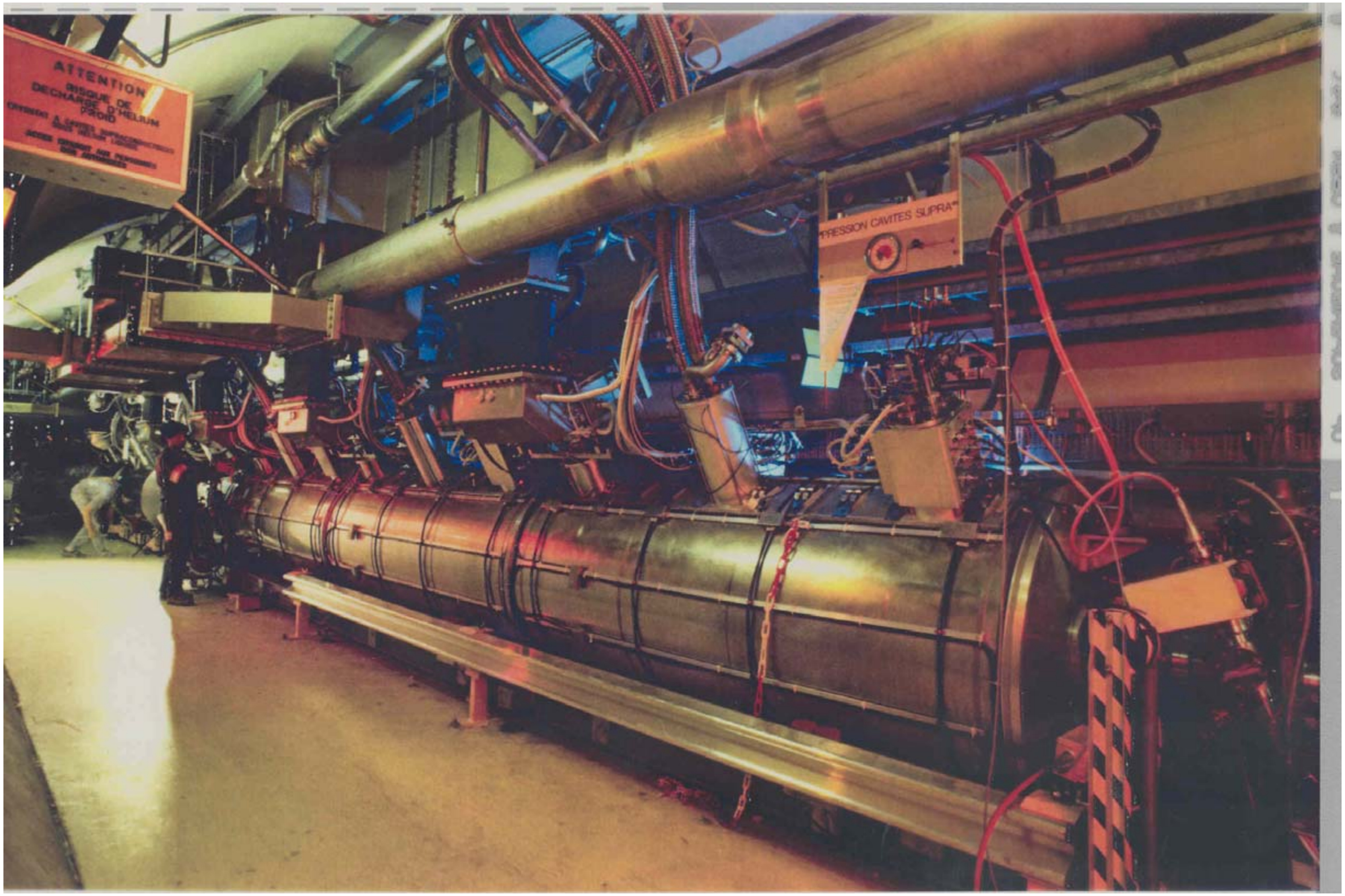




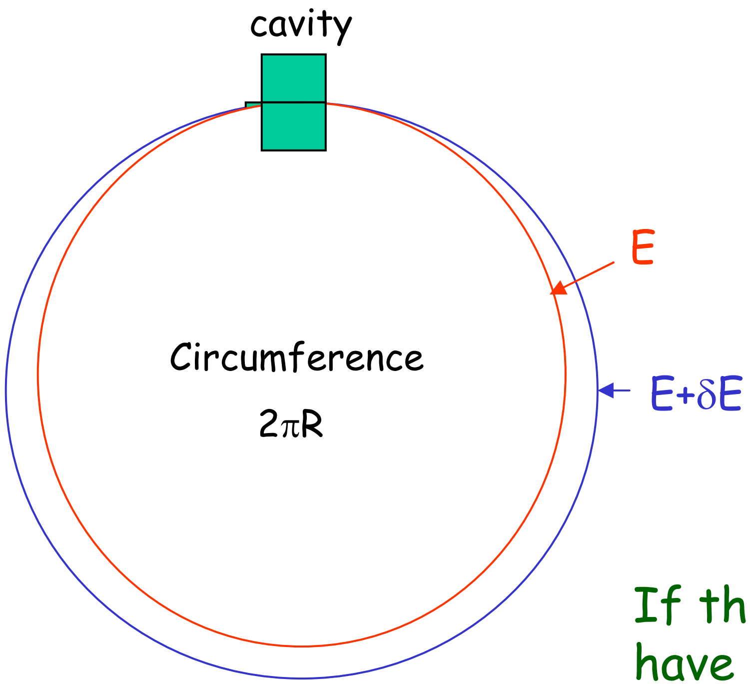




LEP 2: 2x100 GeV with SC cavities



Dispersion Effects in a Synchrotron



If a particle is slightly shifted in momentum it will have a different orbit:

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

This is the "momentum compaction" generated by the bending field.

If the particle is shifted in momentum it will have also a different velocity. As a result of both effects the revolution frequency changes:

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

p=particle momentum

R=synchrotron physical radius

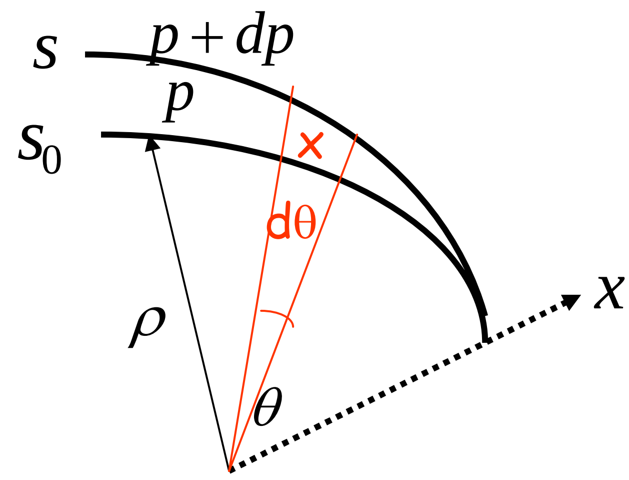
f_r=revolution frequency

Dispersion Effects in a Synchrotron (2)

$$\alpha = \frac{p}{R}\frac{dR}{dp}$$

$$ds_0 = \rho d\theta$$

$$ds = (\rho + x)d\theta$$



The elementary path difference from the two orbits is:

$$\frac{ds - ds_0}{ds_0} = \frac{dl}{ds_0} = \frac{x}{\rho}$$

leading to the total change in the circumference:

$$\int dl = 2\pi dR = \int \frac{x}{\rho} ds_0 = \frac{1}{\rho}\int_m x ds_0 \quad \Rightarrow \quad dR = \langle x \rangle_m$$

$\langle\ \rangle_m$ means that the average is considered over the bending magnet only

Since: $$x = D_x \frac{dp}{p}$$ we get: $$\alpha = \frac{\langle D_x \rangle_m}{R}$$

Dispersion Effects in a Synchrotron (3)

$$\eta = \frac{p}{f_r}\frac{df_r}{dp}$$

$$f_r = \frac{\beta c}{2\pi R} \Rightarrow \frac{df_r}{f_r} = \frac{d\beta}{\beta} - \frac{dR}{R}$$

$$p = mv = \beta\gamma\frac{E_0}{c} \Rightarrow \frac{dp}{p} = \frac{d\beta}{\beta} + \frac{d\left(1-\beta^2\right)^{-\frac{1}{2}}}{\left(1-\beta^2\right)^{-\frac{1}{2}}} = \left(1-\beta^2\right)^{-1}\frac{d\beta}{\beta}$$

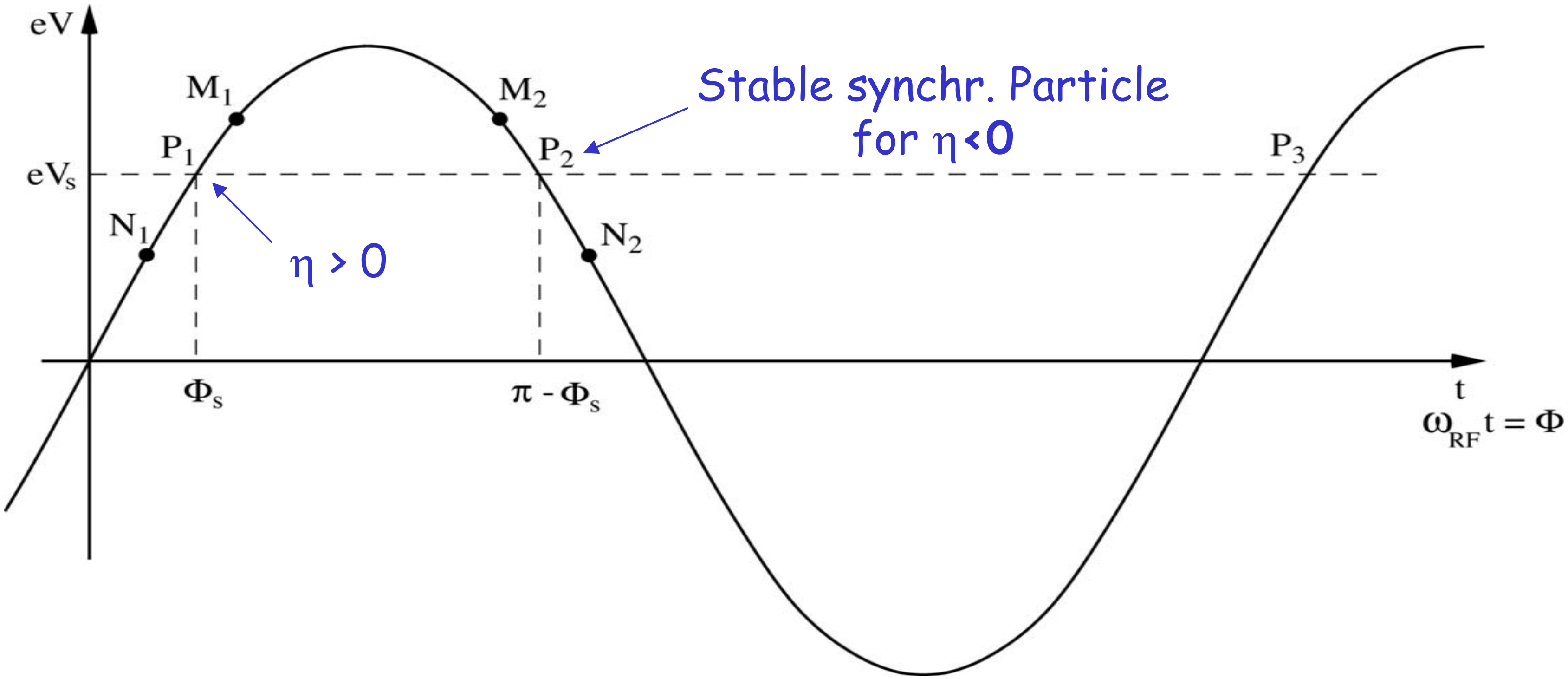
$$\frac{df_r}{f_r} = \left(\frac{1}{\gamma^2} - \alpha\right)\frac{dp}{p} \longrightarrow \eta = \frac{1}{\gamma^2} - \alpha$$

η=0 at the transition energy

$$\gamma_{tr} = \frac{1}{\sqrt{\alpha}}$$

Phase Stability in a Synchrotron

From the definition of η it is clear that below transition an increase in energy is followed by a higher revolution frequency (increase in velocity dominates) while the reverse occurs above transition ($v \approx c$ and longer path) where the momentum compaction (generally > 0) dominates.



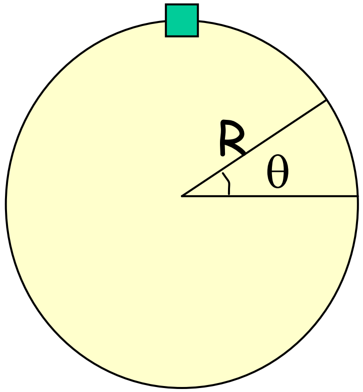
Longitudinal Dynamics

It is also often called " synchrotron motion".

The RF acceleration process clearly emphasizes two coupled variables, the energy gained by the particle and the RF phase experienced by the same particle. Since there is a well defined synchronous particle which has always the same phase ϕ_s, and the nominal energy E_s, it is sufficient to follow other particles with respect to that particle. So let's introduce the following reduced variables:

revolution frequency :	$\Delta f_r = f_r - f_{rs}$
particle RF phase :	$\Delta\phi = \phi - \phi_s$
particle momentum :	$\Delta p = p - p_s$
particle energy :	$\Delta E = E - E_s$
azimuth angle :	$\Delta\theta = \theta - \theta_s$

First Energy-Phase Equation



$$f_{RF} = hf_r \quad \Rightarrow \quad \Delta\phi = -h\Delta\theta \quad with \quad \theta = \int \omega_r dt$$

For a given particle with respect to the reference one:

$$\Delta\omega_r = \frac{d}{dt}\left(\Delta\theta\right) = -\frac{1}{h}\frac{d}{dt}\left(\Delta\phi\right) = -\frac{1}{h}\frac{d\phi}{dt}$$

Since: $\eta = \frac{p_s}{\omega_{rs}}\left(\frac{d\omega_r}{dp}\right)_s$ and $E^2 = E_0^2 + p^2c^2$

$$\Delta E = v_s \Delta p = \omega_{rs} R_s \Delta p$$

one gets:

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}}\frac{d\left(\Delta\phi\right)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}}\dot{\phi}$$

Second Energy-Phase Equation

The rate of energy gained by a particle is:

$$\frac{dE}{dt} = e\hat{V}\sin\phi \, \frac{\omega_r}{2\pi}$$

The rate of relative energy gain with respect to the reference particle is then:

$$2\pi\Delta\left(\frac{\dot{E}}{\omega_r}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

Expanding the left hand side to first order:

$$\Delta\left(\dot{E}T_r\right) \cong \dot{E}\,\Delta T_r + T_{rs}\,\Delta\dot{E} = \Delta E\,\dot{T}_r + T_{rs}\,\Delta\dot{E} = \frac{d}{dt}\left(T_{rs}\,\Delta E\right)$$

leads to the second energy-phase equation:

$$\boxed{2\pi\frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}\left(\sin\phi - \sin\phi_s\right)}$$

Equations of Longitudinal Motion

$$\frac{\Delta E}{\omega_{rs}} = -\frac{p_s R_s}{h\eta\omega_{rs}} \frac{d(\Delta\phi)}{dt} = -\frac{p_s R_s}{h\eta\omega_{rs}} \dot{\phi}$$

$$2\pi \frac{d}{dt}\left(\frac{\Delta E}{\omega_{rs}}\right) = e\hat{V}(\sin\phi - \sin\phi_s)$$

deriving and combining

$$\frac{d}{dt}\left[\frac{R_s p_s}{h\eta\omega_{rs}} \frac{d\phi}{dt}\right] + \frac{e\hat{V}}{2\pi}(\sin\phi - \sin\phi_s) = 0$$

This second order equation is non linear. Moreover the parameters within the bracket are in general slowly varying with time…………………

Small Amplitude Oscillations

Let's assume constant parameters R_s, p_s, ω_s and η:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

with

$$\Omega_s^2 = \frac{h\eta\omega_{rs}e\hat{V}\cos\phi_s}{2\pi R_s p_s}$$

Consider now small phase deviations from the reference particle:

$$\sin\phi - \sin\phi_s = \sin\left(\phi_s + \Delta\phi\right) - \sin\phi_s \simeq \cos\phi_s \Delta\phi$$

(for small $\Delta\phi$)

and the corresponding linearized motion reduces to a harmonic oscillation:

$$\ddot{\phi} + \Omega_s^2 \Delta\phi = 0$$

stable for $\Omega_s^2 > 0$ and Ω_s real

$\gamma < \gamma_{tr}$	$\eta > 0$	$0 < \phi_s < \pi/2$	$\sin\phi_s > 0$
$\gamma > \gamma_{tr}$	$\eta < 0$	$\pi/2 < \phi_s < \pi$	$\sin\phi_s > 0$

Large Amplitude Oscillations

For larger phase (or energy) deviations from the reference the second order differential equation is non-linear:

$$\ddot{\phi} + \frac{\Omega_s^2}{\cos\phi_s}\left(\sin\phi - \sin\phi_s\right) = 0$$

(Ω_s as previously defined)

Multiplying by $\dot{\phi}$ and integrating gives an invariant of the motion:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = I$$

which for small amplitudes reduces to:

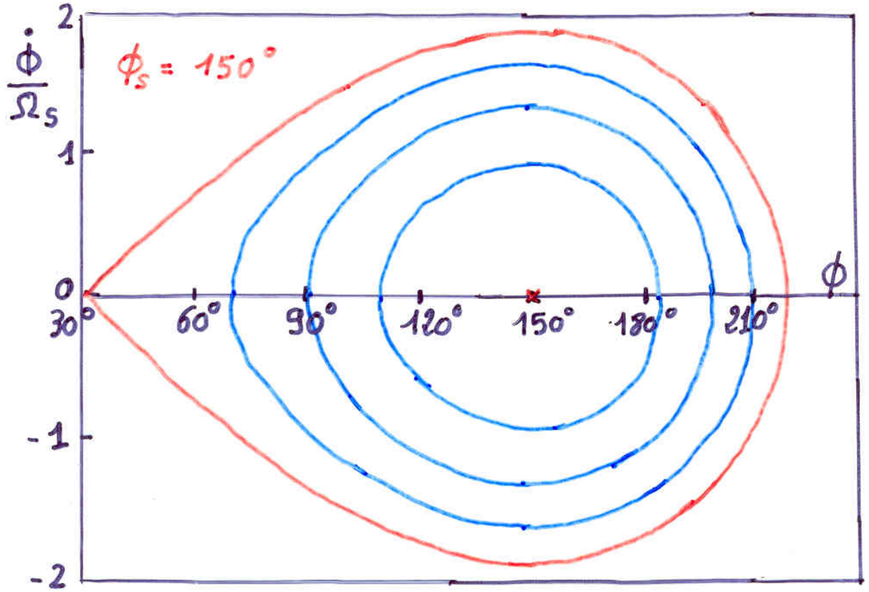
$$\frac{\dot{\phi}^2}{2} + \Omega_s^2\frac{(\Delta\phi)^2}{2} = I$$

(the variable is $\Delta\phi$ and ϕ_s is constant)

Similar equations exist for the second variable : $\Delta E \propto d\phi/dt$

Large Amplitude Oscillations (2)

When ϕ reaches $\pi-\phi_s$ the force goes to zero and beyond it becomes non restoring. Hence $\pi-\phi_s$ is an extreme amplitude for a stable motion which in the phase space($\frac{\dot{\phi}}{\Omega_s}, \Delta\phi$) is shown as closed trajectories.



Equation of the separatrix:

$$\frac{\dot{\phi}^2}{2} - \frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s\right)$$

Second value ϕ_m where the separatrix crosses the horizontal axis:

$$\cos\phi_m + \phi_m\sin\phi_s = \cos(\pi-\phi_s) + (\pi-\phi_s)\sin\phi_s$$

Energy Acceptance

From the equation of motion it is seen that $\dot{\phi}$ reaches an extremum when $\ddot{\phi} = 0$, hence corresponding to $\phi = \phi_s$.

Introducing this value into the equation of the separatrix gives:

$$\dot{\phi}^2_{\max} = 2\Omega_s^2 \left\{ 2 + \left(2\phi_s - \pi \right) \tan \phi_s \right\}$$

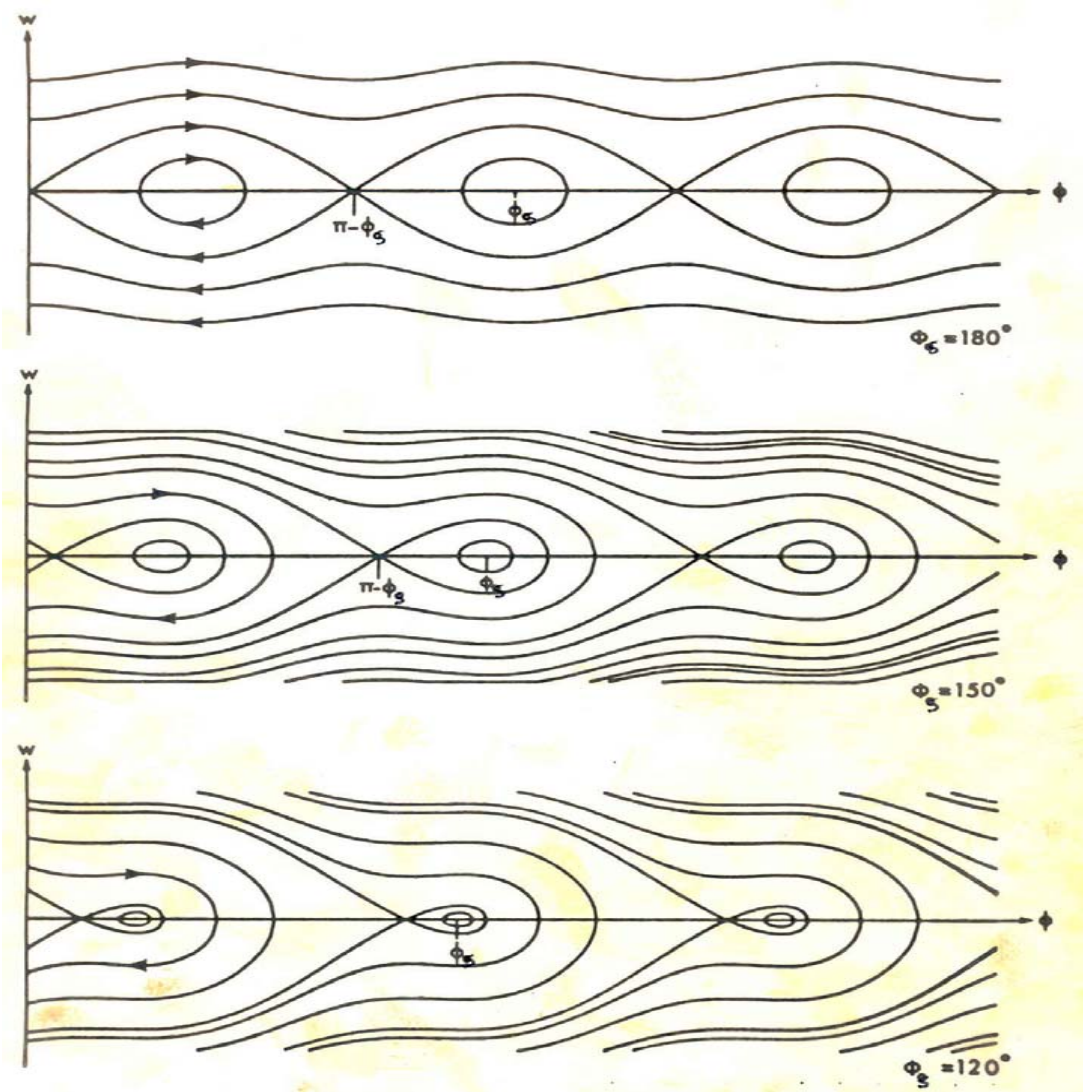
That translates into an acceptance in energy:

$$\left(\frac{\Delta E}{E_s} \right)_{\max} = \mp \beta \left\{ - \frac{e\hat{V}}{\pi h \eta E_s} G(\phi_s) \right\}^{\frac{1}{2}}$$

$$G(\phi_s) = \left[2\cos\phi_s + (2\phi_s - \pi)\sin\phi_s \right]$$

This "RF acceptance" depends strongly on ϕ_s and plays an important role for the electron capture at injection, and the stored beam lifetime.

RF Acceptance versus Synchronous Phase



As the synchronous phase gets closer to 90° the area of stable motion (closed trajectories) gets smaller. These areas are often called "BUCKET".

The number of circulating buckets is equal to "h".

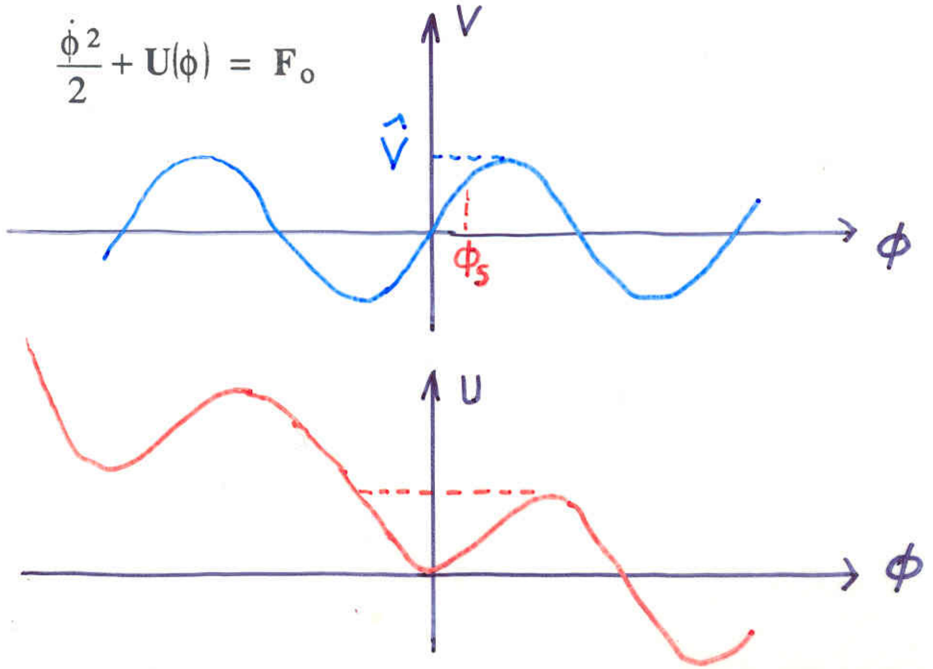
The phase extension of the bucket is maximum for ϕ_s =180° (or 0°) which correspond to no acceleration . The RF acceptance increases with the RF voltage.

Potential Energy Function

The longitudinal motion is produced by a force that can be derived from a scalar potential:

$$\frac{d^2\phi}{dt^2} = F(\phi) \qquad F(\phi) = -\frac{\partial U}{\partial \phi}$$

$$U = -\int_0^{\phi} F(\phi)\, d\phi = -\frac{\Omega_s^2}{\cos\phi_s}\left(\cos\phi + \phi\sin\phi_s\right) - F_0$$



The sum of the potential energy and kinetic energy is constant and by analogy represents the total energy of a non-dissipative system.

Ions in Circular Accelerators

A =atomic number

Q =charge state

$q = Q\,e$

$W = E - E_r$

$P = q\,B\,r$

$E^2 = p^2c^2 + E_r^2$

$E_r = A\,E_0$

$m = \gamma\,m_r$

$P = m\,v$

$E = \gamma\,E_r$

$$E^2 - E_r^2 = (qcBr)^2$$

$$\frac{W}{A}\left(\frac{W}{A} + 2E_0\right) = \left(\frac{Q}{A}\right)^2 (ecBr)^2$$

Moreover:

$$dW = dE = \frac{E^2 - E_r^2}{E}\left[\frac{dB}{B} + \frac{dr}{r}\right]$$

$dr/r = 0$ synchrotron $dB/B = 0$ cyclotron