Multi-Particle Effects: Instabilities

Karlheinz SCHINDL/CERN-AB

Longitudinal Instabilities
- Basics
- "Negative Mass" Instability
- Stability Diagram and Landau Damping
- Longitudinal Stability Criterion
- Impedance (resonator)
- Line spectra: single particle, single bunch
- Bunched beam longitudinal instability:
  - one bunch; many bunches
- Higher-order coupled-bunch modes
- Microwave instability
- Cures

Transverse Instabilities
- Fields and forces
- Transverse coupling impedances
- Spectrum of beam signals
- Instability of un-bunched beam
- Bunched beam: Head-Tail instability
  - zero and non-zero chromaticity
- Many bunches – long and short
- Resistive wall instability
- Transverse wake fields
- Cures

Further Reading:

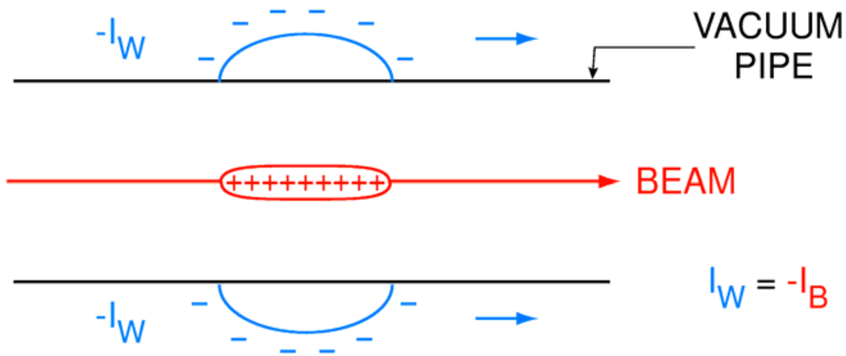
A. Hofmann, Single beam collective phenomena – longitudinal, CAS Erice, 1976, CERN 77-13, p. 139

J. Gareyte, Observation and correction of instabilities in circular accelerators, CERN SL/91-09 (AP), Joint US-CERN Accelerator School, Hilton Head Island, USA, 1990

F. Pedersen, Multi-bunch instabilities, CERN PS 93-36 (RF), Joint US-CERN Accelerator School, Benaldamena, Spain 1992

A.W. Chao, Physics of collective beam instabilities in high energy accelerators, John Wiley&Sons, New York, 1993

Longitudinal Instabilities - Basic Mechanism



Wall current I_w due to circulating bunch
Vacuum pipe not smooth, I_w sees an **IMPEDANCE** (resistive, capacitive, inductive)

Impedance $\mathbf{Z} = \mathbf{Z_r} + \mathbf{iZ_i}$

Induced voltage $\mathbf{V} \sim \mathbf{I_w}\,\mathbf{Z} = -\mathbf{I_B}\,\mathbf{Z}$

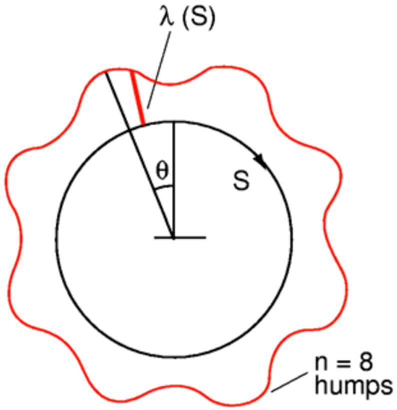
V acts back on the beam ⇨ INSTABILITIES INTENSITY DEPENDENT

General Scheme to investigate instabilities

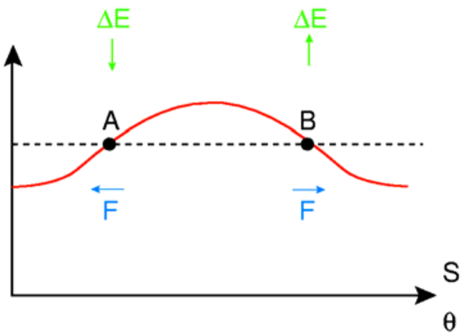
Step 1: Start with a **nominal particle distribution** (i.e. longitudinal position, density)
Step 2: Compute **fields** and **wall currents** induced by a **small perturbation** of this nominal distribution, and determine **forces acting back on the beam**
Step 3: Calculate **change of distribution** due to these forces

If Initial Small Perturbation → INCREASED? INSTABILITY
→ DECREASED? STABILITY

"Negative Mass" Instability - Qualitative



Line density modulation
"mode" with n=8 humps



Zooming in one modulation

Un-bunched (=coasting) beam in a proton/ion ring, travels around ring with angular frequency ω_0
Line density $\lambda(s)$ [particles/m]
is modulated around the synchrotron

WILL THE HUMPS INCREASE
OR ERODE?

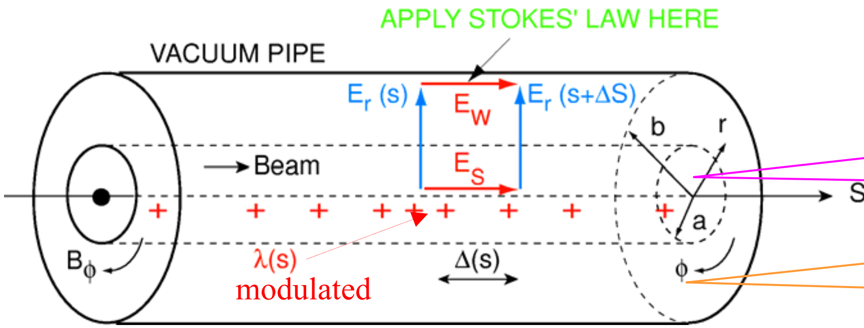
The **self-force F** (proportional to $-\partial\lambda/\partial s$)

- **Increases** energy of particles in B
- **Decreases** energy of particles in A

$\gamma < \gamma_t$: if $\Delta E \uparrow$ then $\Delta\omega_0 \uparrow$ — A and B move away from the hump eroding the mountain — **STABLE**

$\gamma > \gamma_t$: if $\Delta E \uparrow$ then $\Delta\omega_0 \downarrow$ — A and B move towards the hump enhancing the mountain — **UNSTABLE**

Negative Mass Instability: Fields Created by Beam



a…beam radius, b….pipe radius

For small modulations of $\lambda(s)$

r<a inside the beam

$$E_r = \frac{e\lambda}{2\varepsilon_0\pi}\frac{r}{a^2} \qquad B_\varphi = \frac{\mu_0}{2\pi}ec\lambda\beta\frac{r}{a^2}$$

r≥a outside the beam

$$E_r = \frac{e\lambda}{2\varepsilon_0\pi}\frac{1}{r} \qquad B_\varphi = \frac{\mu_0}{2\pi}ec\lambda\beta\frac{1}{r}$$

apply Stokes' law: $\oint_{\text{Line}} \vec{E}d\vec{l} = -\frac{\partial}{\partial t}\int_{\text{Surface}} \vec{B}d\vec{\sigma} = -\frac{\partial}{\partial t}\Delta s\int_0^b B_\varphi dr$ (Faraday's law of induction)

With $\frac{\partial\lambda}{\partial t} = -\frac{\partial\lambda}{\partial s}\frac{ds}{dt} = -\beta c\frac{\partial\lambda}{\partial s}$ and $g_0 = 1 + 2\ln(b/a)$, one gets

Pipe inductance per m

$$E_s = -\frac{eg_0}{4\varepsilon_0\pi}\frac{1}{\gamma^2}\frac{\partial\lambda}{\partial s} + \frac{L}{2R\pi}e\beta^2c^2\frac{\partial\lambda}{\partial s}$$

field seen by the beam

longitudinal "**space charge**" field: "**capacitive**"

Field due to inductive wall: "**inductive**"

Negative Mass Instability: Field Acting Back on Beam

$\lambda(s)$ has n humps and rotates with Ω **near** but **not exactly** $\mathbf{n\omega_0}$

$\lambda = \lambda_0 + \lambda_1 e^{i(n\Theta - \Omega t)}$, $I + I_0 + \mathbf{I_1} e^{i(n\Theta - \Omega t)}$ instantaneous density λ_1 and current $\mathbf{I_1}$

$$\underbrace{\mathbf{U_s}}_{\text{voltage per turn}} = \underbrace{-\mathbf{I_1}\, e^{i(n\Theta - \Omega t)}}_{\text{(small) AC component}} \times \underbrace{\mathbf{Z(\Omega)}}_{\text{longitudinal impedance}}$$

$\mathbf{U_s}$ perturbs the motion of the pattern and leads to a complex frequency shift $\Delta\Omega = \Delta\Omega_r + \mathbf{i}\Delta\Omega_i$

$\Omega = \mathbf{n\omega_0} + \Delta\Omega$ slightly perturbed frequency

A SHORTCUT TO CALCULATE $\Delta\Omega$

$$\underbrace{\left[\frac{E_0\beta^2\gamma}{2\pi\eta h f_0^2 e}\right]}_{\text{“m”}}\ddot{\varphi} + V_0\varphi = 0$$

We make use of the equation of **small-amplitude synchrotron oscillations** in a **stationary bucket**

This "**mass**" becomes **negative** above transition (η<0) and the motion unstable

$$\ddot{\varphi} + \underbrace{\left[\frac{e\eta h V_0 \omega_0^2}{2\pi E_0 \beta^2\gamma}\right]}_{\omega_s^2}\varphi = 0$$

ω_s^2 synchrotron frequency

V_0 …voltage per turn
f_0 ….revolution frequency
η….$1/\gamma^2 - 1/\gamma_t^2$
E_0…particle rest energy
h….harmonic number.

Negative Mass Instability: Shortcut to Compute $\Delta\Omega$

- Replace ω_s by $\Delta\Omega$
- Replace $\mathbf{hV_0}$ by **beam-induced voltage** $\mathbf{i\,n\,Z\,I_0}$ with $\mathbf{Z = Z_r + i\,Z_i}$ **complex impedance**

$$\boxed{(\Delta\Omega)^2 = (\Omega - n\omega_0)^2 = -i\frac{e\eta\omega_0^2 n I_0}{2\pi\beta^2 E_0\gamma}\ (\mathbf{Z_r + iZ_i})}$$

Complex Frequency shift required to sustain "**self-consistent**" modulation

$$I(t,\Theta) = I_0 + I_1 \underbrace{e^{\Delta\Omega_i t}}_{\text{growth or damping}} e^{i(n\theta - (n\omega_0 - \Delta\Omega_r)t)}$$

frequency shift

Instantaneous current with $\Delta\Omega = \Delta\Omega_r + i\Delta\Omega_i$

- $\mathbf{Z_r = 0}$: Vacuum pipe ideal conductor

From $U_s = -I_1 e^{i(n\Theta - \Omega t)}\mathbf{Z}$ and $Z_0 = 1/\varepsilon_0 c = 377\ \Omega$

$$\mathbf{Z_i} = \frac{n g_0 Z_0}{2\gamma^2\beta} - n\omega_0 L$$

space charge impedance

inductive impedance

- $\mathbf{Z_r \neq 0}$: realistic resistive vacuum pipe

$\mathbf{\Delta\Omega_i \neq 0}$

always one unstable solution

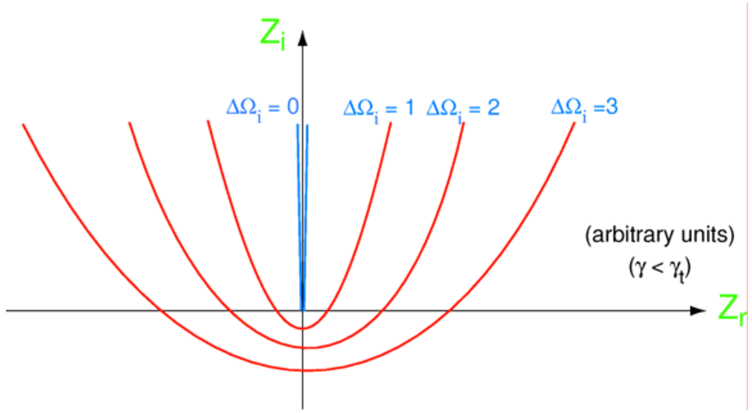
Z_i	$\gamma < \gamma_t\ (\eta > 0)$	$\gamma > \gamma_t\ (\eta < 0)$ $(m < 0)$
> 0 (capacitive)	$\Delta\Omega_i = 0$ STABLE	$\Delta\Omega_i \neq 0$ UNSTABLE
< 0 (inductive)	$\Delta\Omega_i \neq 0$ UNSTABLE	$\Delta\Omega_i = 0$ STABLE

Stability Diagram

- Relates (complex) growth rate $\Delta\Omega$ to (complex) impedance Z

$$(\Delta\Omega)^2 = -i\,\xi\,(Z_r + iZ_i) = \xi\,(Z_i - iZ_r) = (\Delta\Omega_r + i\Delta\Omega_i)^2$$

- Plot contours $\Delta\Omega_i = \text{const}$ (= equal growth rate) into Z_r, Z_i plane. Equating real and imaginary parts yields **parabolae** for $\Delta\Omega_i = \text{const}$ ⇨ $Z_r = 2\Delta\Omega_i\sqrt{Z_i/\xi + \Delta\Omega_i^{\,2}/\xi^2}$



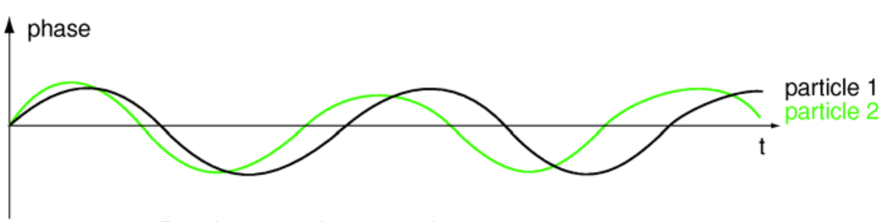
Stability Diagram

For any $Z_r \neq 0$ the unbunched beam is subject to the negative mass instability and is **unstable even at low intensity!**

Is there a way out?

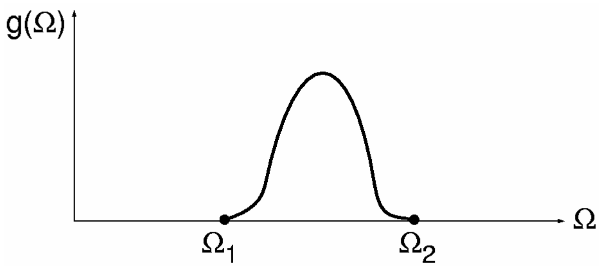
YES: LANDAU DAMPING

In real machines, the beam has an **energy spread**, so individual particles move with **different oscillation frequencies** around the ring → the **coherent motion becomes confused** and may **collapse faster** than the **rise time of the instability**



Decoherence of two particles

Landau Damping - Basic Idea



N particles (oscillators), each **resonating** at a frequency between Ω_1 and Ω_2 with a density $g(\Omega)$

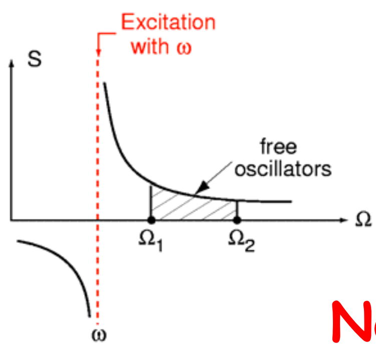
$$\int_{\Omega_1}^{\Omega_2} g(\Omega)\, d\Omega = 1$$ normalization

$$X = \frac{1}{\Omega^2 - \omega^2} e^{i\omega t} = \frac{1}{(\Omega-\omega)\underbrace{(\Omega+\omega)}_{\sim 2\Omega_0}} e^{i\omega t}$$

Response $\mathbf{X}$ of an individual oscillator with frequency Ω to an external excitation with ω

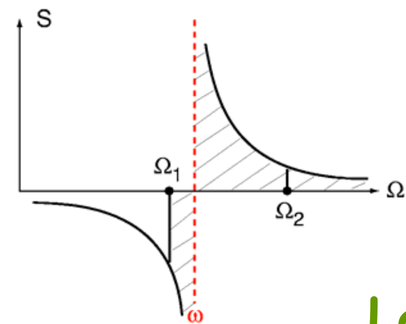
$$S = \frac{N}{2\Omega_0} \int_{\Omega_1}^{\Omega_2} \frac{i\,\frac{dg(\Omega)}{d\Omega}}{\Omega - \omega} d\Omega \cdot e^{i\omega t}$$

Coherent response of the beam obtained by summing up the single-particle responses of the n oscillators



External excitation is **outside the frequency range** of the oscillators

No damping

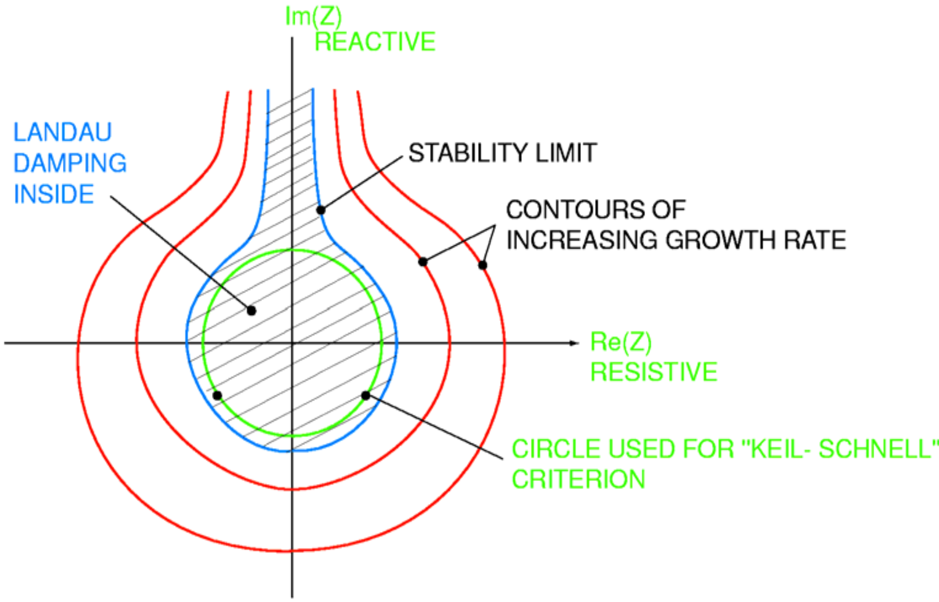


External excitation is **inside the frequency range** of the oscillators

The integral has a **pole at $\Omega = \omega$**

Landau damping

Landau Damping and Stability Diagram



The evaluation of the **integral with the pole** at $\Omega = \omega$ shows that **Landau Damping** only **works** if **coherent frequency** of the **external excitation** lies **inside the frequency spread** of the oscillators. The **stability diagram** has then a **stable region!**

Stability Diagram with Landau Damping

The form of the "bottle" depends on $g(\Omega)$; for most distributions, a **circle** can be inscribed, giving a **handy approximation** for the **longitudinal stability limit of un-bunched beams**

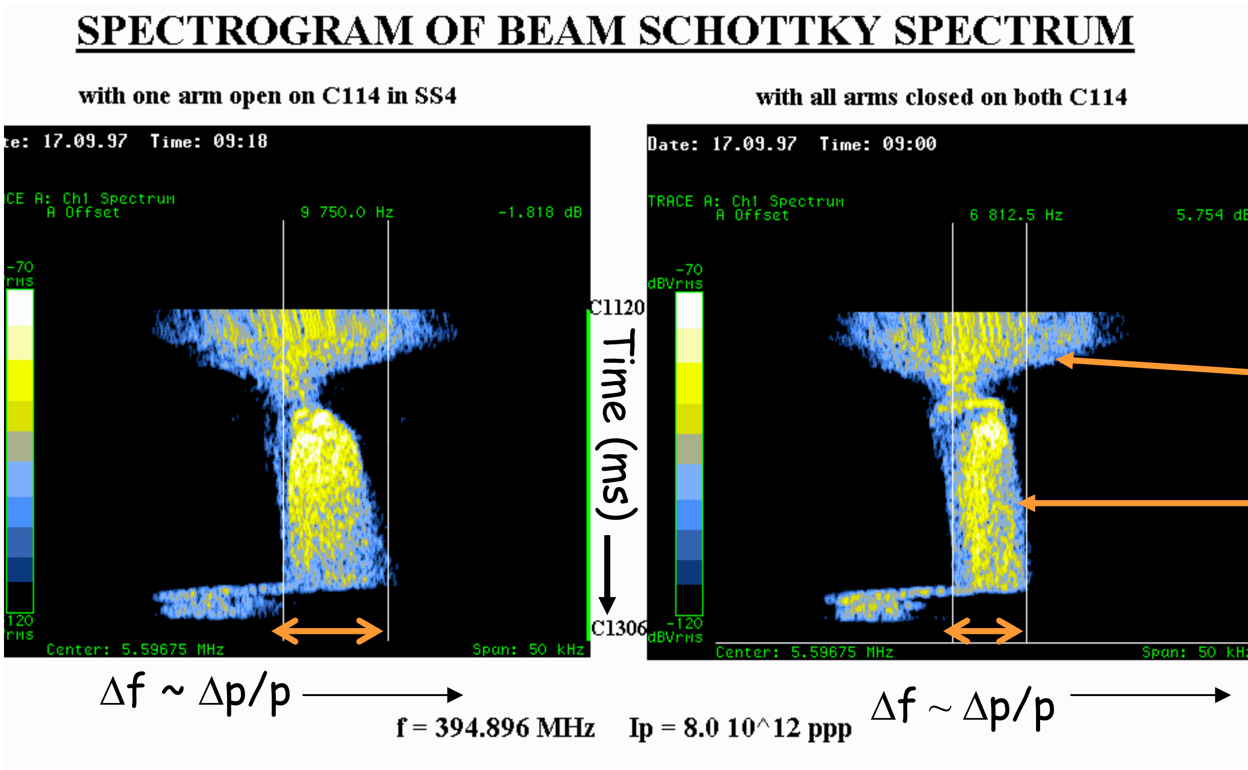
$$\left|\frac{Z}{n}\right| \leq F \frac{m_0 c^2 \beta^2 \gamma |\eta|}{e} \frac{(\Delta p/p)^2}{I_0}$$

KEIL-SCHNELL CRITERION

Coasting Beam Longitudinal Instability excited by Narrow-Band Resonator: Example at CERN PS (LHC Beam)

Gap open
Z High
Beam
Unstable

180 ms



Short-circuited
Z Low
Beam
stable

beam debunching

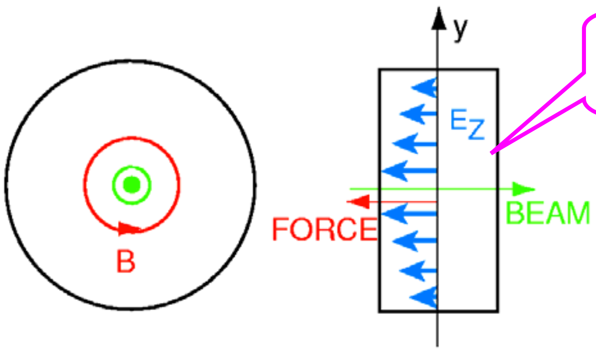
beam debunched

Blow-up:
Large Δp/p

No blow-up:
small Δp/p

A **narrow-band resonator** (114 MHz cavity) drives a **longitudinal coasting beam instability** if the gap short circuit is open (left). Several neighbouring modes are driven, resulting in **increased momentum spread**.
Horizontal: Δf proportional to **Δp/p** (**"Schottky" scan** on a spectrum analyser)
Vertical: **time** moving downwards, total 180 ms.

Impedance of a Resonator

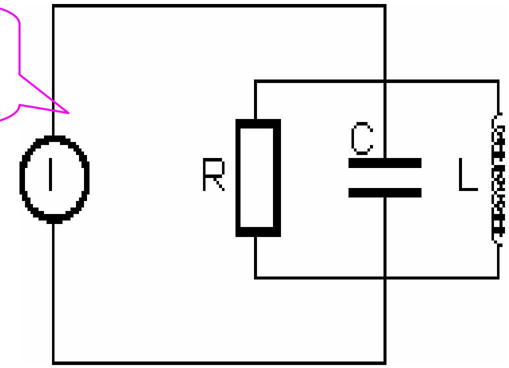


Pill-box cavity

$$\omega_r = \frac{1}{\sqrt{LC}}$$ resonance frequency

$$Q = R\sqrt{\frac{C}{L}} = \frac{R}{\omega_r L}$$ quality factor

Equivalent RLC circuit



$$\ddot{V} + \frac{\omega_r}{Q}\dot{V} + \omega_r^2 V = \omega_r \frac{R}{Q}\dot{I}$$

Differential equation of RLC circuit (current I represents the beam)

$$V(t) = V_0 e^{-\alpha t}\cos\left[\omega_r\sqrt{1 - 1/4Q^2}\, t + \varphi\right]$$

Solution: **damped oscillation** with $\alpha = 1/\tau = \omega_r/2Q$

HOW TO COMPUTE IMPEDANCE?

- ❑ **Excite** RLC circuit with $\mathbf{I = I_0\, e^{i\omega t}}$, $(-\infty < \omega < \infty)$
- ❑ Look for solutions $\mathbf{V(t) = V_0\, e^{i\omega t}}$ in the differential equation:

$$-\omega^2 V_0\, e^{i\omega t} + i\frac{\omega\omega_r}{Q} V_0\, e^{i\omega t} + \omega_r^2 V_0\, e^{i\omega t} = i\frac{\omega_r \omega R}{Q} I_0\, e^{i\omega t}$$

$$\boxed{\Rightarrow Z(\omega) = \frac{V_0}{I_0} = R\frac{1}{1 + iQ\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}}}$$

Impedance of Longitudinal Resonator

V_0 is **complex** since in general **not in phase with** exciting current I_0

⇨ **Z is complex and a function of** ω

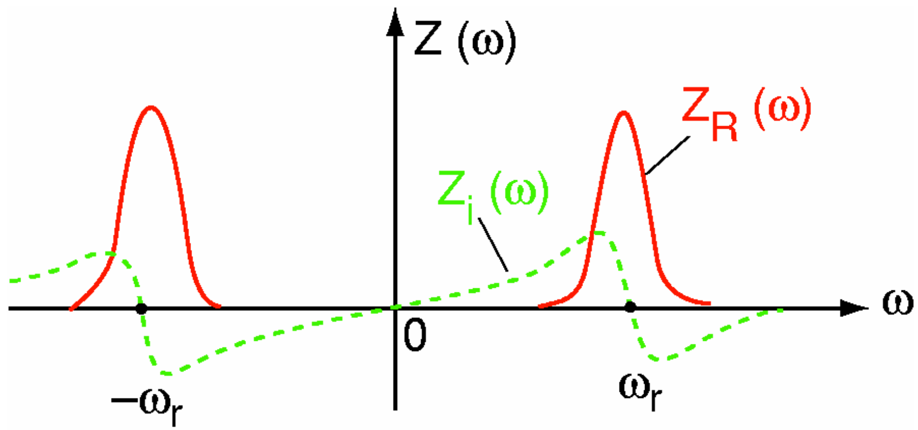
Impedance of a Resonator

$$Z(\omega) = Z_r(\omega) + iZ_i(\omega) = R\,\frac{1 - iQ\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}}{1 + \left[Q\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}\right]^2}$$

$Z_r(\omega) = Z_r(-\omega)$ (even)

$Z_i(\omega) = - Z_i(-\omega)$ (odd)

Longitudinal impedance of a resonator with resonance frequency ω_r



$$Z(\omega) \approx R_s\,\frac{1 - i2Q\dfrac{\Delta\omega}{\omega_r}}{1 + \left(2Q\dfrac{\Delta\omega}{\omega_r}\right)^2}$$

Impedance of a narrow-band ("high-Q") Cavity

with $\Delta\omega = n\omega_0 - \omega_r$, R_S = **"shunt impedance"**

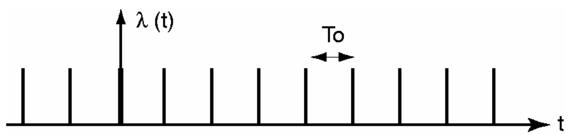
The **excitation signal** in such a cavity **decays slowly**: the **field induced** by the beam is **memorized for many turns**

Longitudinal Spectrum – Single Particle and Bunch

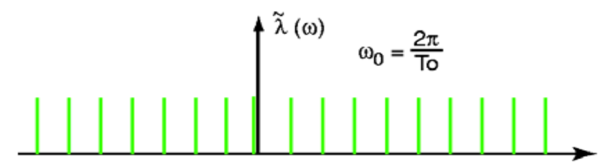
current monitor signal

Spectrum

SINGLE PARTICLE



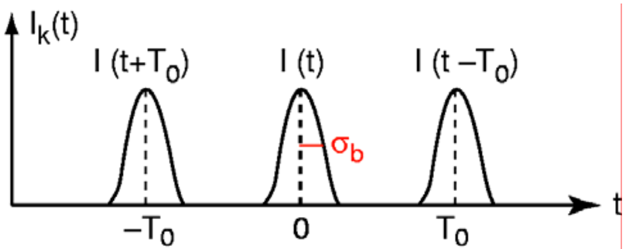
Fourier series



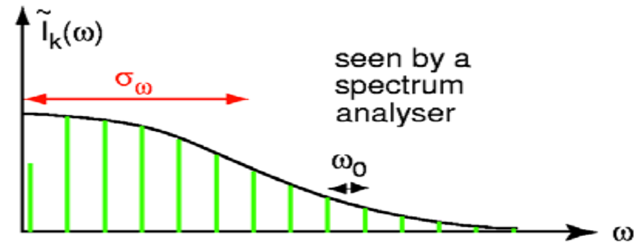
$$\lambda(t) = \frac{e}{\beta c} \sum_{\ell=-\infty}^{+\infty} \delta(t - \ell T_0)$$

$$\lambda(t) = \frac{e\omega_0}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$

SINGLE BUNCH



Fourier transform



$$I_k(t) = \sum_{k=-\infty}^{+\infty} I(t - kt_0)$$

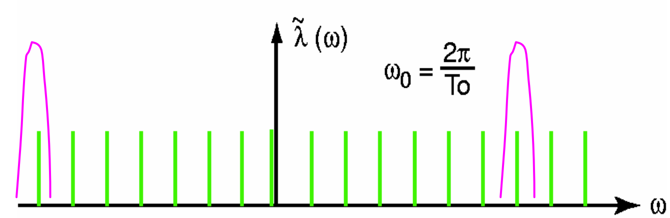
$$I_k(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t)$$

$$\tilde{I}(\omega) = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} I_k(t)\cos(n\omega_0 t)dt$$

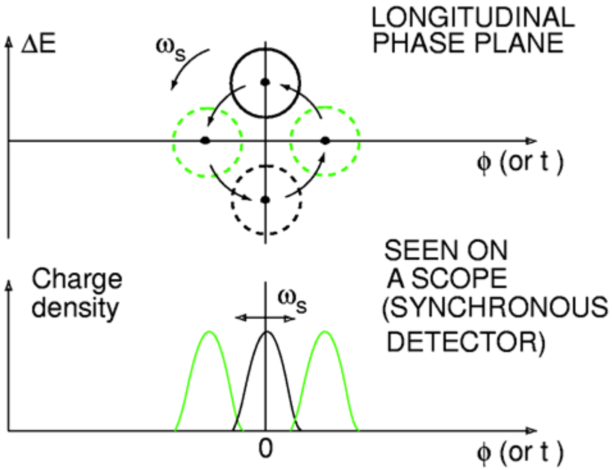
$\sigma_\omega \sim 2\pi/\sigma_b$: the **shorter** the bunch, the **wider** the spectrum

SPECTRUM AND IMPEDANCE

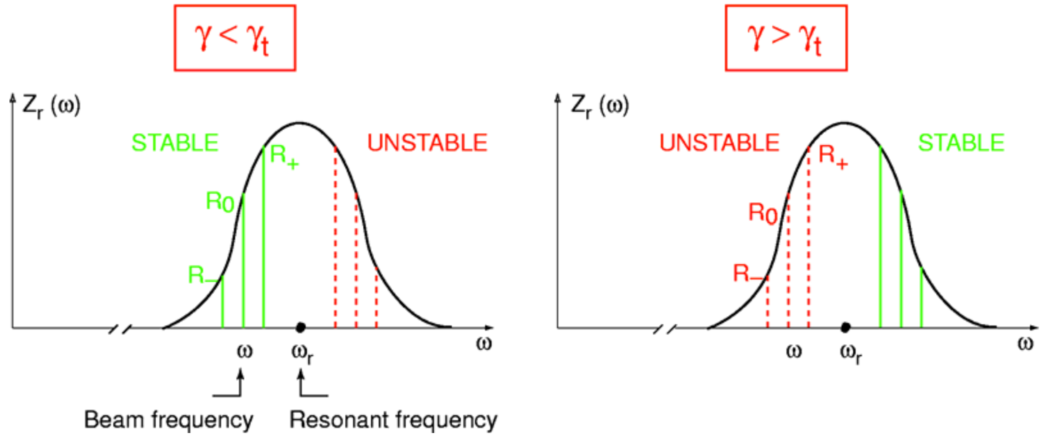
Narrow-band impedance $Z(\omega)$ driving a **single mode** (here 7 ω_0)



Single Bunch + Resonator: "Robinson" Instability



Bunch sees **resonator impedance** at $\omega_r \cong \omega_0$



"Dipole" mode or
"Rigid Bunch" mode

A single bunch rotates in longitudinal phase plane with ω_s:
its **phase** ϕ and **energy** ΔE also vary with ω_s

$\omega < \omega_r$

Whenever $\Delta E>0$:	Whenever $\Delta E>0$:
• ω increases (below transition)	• ω decreases (above transition)
• sees larger real impedance R_+	• sees smaller real impedance R_+
• more energy taken from beam	• less energy taken from beam
➢ STABILIZATION	➢ INSTABILITY

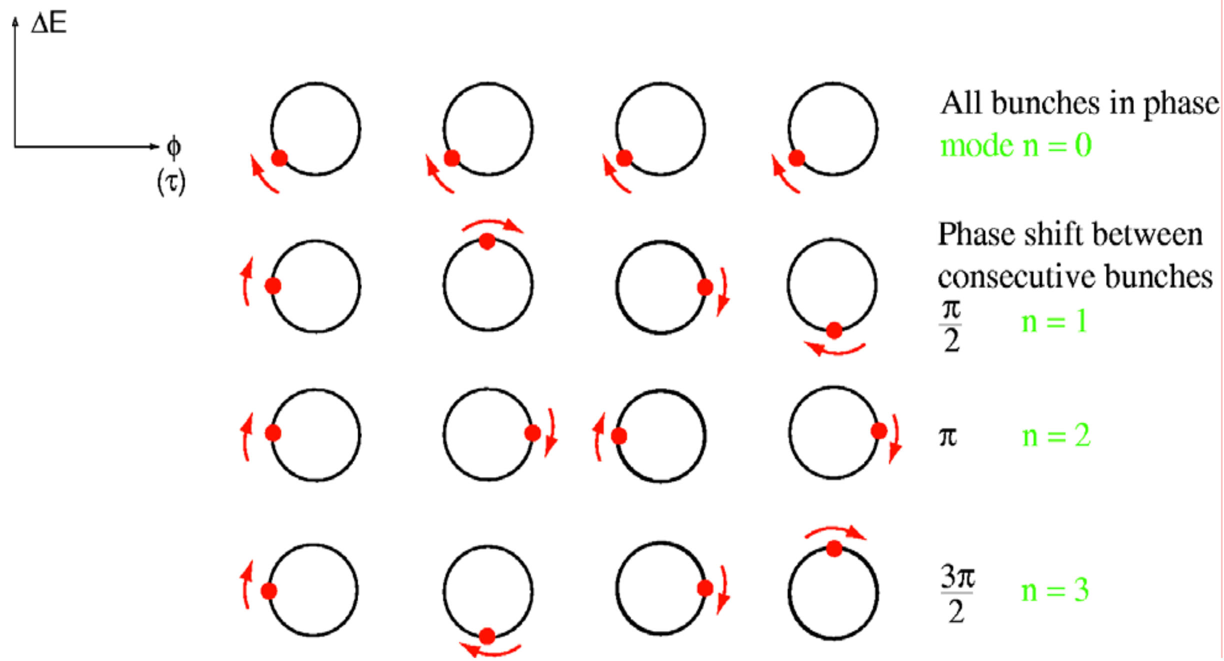
$\omega > \omega_r$

UNSTABLE STABLE

Longitudinal Instabilities with Many Bunches

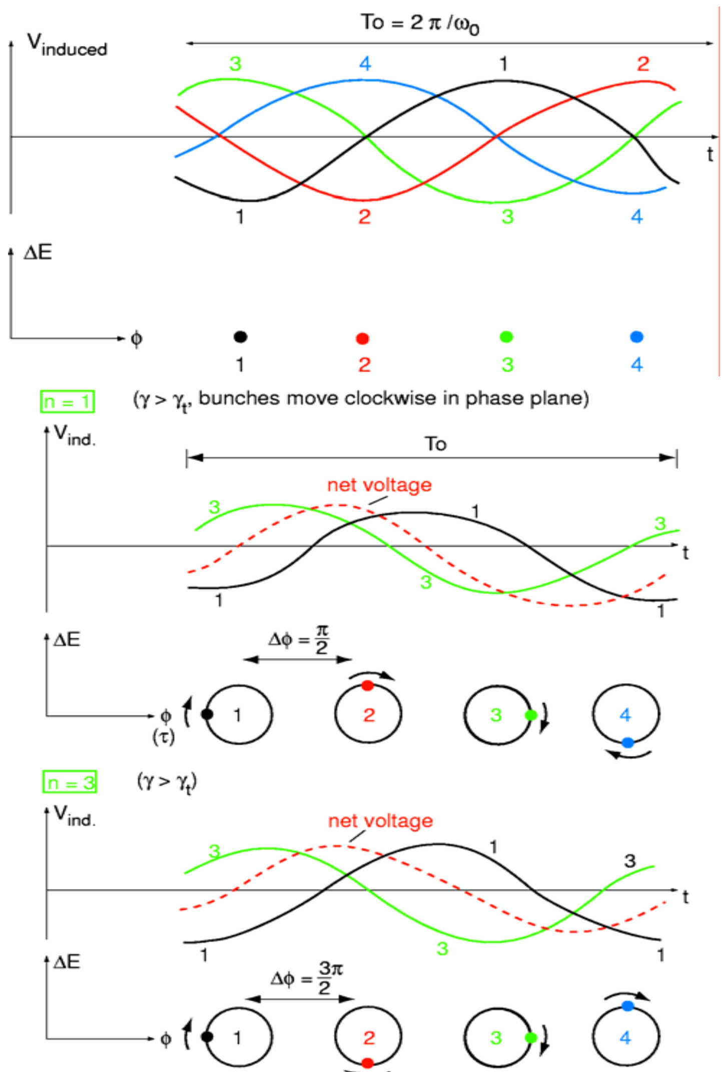
- ❑ Fields induced in resonator remain long enough to influence subsequent bunches
- ❑ Assume M = 4 bunches performing synchrotron oscillations

Coupled-Bunch Modes n



- ❑ **Four possible phase shifts** between **four bunches**
- ❑ **M bunches: phase shift** of coupled-bunch mode **n**: $2\pi\frac{n}{M},\ 0 \le n \le M-1 \Rightarrow$ M modes

Coupled-Bunch Modes and Stability



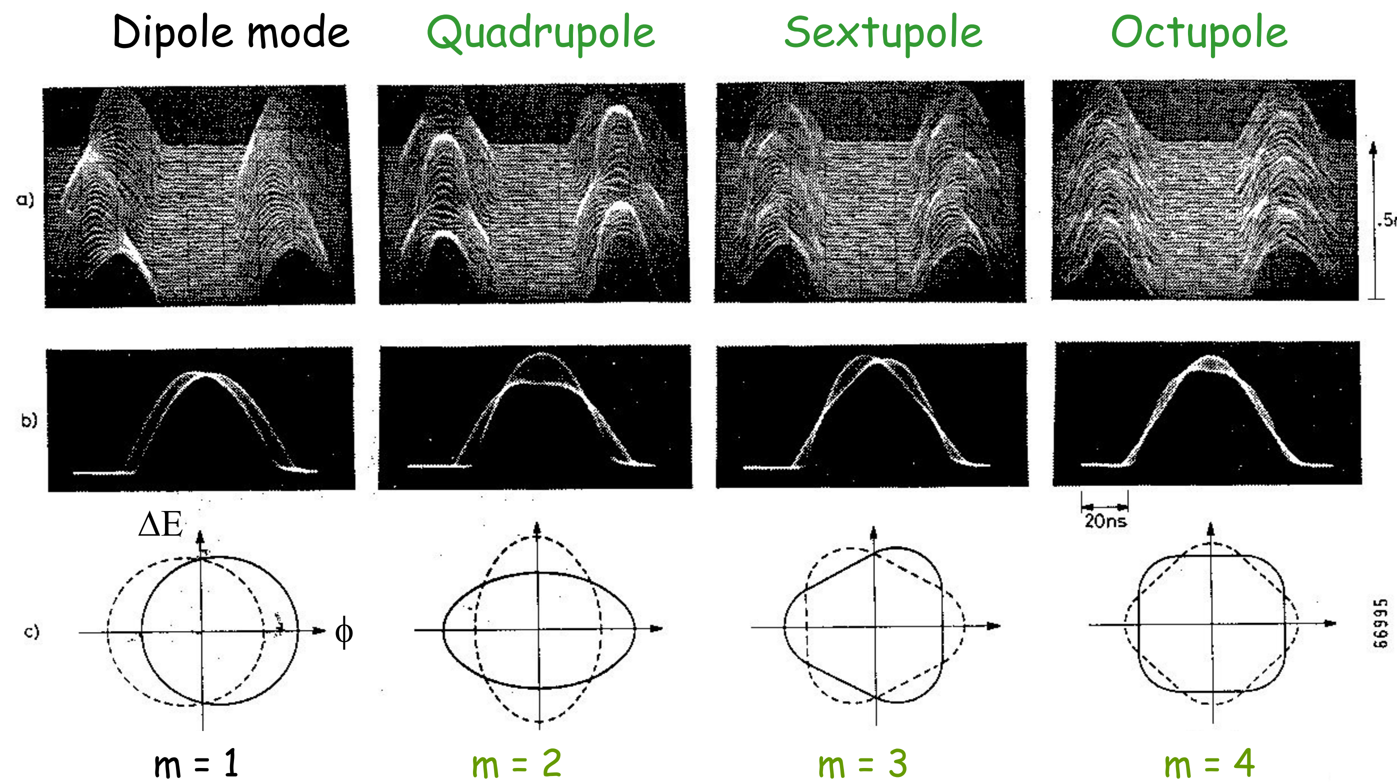
M = 4 bunches, resonator tuned at ω_0

Four stationary buckets (no synchrotron oscillations)
Voltages induced by bunches 2 and 4 **cancel**
Voltages induced by bunches 1 and 3 **cancel**
➔ NO EFFECT

Voltages induced by bunches 2 and 4 cancel, but
bunches 1 and 3 **induce a net voltage**
Bunch 2 **accelerated**, bunch 4 **decelerated**
Synchrotron oscillation amplitude increases
➔ **UNSTABLE**

Voltages induced by bunches 2 and 4 cancel, but
bunches 1 and 3 **induce a net voltage**
Bunch 2 **accelerated**, 4 **decelerated**
Same as n=1
Synchrotron oscillation amplitude decreases
➔ **STABLE**

Coupled Bunch Modes, Dipole & Higher Order



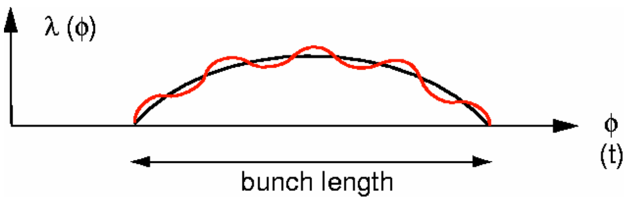
Mountain-range display during one synchrotron oscillation period (0.5 ms)

Signals superimposed

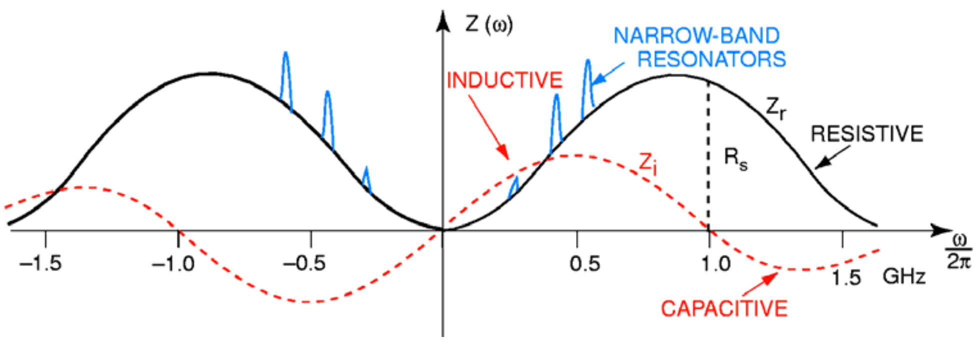
Mode pattern in longitudinal phase plane

Dipole (m=1) and higher-order (m=2,3,4) modes in a synchrotron with 5 bunches
Two adjacent bunches shown. *Note phase shifts between adjacent bunches*

Longitudinal Microwave Instability



- **High-frequency** density modulation along the bunch
- **wave length « bunch length** (frequencies **0.1-1 GHz)**
- **Fast growth rates** – even **leptons** concerned
- Generated by "**BROAD-BAND" IMPEDANCE**



All elements in a ring are "lumped" into a low-Q resonator yielding the impedance

$$\boxed{Z(\omega) = R_s \frac{1 - iQ\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}}{1 + \left(Q\dfrac{\omega^2 - \omega_r^2}{\omega\omega_r}\right)^2} \qquad Q \approx 1 \qquad \omega_r \approx 1\ \text{GHz}}$$

For small ω

$$Z(\omega) \approx i\frac{R_s\omega}{Q\omega_r} = i\frac{R_s}{Q}\overbrace{\frac{\omega}{\omega_0}}^{n}\frac{\omega_0}{\omega_r} = i\frac{R_s}{Q}\frac{\omega_0 n}{\omega_r}$$

and with (p. 11) $Q = \dfrac{R_s}{\omega_r L}$

$$\boxed{\left|\frac{Z}{n}\right|_0 = L\omega_0}$$

"Impedance" of a synchrotron in Ω

- This **inductive impedance** is caused mainly by **discontinuities** in the beam pipe
- If **high**, the machine is **prone to instabilities**
- Typically **20…50 Ω** for **old** machines
- **< 1 Ω** for **modern** synchrotrons

Microwave Instability - Stability Limit

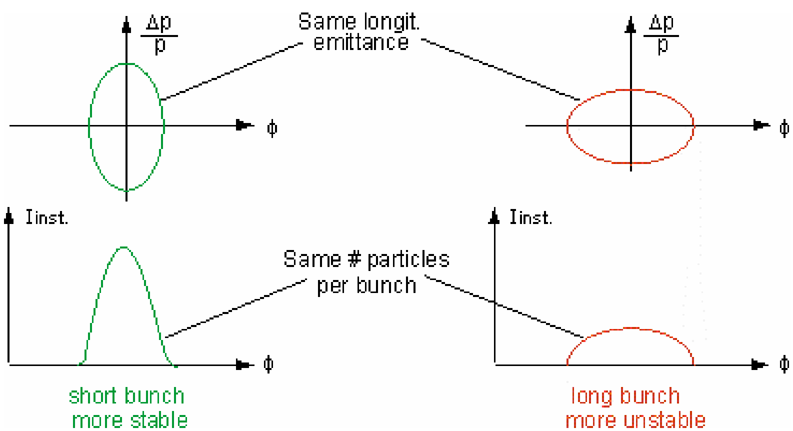
- The **Broad-Band Impedance** with **Q=1** has little memory
  - **No coupling** between **consecutive** bunches
  - Microwave instability is a **single bunch effect**
- leading to **longitudinal bunch blow-up**
- In lepton machines also called **"turbulent bunch lengthening"**

STABILITY LIMIT: Apply **Keil-Schnell criterion for unbunched beams** to **instantaneous current and momentum spread**

$$\left|\frac{Z}{n}\right| \leq F\frac{m_0 c^2 \beta^2 \gamma |\eta|}{e}\left[\frac{(\Delta p/p)^2}{I}\right]_{\text{instant.}}$$

KEIL-SCHNELL-BOUSSARD CRITERION

protons: F ~ 0.65 leptons: F ~ 8



For both **bunch population** and **longitudinal emittance equal**, **short** bunches are **more stable** than **long** ones

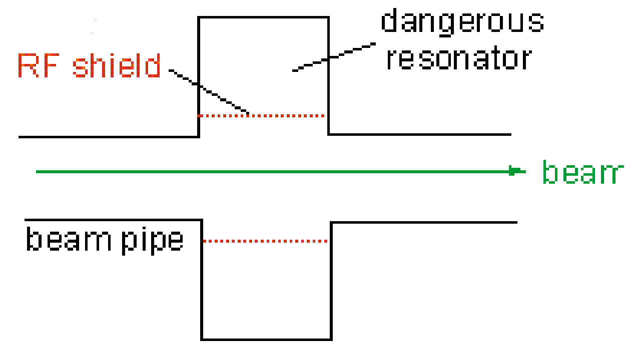
Longitudinal Instabilities - Cures

- **Robinson Instability**, generated by main RF cavities:
  Tune cavity resonance frequency ω_r relative to bunch frequency $h\omega_0$

for $\gamma < \gamma_t$	$h\omega_0 < \omega_r$
for $\gamma > \gamma_\tau$	$h\omega_0 > \omega_r$

- **Cavities "Parasitic" Modes** are damped by **"Higher Order Mode Dampers"(HOM):** the unwanted mode is picked up by an antenna and sent to a damping resistor.

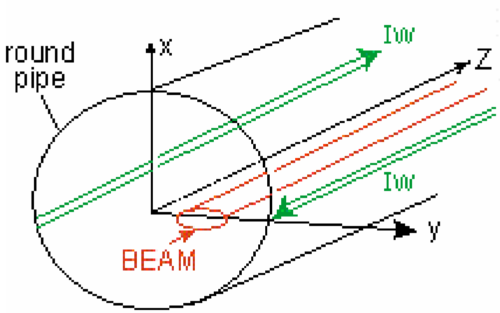
- **Unwanted Resonators in beam pipe: RF shield** protects the beam **mimicking a smooth beam pipe**

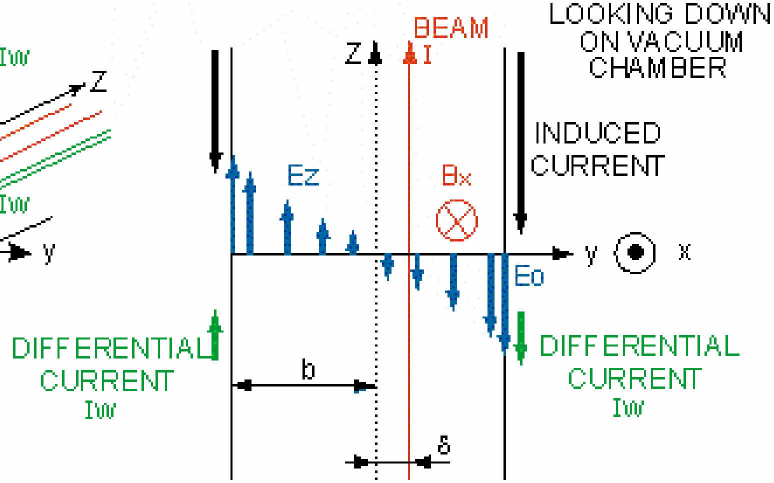


- **Microwave Instabilities:** Reduce Broad-Band Impedance by smooth changes in beam pipe cross section and shielding cavity-like objects. Large $\Delta p/p$ helpful (Landau damping) but costly in RF voltage.

- **Feedback systems:** The **beam phase** or **amplitude** deviation is **measured** with a **synchronous detector** and **corrected** in an **accelerating gap** covering the bandwidth
  - ⇨ **In-phase** $(n=0)$ **dipole mode** tackled by "**phase loop**" locking beam phase to RF phase
  - ⇨ **Coupled-bunch** $(n\neq0)$ **instabilities** are controlled by feedback loops either **tackling each of M bunches** or **each mode n** (out of M) individually; **bandwidth ~ ½ Mω_0**

Transverse Beam Instabilities - Fields and Forces





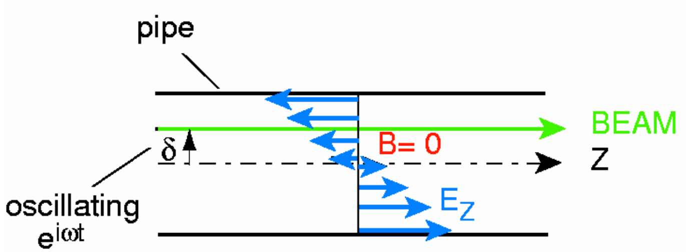
δ...displacement of beam in y oscillating with $e^{i\omega t}$

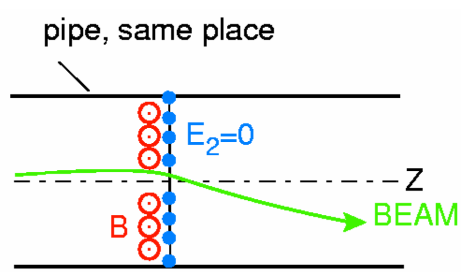
To sustain the **differential wall current I_w** a longitudinal electric field **E_z varying across the aperture** is required

➔ $E_z = E_0(y/b)\, e^{i\omega t}$ in the median plane x = 0

From $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$ one gets $\frac{\partial B_x}{\partial t} = -\frac{\partial E_z}{\partial y} = -\frac{E_0}{b} e^{i\omega t}, \quad B_x = \frac{i}{\omega}\frac{E_0}{b} e^{i\omega t}$

Phase-shifted with respect to exciting beam oscillation





t=0, **excitation** by displaced beam

t = (1/4) (2π/ω), **deflection**

Transverse Coupling Impedance

$$\boxed{Z_T(\omega) = i\frac{\oint[\vec{E}+\vec{v}\times\vec{B}]_t\, ds}{\beta I \delta}} = \frac{\textbf{Deflecting field} \text{ integrated around ring}}{\textbf{Dipole moment} \text{ of exciting current}} \quad [\Omega/m]$$

phase shift between dipole moment $I\delta$ and deflecting field

Transverse Impedance Z_T vs. Longitudinal Impedance Z_L

Relation between Z_T and Z_L
(Z_L longitudinal impedance called Z so far), for a **resistive round pipe:**

$$\boxed{Z_T(\omega) \cong \frac{2c}{b^2}\frac{Z_L}{\omega}}$$

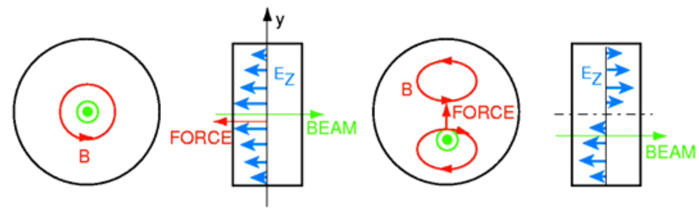
Handy approximate relation between Z_T and Z_L

	Z_L	Z_T
unit	Ω	Ω/m
Symmetry real part	even	odd
Symmetry imaginary part	odd	even
Orders of magnitude for synchrotrons	~ Ω	MΩ/m

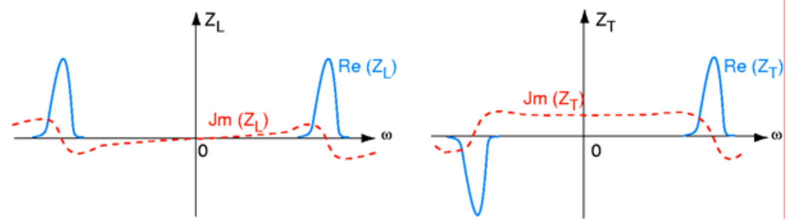
Transverse and Longitudinal Impedances

Z_L | Z_T

Resonator-type object
Fields and Forces



Resonator-type object
Impedance



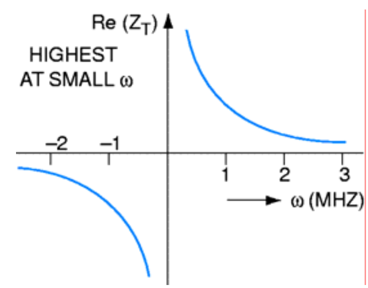
Resistive Wall

R….machine radius
ρ….vacuum chamber resistivity
δ….wall thickness

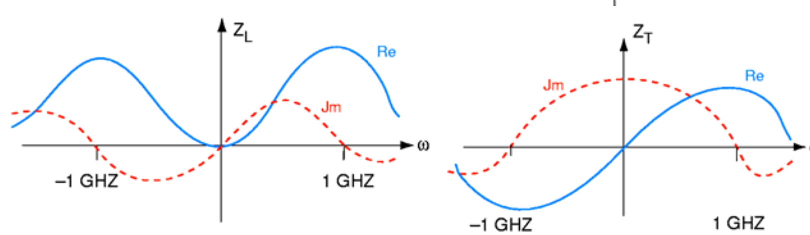
$$\mathrm{Re}(Z_T) = \frac{2cR}{\omega b^3}\frac{\rho}{\delta}\ (\text{low } \omega)$$

$$\boxed{\mathrm{Re}(Z_L) = \frac{R}{b}\frac{\rho}{\delta}}$$

independent of ω



Broad-Band (with Q=1)

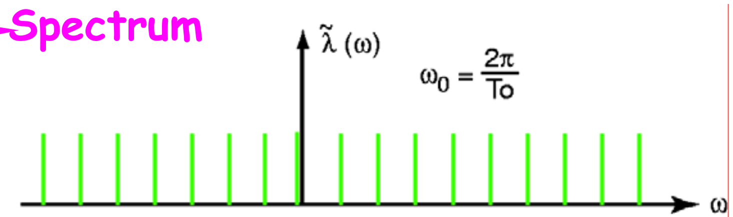


Transverse Beam Signals - Time and Frequency

Single particle on central orbit - longitudinal signal

$$\lambda(t) = \frac{e\omega_0}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in\omega_0 t}$$

Spectrum



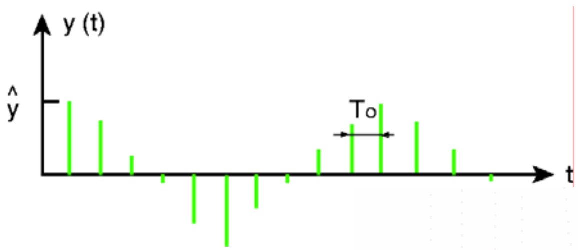
Single particle, oscillating transversally

$$y = \hat{y}\cos(\omega_\beta t + \varphi)$$

$$\omega_\beta = Q\omega_0 = (k + q)\omega_0$$

fractional tune

Position monitor signal for $q \sim 0.1$

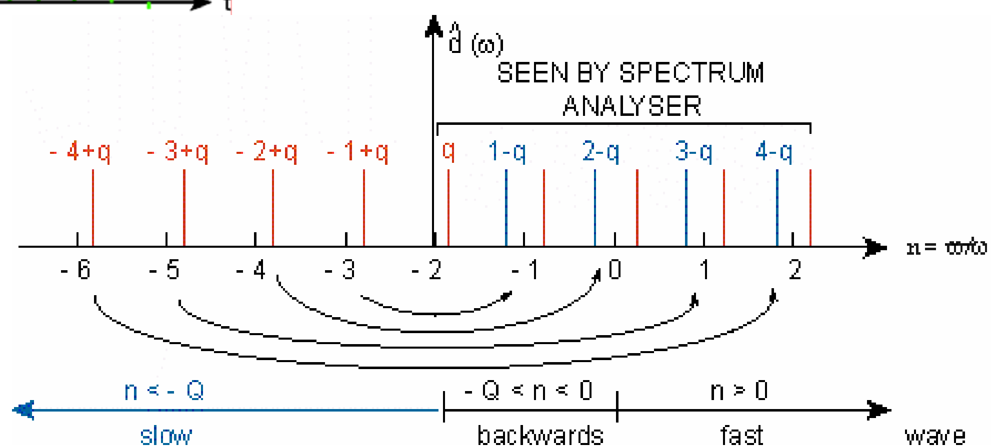


$$d(t) = \frac{e\hat{y}\omega_0}{2\pi} \sum_{n=-\infty}^{+\infty} \cos[(n+Q)]\omega_0 t + \varphi$$

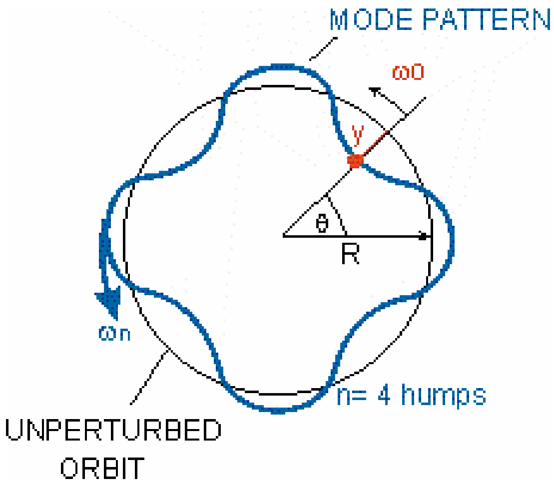
Spectrum $\hat{d}(\omega)$

- constant amplitude
- lines at $(n+Q)\omega_0$, n any integer

Example: Q = 2.25
(q = 0.25)



Transverse Instabilities – Unbunched Beam



MODE: particles are arranged around the synchrotron with a strict correlation between transverse particle positions.

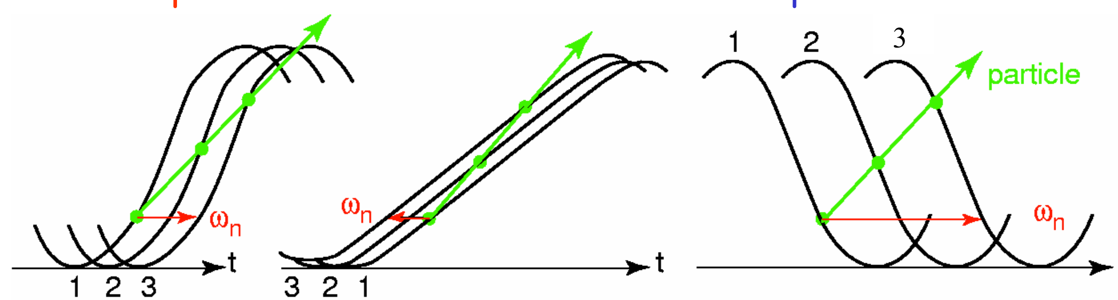
The mode shown is **n=4**, a snapshot at t=0.

A **single particle** always rotates with **revolution frequency** ω_0 but **the pattern n rotates with** $\omega_n \neq \omega_0$

Rotation frequency of mode pattern

$$\boxed{\omega_n = \left(1 + \frac{Q}{n}\right)\omega_0}$$

	$n < -Q$	$-Q < n < 0$	$n > 0$
Pattern moves	slower than particle	backwards	faster than particle



Slow wave **Backwards wave** **Fast wave**

Unbunched Beam – Transverse Growth Rate

Only one mode n (one single line) grows, so only Z_T around frequency $(\mathbf{Q}+\mathbf{n})\omega_0$ relevant

- Assume $e(\vec{E}+\vec{v}\times\vec{B})_T$ constant around the ring for a given y

$$F(t)=e(\vec{E}+\vec{v}\times\vec{B})_T=-i\frac{e\beta IZ_T}{2R\pi}y(\theta(t)) \qquad Z_T=i\frac{\int_0^{2\pi R}(\vec{E}+\vec{v}\times\vec{B})_T\,ds}{\beta yI}$$

- A particle's betatron amplitude $y(t)$ satisfies

$$\ddot{y}+Q^2\omega_0^2 y=\frac{\text{Force}}{m_0\gamma}=-i\frac{e\beta IZ_T}{2\pi Rm_0\gamma}y \quad \text{where} \quad y=y_n e^{i(Q\omega_0 t-n\theta_0)}$$

$$\ddot{y}+\underbrace{(Q\omega_0+\Delta\Omega)^2}_{\approx Q^2\omega_0{}^2+2Q\omega_0\Delta\Omega}y=0 \quad \Rightarrow \Delta\Omega=i\frac{e\beta Z_T I}{4Q\pi\omega_0 R\gamma m_0},$$

- With $\omega_0 R=\beta c$ and $\gamma m_0=E/c^2$

$$\boxed{\Delta\Omega=i\frac{cZ_T I}{4\pi QE/e}}$$

Transverse growth rate, **un-bunched beam**, Z_T constant around the ring

- Single particle oscillation changed to

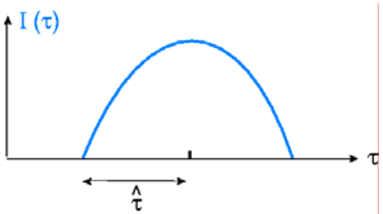
$$y(t)=y_n\,e^{i[(Q\omega_0+\Delta\Omega)t-n\theta_0]}$$

Unstable if $\mathbf{Im(\Delta\Omega)<0}$

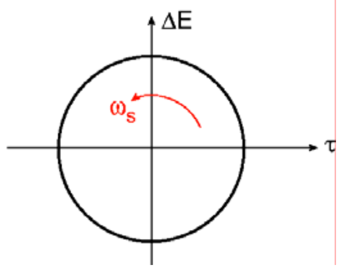
- $\mathbf{Re\,[Z_T((Q+n)\omega_0)]<0}$
- $\mathbf{(Q+n)<0}$ **slow waves!**

For **unbunched** beam, only **slow wave unstable (applies also for bunched beam)**

Transverse Instabilities – Bunched Beams



Bunch shape observed with current monitor



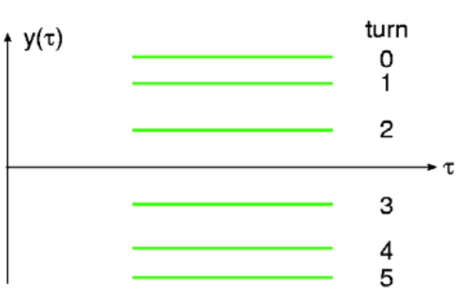
All particles perform synchrotron oscillations – their energy changes with frequency ω_s

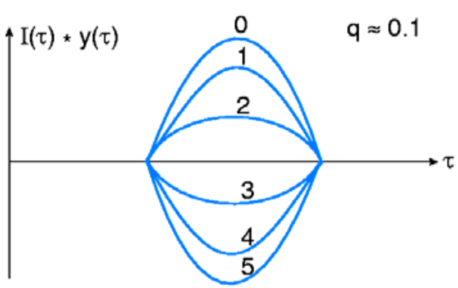
ZERO CHROMATICITY

$$\xi = \frac{dQ}{Q} \Big/ \frac{dp}{p} = 0$$

All particles have same betatron tune Q – even with changing energies

RIGID BUNCH MOTION (m=0) [A. SESSLER ~1960]

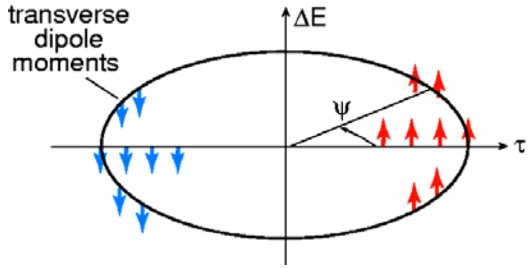




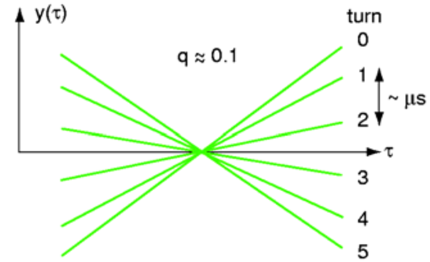
All particles in the bunch start at t=0 with same betatron phase. Although synchrotron motion sweeps them back and forth and changes their energy, they all oscillate in phase

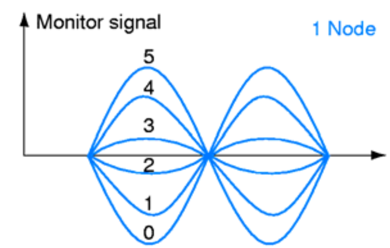
transverse position $y(\tau)*$current $I(\tau)$ = position monitor signal

Transverse Instabilities - Head-Tail Modes



Head-Tail Mode m=1





Arrange initial betatron phases so as to have dipole moments **up** near the **head** of the bunch
down near the **tail**

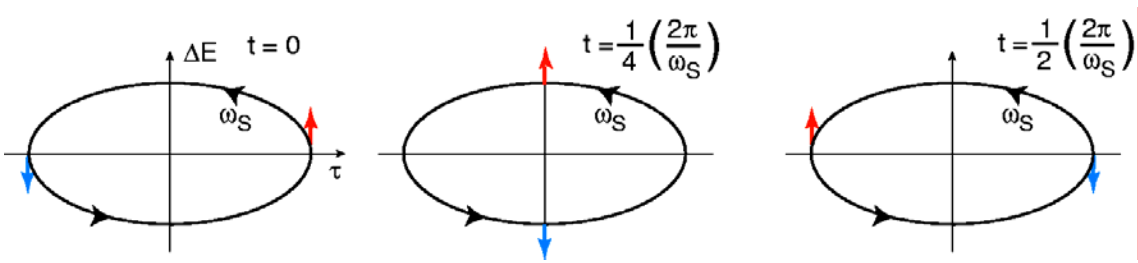
Mode pattern described by $e^{i\psi}$ in longitudinal phase plane

On a slower timescale (~ms): the pattern rotates with ω_s

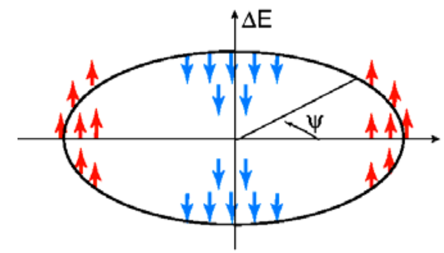
Initial condition (as above)

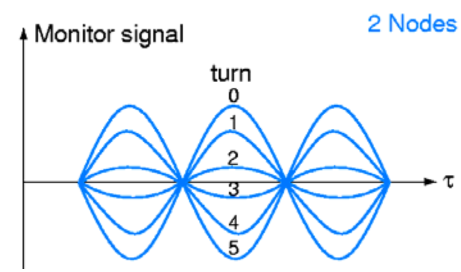
ups and downs superimposed: signal = 0

ups and downs exchange places



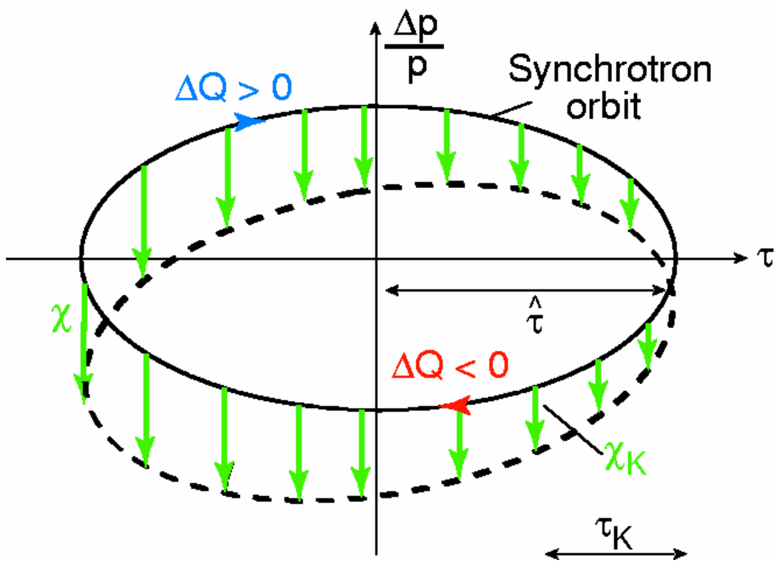
Head-Tail Mode m=2





- **2 nodes**
- pattern described by $e^{i2\psi}$
- pattern rotating with $2\omega_s$

Head-Tail Modes with Non-Zero Chromaticity



$\xi \neq 0$: Q varies along the synchrotron orbits

In the sketch, one assumes

$$\xi = \frac{dQ/Q}{dp/p} > 0,$$

$$\gamma < \gamma_t \left[\eta = \frac{1}{\gamma_t^2} - \frac{1}{\gamma^2} < 0 \right]$$

χ_k....betatron phase slip after k machine turns

χ..... betatron phase slip between head and tail

T_0.....revolution time

$\hat{\tau}$ .....half bunch length

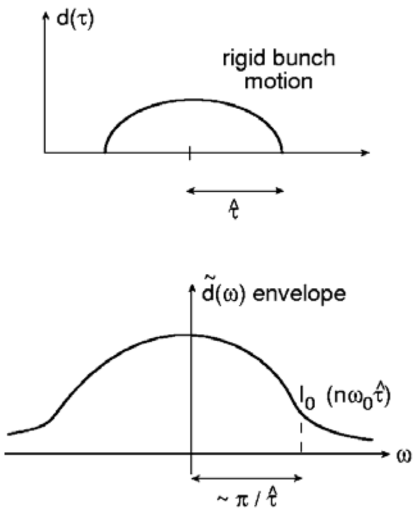
The pattern (**"mode"**) can be kept **stationary** if the particles' betatron phases are **arranged** as in the figure

Total **phase shift between head and tail**

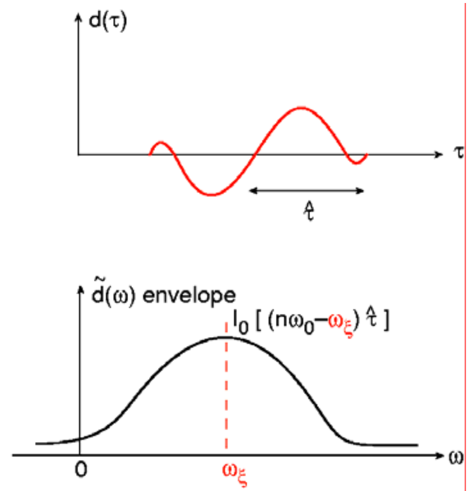
$$\boxed{\chi = \frac{\xi}{\eta} Q \omega_0 \times 2\hat{\tau}}$$

Head-Tail Phase Shift Changes Bunch Spectrum

$\xi=0 \quad (\chi=0)$



$\xi\neq0 \quad (\chi\neq0)$



Example: Mode m=0

Head-tail mechanism
discovered by
C. Pellegrini, M. Sands
end 60ies

"Standard model"
F.Sacherer mid-70ies

The **shorter** the **bunch length** $\hat{\tau}$, the **larger** the **width** of the spectrum

The **wiggly signal** passes through a position monitor which sees
- during bunch passage time $2\hat{\tau}$
- a **phase shift** of χ radians
- the monitor (or an impedance) "sees" an **additional frequency**

	η	ξ	ω_ξ
$\gamma < \gamma_t$	< 0	> 0	< 0
		< 0	> 0
$\gamma > \gamma_t$	> 0	> 0	> 0
		< 0	< 0

$\omega_\xi = 0$

Chromaticity Frequency ω_ξ

$$\omega_\xi = \frac{\xi}{\eta} Q \omega_0$$

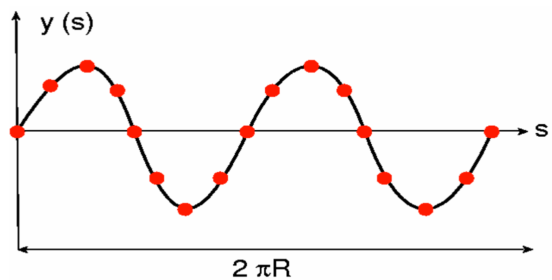
Transverse Instabilities – Many Bunches

Transverse positions of bunches arranged to form a **pattern of n waves** around the synchrotron

➢ **Coupled-bunch mode n**

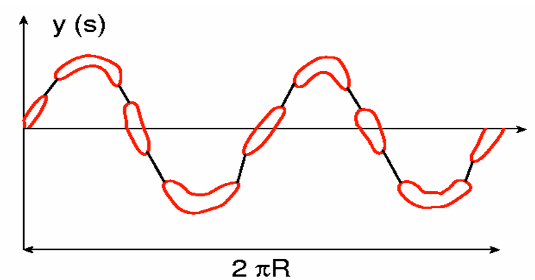
With M bunches, bunch-to-bunch **betatron phase shift $2\pi n/M$**

short bunches

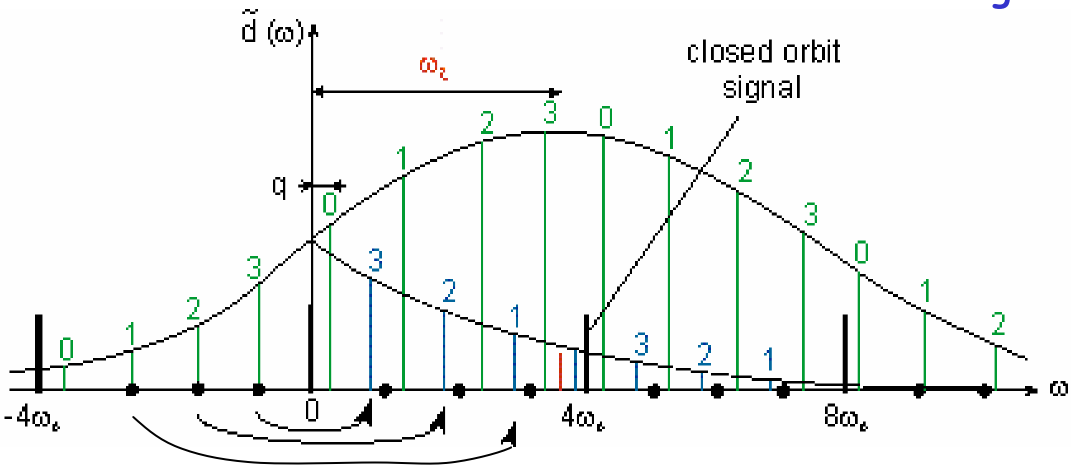


- n=2 (waves), M=16 (bunches)
- bunch-to-bunch betatron phase shift $\pi/4$
- Head-tail phase shift small
- **behaves like coasting beam**

long bunches



- n=2, M=8
- bunch-to-bunch betatron phase shift $\pi/2$
- Head-tail phase shift χ large
- can only be sustained with a certain value $\chi \neq 0$



Spectrum for

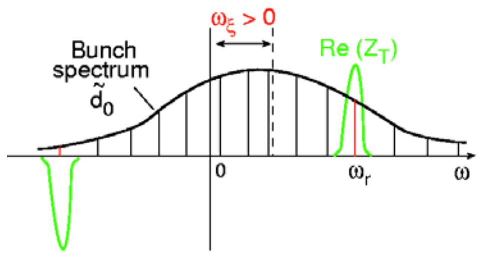
- M=4 bunches
- m=0 nodes within the bunch
- q = 0.25
- coupled-bunch modes n=0,1,2,3

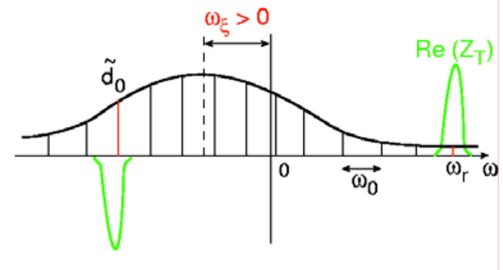
Bunched Beam Transverse Stability vs. Impedance

Narrow-Band Resonator

- only two spectral lines contribute to the sum
- **Fields stored long enough** to act on subsequent bunches during several turns

Reminder: $\mathrm{Re}[Z_T(\omega)] = -\mathrm{Re}[Z_T(-\omega)]$





$\omega_\xi > 0$

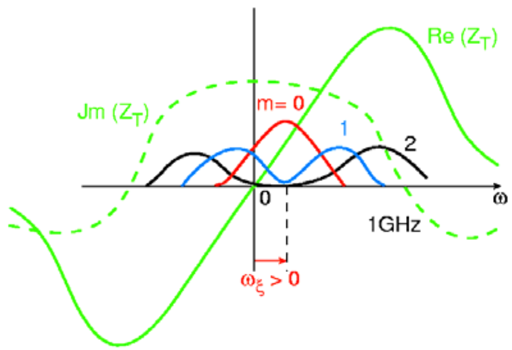
$\Sigma\, \mathrm{Re}[Z_T]\, d_0^2 > 0$ → stable

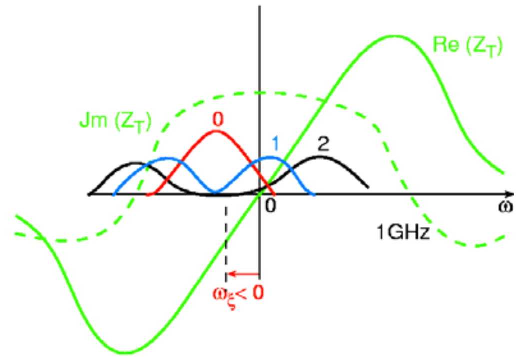
$\omega_\xi < 0$

$\Sigma\, \mathrm{Re}[Z_T]\, d_0^2 < 0$ → unstable

Broad-Band Resonator

- extends to ~GHz
- thus spectral lines very dense
- spectrum envelopes d_0, d_1, d_2 for modes m=0, m=1, m=2 shown
- Quality factor **Q low** → **fields not stored long enough** to influence subsequent bunches





$\Sigma\, \mathrm{Re}[Z_T]\, d_0^2 > 0$ → stable

$\Sigma\, \mathrm{Re}[Z_T]\, d_0^2 < 0$ → unstable

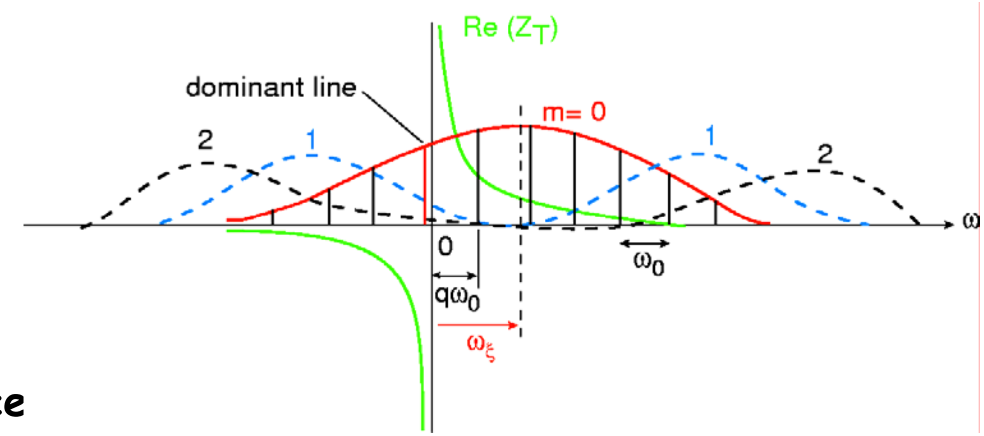
For any "normal" transverse impedance

$\gamma < \gamma_t$: set $\xi < 0$ ($\omega_\xi > 0$) to stabilize beam
$\gamma > \gamma_t$: set $\xi > 0$ ($\omega_\xi > 0$) to stabilize beam

Resistive Wall Transverse Instability

$$\mathrm{Re}(Z_T) = \frac{2cR}{\omega b^3}\frac{\rho}{\delta}\ (\mathrm{low}\,\omega)$$

ρ...resistivity of beam pipe
δ...wall thickness (low frequency)



- **not a "normal" transverse impedance**
- dominant line at Re(Z_T) most negative at very low frequency
- dominant mode normally m=0 but cannot be stabilized by setting $\omega_\xi > 0$
- **setting Q above an integer (q < 0.5)** puts dominant line near the origin but at Re(Z_T) > 0 thus stabilizing the beam

For the resistive wall impedance, fractional tune $q < 0.5$ preferable (A.Sessler 60ies)

Further increasing ω_ξ (by varying ξ with sextupoles) may drive the hump of m=1, 2 etc. onto this dominant line, thus switching from one mode to the next.

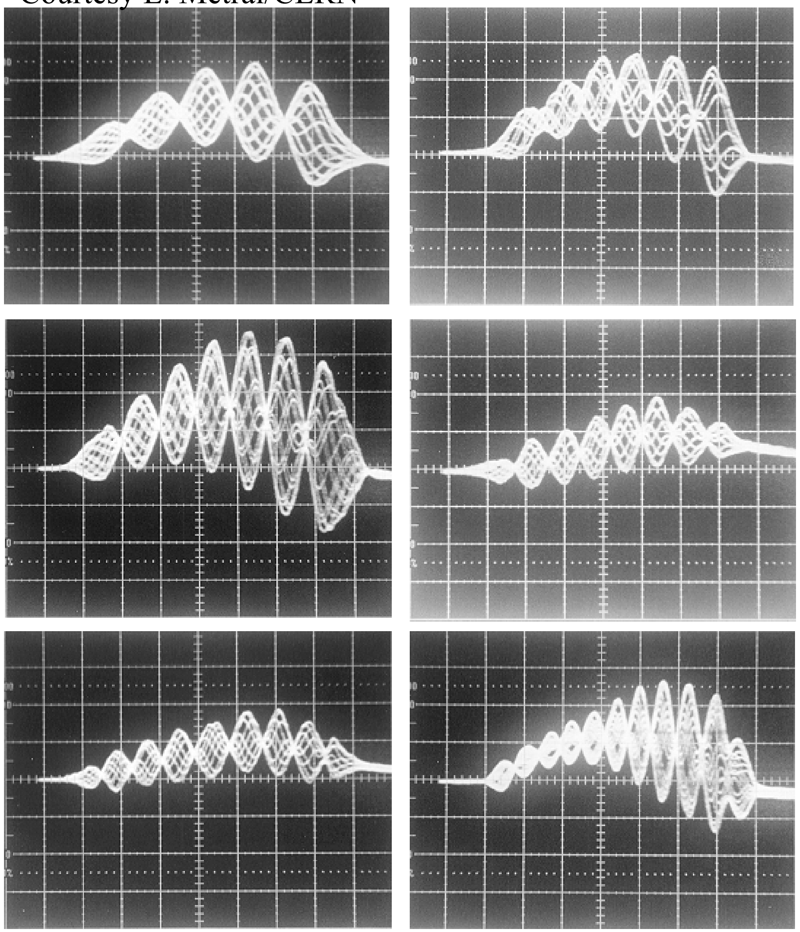
Horizontal Head-Tail Instabilities in CERN PS

Courtesy E. Metral/CERN

m=4, 5

m=6, 7

m=8, 10



20 ns/div

A single bunch with ~10^{12} protons and ~150 ns length on the 1.4 GeV injection plateau in the CERN PS (below transition energy)

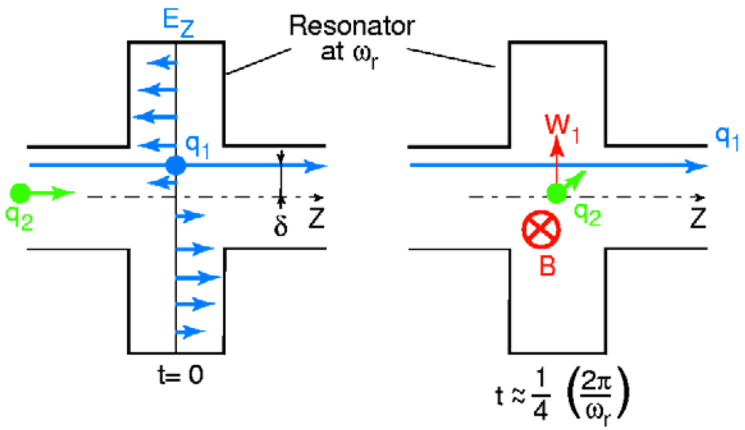
Head-tail mode numbers m=4,…,9 are generated by changing horizontal chromaticity ξ_h from -0.5 (m=4) to -1.3 (m=10). The natural chromaticity, $\xi_h = -0.9$, yields m=6 (6 nodes). For all pictures, $\omega_\xi > 0$, which normally stabilizes the beam, but not in this case.

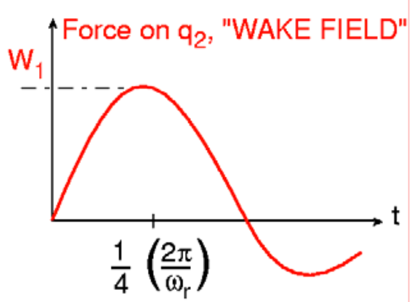
→ The impedance responsible for this horizontal instability is the resistive wall impedance

Transverse Wake Fields

Instead of treating instability dynamics in the frequency domain as done so far, one can do it in the time domain by using "Wake Fields"

What is a Wake Field?
Point charge q_1 passes through a resonator with a transverse displacement δ.
The induced Wake field W will act on the subsequent charge q_2.



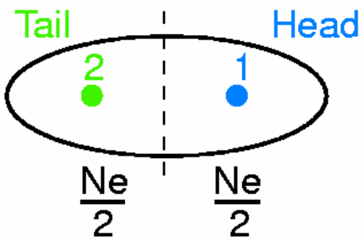


RLC-circuit (p. 11)

$W = W_1 e^{-\alpha t} \sin S\omega_r t$
with
$\alpha = \omega_r/(2Q)$
$S = (1 - Q^2/4)^{1/2}$

The Wake Field concept is very useful for impedances with short memory where the fields do not act on subsequent bunches but only on particles within the same bunch (single-bunch effects). Example: broad-band impedance (low-Q resonator)

Transverse Wake Fields – A Simple Model



Approximate bunch by just two superparticles
"head" (1) and "tail" (2) with Ne/2 charges each

Model by A. Chao

If **head** is displaced by δ, force on particle in **tail** is

$$f = e\frac{Ne}{2}W_1\delta$$

Both **head** (y_1) and **tail** (y_2) oscillate with same betatron frequency ω_β

same frequency

$$y_1 = \delta\cos\boxed{\omega_\beta}t$$

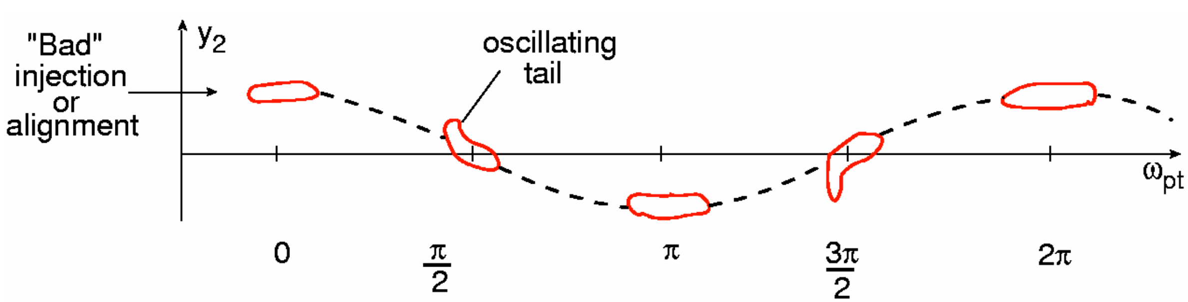
$$\ddot{y}_2 + \boxed{\omega_\beta}^2 y_2 = \frac{f}{m_0\gamma} = \frac{Ne^2W_1}{2m_0\gamma}\boxed{y_1}$$

Excitation on right-hand side **has same frequency**

$$\Rightarrow y_2 = \delta\left[\cos\omega_\beta t + \underbrace{\frac{Ne^2W_1}{4\omega_\beta m_0\gamma}t\sin\omega_\beta}_{}t\right]$$

tail amplitude y_2 **grows linearly** with time

SLAC 50 GeV Electron Linac



Observation: Tail amplitude increasing along the Linac – caused by misalignments

Transverse Instabilities - Cures

- As for longitudinal impedances: damp unwanted HOM's, protect beam by RF shields

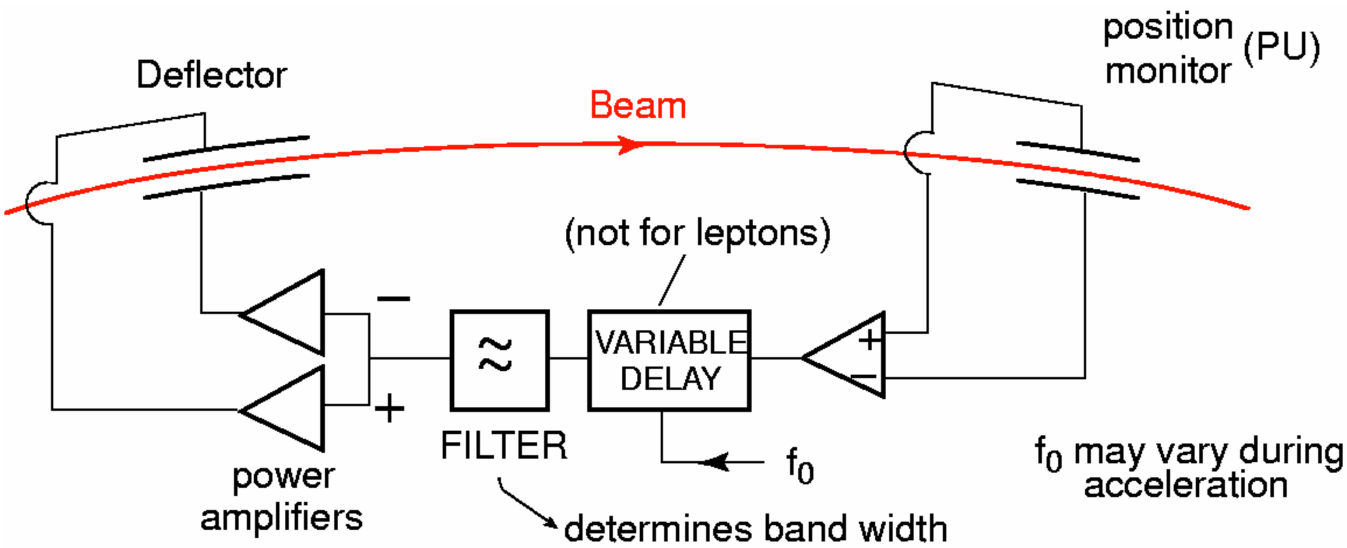
- For "normal" transverse impedances, operate with a slightly positive chromaticity frequency ω_ξ → for $\gamma < \gamma_t$ set $\xi < 0$ (by sextupoles)
  → for $\gamma > \gamma_t$ set $\xi > 0$

$$\boxed{\omega_\xi = \frac{\xi}{\eta} Q \omega_0}$$

- For the resistive wall impedance:
  - operate machine with a betatron tune just above an integer
  - use highly conductive vacuum pipe material to reduce $\mathrm{Re}(Z_T)$ and growth rate

- Landau damping also works in the transverse plane; a betatron frequency spread $\Delta\omega_\beta$ is generated by octupoles (betatron tune depends on oscillation amplitude)

Transverse Instabilities - Feedback



- a **position error in PU** must result in an **angle error in the deflector** which is (partially) corrected there
- **betatron** **phase** from PU to deflector ~ **(2n+1)** $\pi/2$
- **electronic delay** ≡ **beam travel time** from PU to deflector
- **Bandwidth**: ~ a few **10 kHz to a few MHz** if only **resistive wall**
  ~ up to **half the bunch frequency** with **bunch-by-bunch feedback**