Introduction to Transverse Beam Optics

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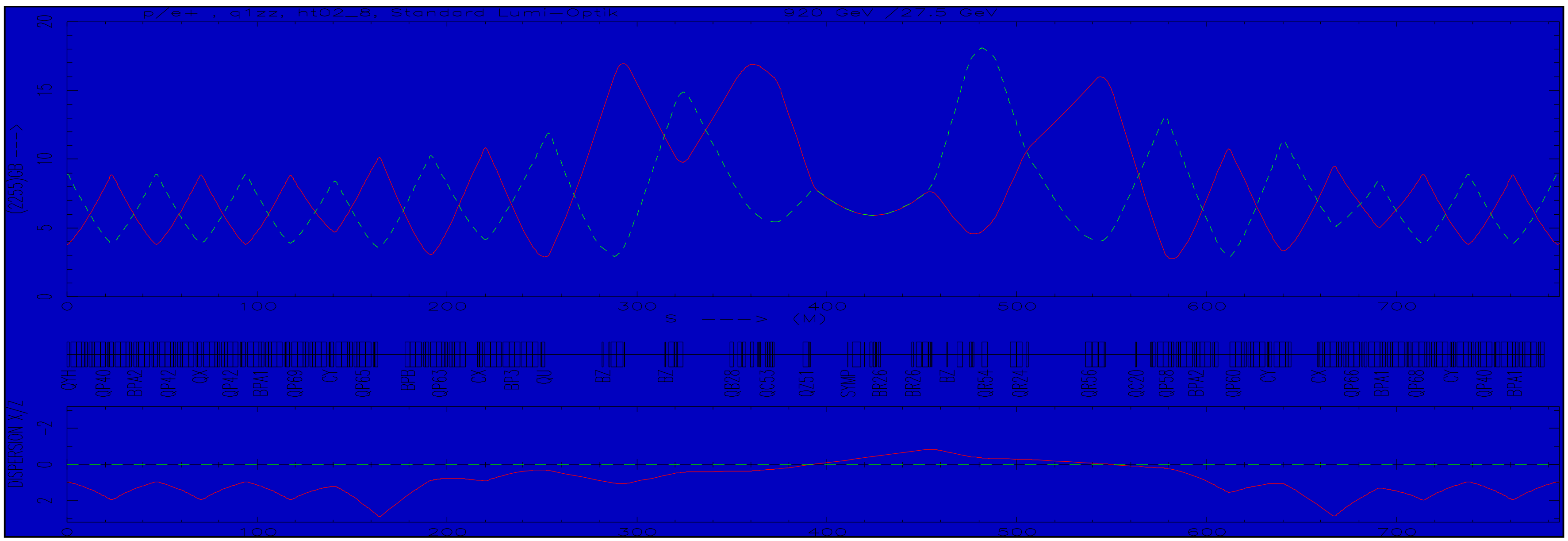
Part I: Lattice Elements and Equation of Motion



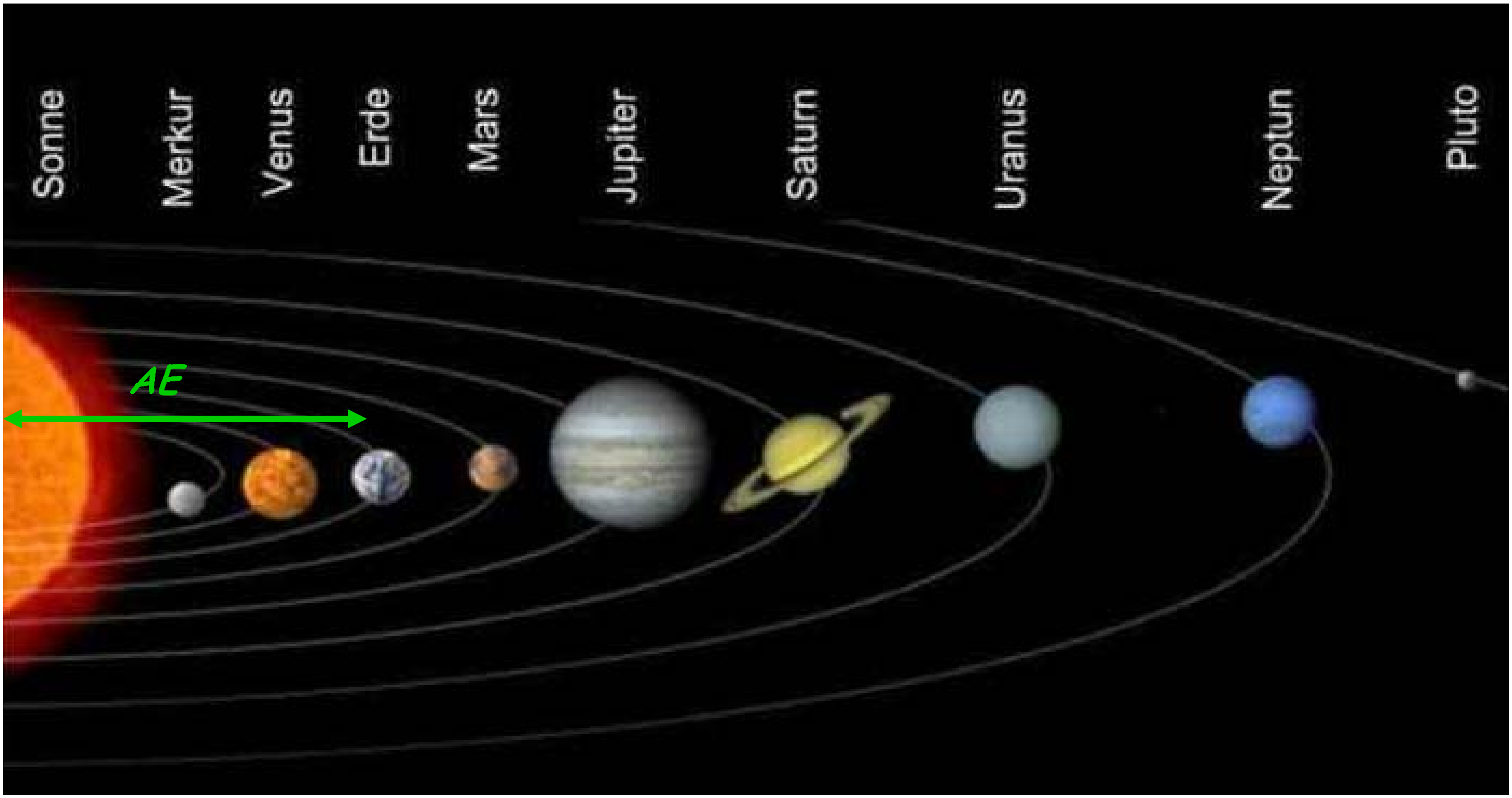
Lattice and Beam Optics of a typical high energy storage ring

Largest storage ring: The Solar System

astronomical unit: average distance earth-sun

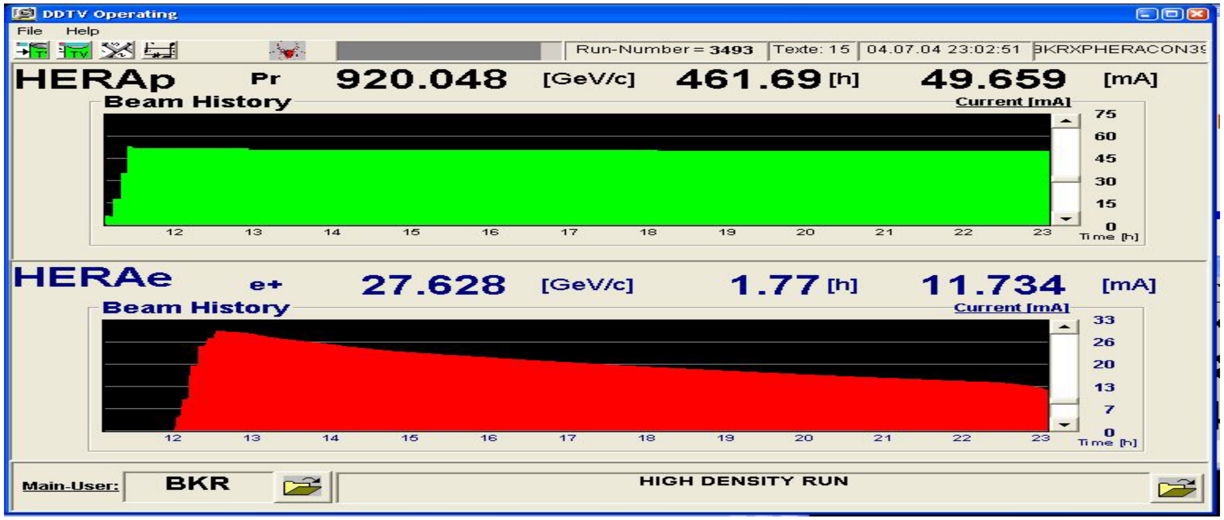
$1AE \approx 150 * 10^6$ km

Distance Pluto-Sun $\approx$ 40 AE



Luminosity Run of a typical storage ring:

HERA storage ring: Protons accelerated and stored for 12 hours
distance of particles travelling at about $v \approx c$
$L = 10^{10}-10^{11}$ km
... several times Sun-Pluto and back



→ *guide the particles on a well defined orbit („design orbit")*
→ *focus the particles to keep each single particle trajectory within the vacuum chamber of the storage ring, i.e. close to the design orbit.*

Transverse Beam Dynamics:

0.) Introduction and basic ideas

„ ... in the end and after all it should be a kind of circular machine“

→ need transverse deflecting force

Lorentz force

$$\vec{F} = q * (\cancel{\vec{E}} + \vec{v} \times \vec{B})$$

typical velocity in high energy machines: $v \approx c \approx 3 * 10^8 \, m/s$

old greek dictum of wisdom:

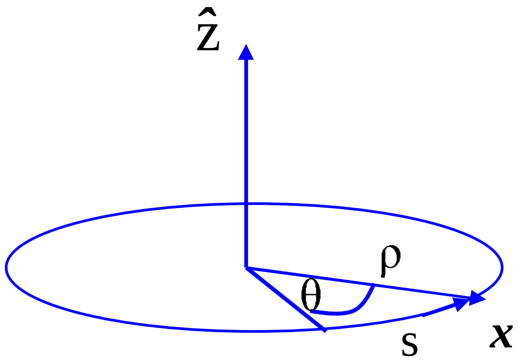
if you are clever, you use magnetic fields in an accelerator where ever it is possible.

But remember: magn. fields act allways perpendicular to the velocity of the particle

→ only bending forces, → no „beam acceleration“

The ideal circular orbit

consider a magnetic field B is independent of the azimuthal angle θ



circular coordinate system

condition for circular orbit:

Lorentz force $$F_L = e * v * B$$

centrifugal force $$F_{Zentr} = \frac{\gamma\, m_0\, v^2}{\rho}$$

ideal condition for circular movement:

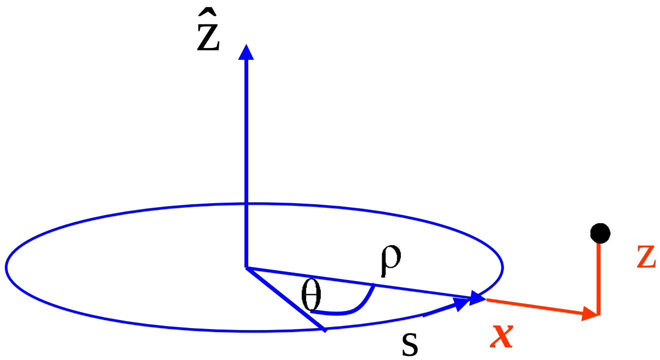
$$\frac{\gamma m_0 v^{\cancel{2}}}{\rho} = e * \cancel{v} * B$$

$$\boxed{\frac{p}{e} = B * \rho}$$

On a circular orbit, the momentum of the particle is related to the guide field B and the radius of curvature ρ.

Focusing Forces:
I.) the principle of weak focusing

still: consider a magnetic field B is independent of the azimuthal angle θ



circular coordinate system

stability of the particle movement: small deviations of particle from ideal orbit ↔ restoring forces

$$e\,v\,B_z(r)\begin{cases} < \dfrac{\gamma\, m_0\, v^2}{r} & \textbf{for} \quad r < \rho \\ > \dfrac{\gamma\, m_0\, v^2}{r} & \textbf{for} \quad r < \rho \end{cases}$$

$$F_{rest} = \frac{m\,v^2}{r} - e\,v\,B$$

* *introduce a gradient of the magnetic field*

$$evB_z = ev\,(B_0 + \frac{\partial B_z}{\partial r} * x) = evB_0\left(1 + \frac{\partial B_z}{\partial r}\frac{x}{B_{0z}}\right)$$

$$= ev\,B_0\left\{1 - n * \frac{x}{\rho}\right\}$$

field gradient „n“, by definition:

$$n = -\frac{\rho}{B_0}\frac{\partial B_z}{\partial r}$$

* *develop for small x*

$$r = \rho + x = \rho(1 + \frac{x}{\rho})$$

$$\frac{mv^2}{r} \approx \frac{mv^2}{\rho}(1 - \frac{x}{\rho})$$

→ *restoring force:*

$$F_{rest} = \frac{mv^2}{\rho}(1 - \frac{x}{\rho}) - evB(1 - \frac{nx}{\rho})$$

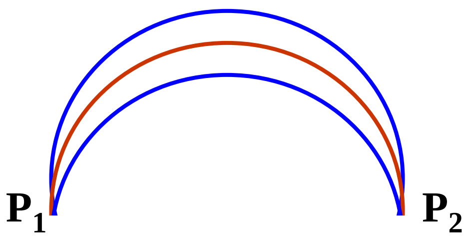
$$= \frac{p}{\rho}v(1 - \frac{x}{\rho}) - evB(1 - \frac{nx}{\rho})$$

$$= -evB * \frac{x}{\rho}(1 - n)$$

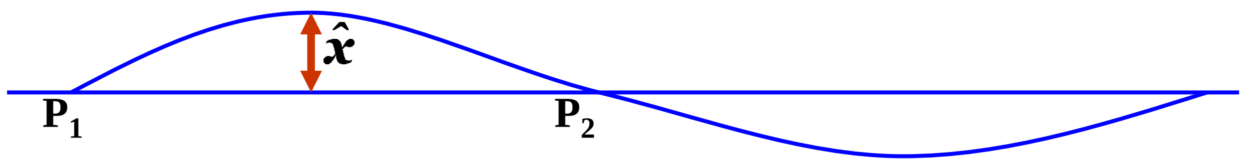
$$F_{rest} = -evB * \frac{x}{\rho}(1-n)$$

condition for focusing in the horizontal plane:
n < 1

Nota Bene: the condition does not exclude n = 0.
there is focusing even in a homogenous field.



„Geometric focusing“ in a homogeneous field:
consider three particles, starting at the same point with different angles



Problem: amplitude of betatron oscillation in this case $\hat{x} \approx \alpha * \rho$

$$\left.\begin{array}{l} \alpha \approx 1\ mrad \ \textit{for a particle beam} \\ \rho \approx \textit{several } 100m \end{array}\right\} \quad \hat{x} \approx 1\,m$$

weak focusing in the vertical plane:

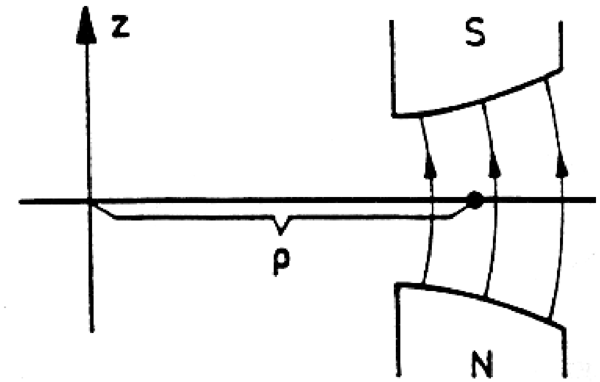
restoring force in „z“ $F_z \propto -z$

we need a horizontal magnetic field component: $B_x = -const * z$

... or a negative horizontal field gradient $\frac{\partial B_x}{\partial z} = -const$

Maxwells equation: $\vec{\nabla} \times \vec{B} = 0$

$$\frac{\partial B_x}{\partial z} = \frac{\partial B_z}{\partial x} = \frac{\partial B_z}{\partial r} < 0$$



typical pole shape in a combined function ring

the vertical field component has to decrease with increasing radius

$$\frac{\partial B_z}{\partial r} < 0 \Leftrightarrow n > 0$$

Comments on weak focusing machines:

**** magnetic field is independent of the azimuthal angle θ, focusing gradient is included in the dipole field***

**** stability of the particle movement in both planes requires***

$$0 < n < 1$$

**** equation of motion (*** *see appendix* ***):***

$$\ddot{x} + \omega_0^2 (1 - n) x = 0$$

$$\ddot{z} + \omega_0^2 n * z = 0$$

Problem: we get less than one transverse oscillation per turn → large oscillation amplitudes

Separate the focusing gradients from the bending fields to obtain n >>1

Example HERA:

$g = 98\ T/m$ **at** $p = 920\ GeV/c$

$$n \approx 12420$$

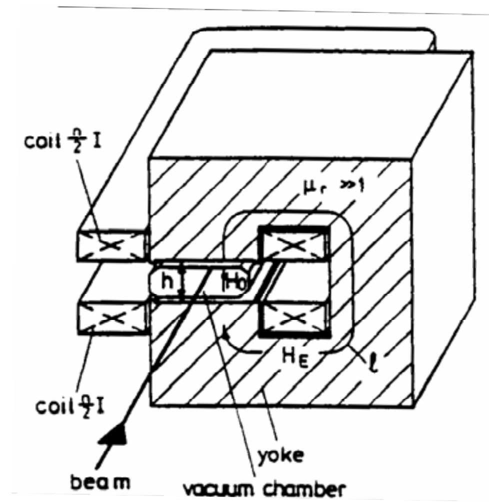
II.) Accelerator Magnets

Separate Function Machines:

Split the magnets and optimise them according to their job: bending, focusing etc

Dipole Magnets:

homogeneous field created by two flat pole shoes



calculation of the field:

3rd Maxwell equation for a static field: $\vec{\nabla} \times \vec{H} = \vec{j}$

according to Stokes theorem:

$$\int_S (\vec{\nabla} \times \vec{H})\vec{n}\, da = \oint \vec{H}\, d\vec{l} = \int_S \vec{j} \cdot \vec{n}\, da = N \cdot I$$

N*I = number of windings times current per winding

$$\oint \vec{H}\, d\vec{l} = H_0 * h + H_{Fe} * l_{Fe}$$

in matter we get with $\mu_r \approx 1000$

$$\oint \vec{H}\, d\vec{l} = H_0 * h + \frac{H_0}{\mu_r} * l_{Fe} \approx H_0 * h$$

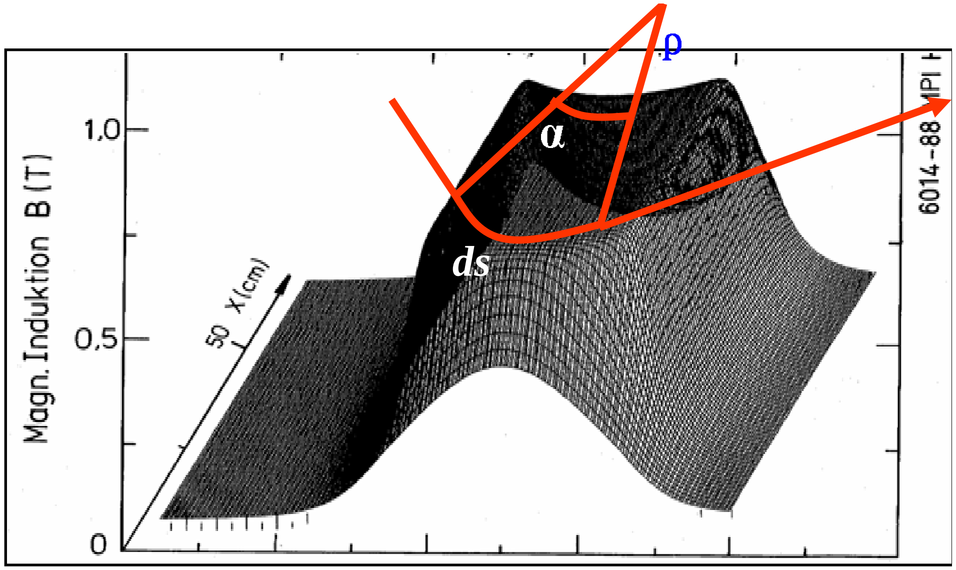
magnetic field of a dipole magnet:

$$B_0 = \frac{\mu_0 n I}{h}$$

radius of curvature

... remember $p/e = B*\rho$

$$\frac{1}{\rho}\left[m^{-1}\right] = \frac{e \cdot B_0}{p} = 0.2998\,\frac{B_0\,[T]}{p\,[GeV/c]}$$



field map of a storage ring dipole magnet

bending angle of a dipole magnet: $\alpha = \frac{ds}{\rho} \approx \frac{dl}{\rho}$

for a circular machine require $\alpha = \frac{\int B\,dl}{B*\rho} = 2\pi \rightarrow \int B\,dl = 2\pi * \frac{p}{q}$

hard edge approximation:

define the effective length of a magnet by

$$\boxed{B_0 \cdot l_{eff} := \int_{-\infty}^{+\infty} B\,dl}$$

typically we get $l_{eff} \approx l_{iron} + 1.3 * h$

Example HERA:

920 GeV Proton storage ring:
N = 416
l = 8.8m,
q = +1 e

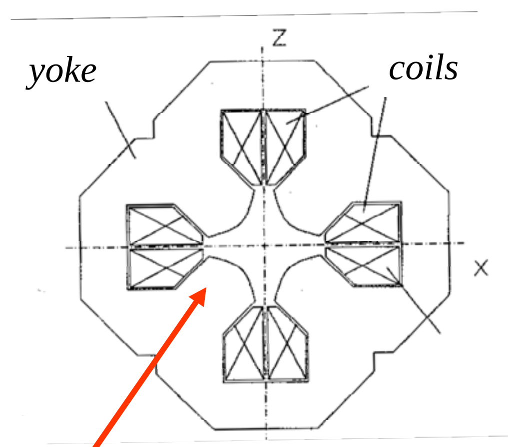
Nota bene: for high energy particles we can set ...

$$E \approx p \cdot c$$

$$\int B dl \approx N * l * B = 2\pi \quad p / q$$

$$B \approx \frac{2\pi * 920 * 10^{9} eV}{416 * 3 * 10^{8} \frac{m}{s} * 8.8m * e} \approx 5.15 \quad Tesla$$

Quadrupole Magnets:



required: linear increasing magnetic field

$$B_z = -g \cdot x \qquad B_x = -g \cdot z$$

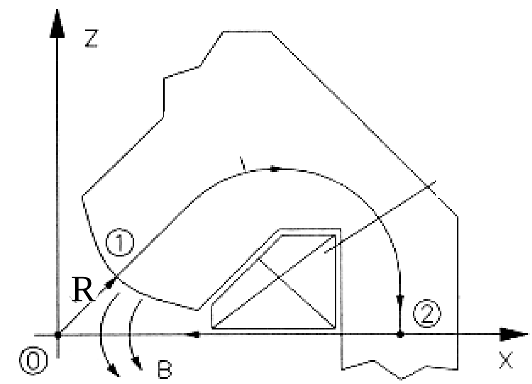
at the location of the particle trajectory: no iron, no current

$$\vec{\nabla} \times \vec{B} = 0 \quad \rightarrow \quad \vec{B} = -\vec{\nabla} V$$

the magnetic field can be expressed as gradient of a scalalr potential !

$$V(x,z) = g \cdot xz$$

equipotential lines (i.e. the surface of the iron contour) = hyperbolas



calculation of the field:

$$\oint \vec{H} \cdot d\vec{s} = nI$$

$$\oint \vec{H} \cdot d\vec{s} = \int_0^R H(r)dr + \cancel{\int_1^2 \vec{H}_{Fe}\, d\vec{s}} + \cancel{\int_2^0 \vec{H} \cdot d\vec{s}}$$

calculation of the quadrupole field:

$$\oint \vec{H} \cdot d\vec{s} = \int_0^R H(r)dr = \int_0^R \frac{B(r)dr}{\mu_0} = n * I$$

$$B(r) = -g * r, \qquad r = \sqrt{x^2 + z^2}$$

gradient of a quadrupole field:

$$g = \frac{2\mu_0 n I}{R^2}$$

remember: normalised dipole strength:

$$\frac{1}{\rho} = \frac{B_0}{p/e}$$

normalised quadrupole strength:

$$k = \frac{g}{p/e}$$

focal length:

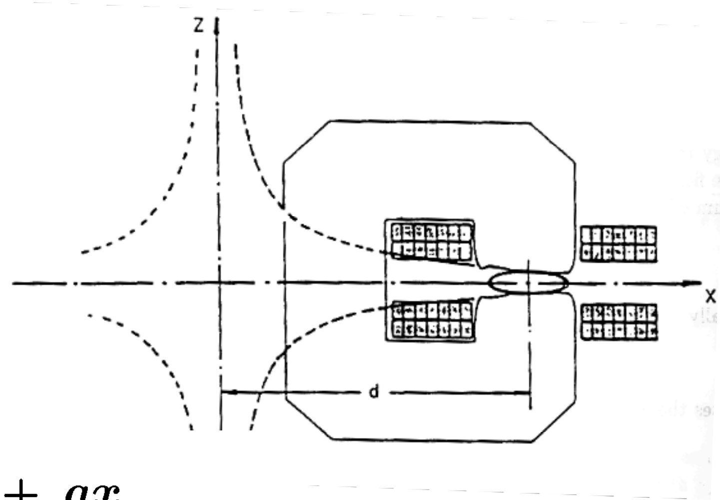
$$f := \frac{1}{k*l}$$

Example of a sparated function machine: heavy ion storage ring TSR

„Synchrotron Magnet“:

combines the homogeneous field of a dipole with a quadrupole gradient

potential: $$V(x,z) = -B_0 z + g \cdot xz$$



advantage: lattice with high compactness

disadvantage: strong correlation of momentum (via dipole field) and beam optics.
→ poor flexibility

field index:

$$B_z = \frac{\partial V}{\partial z} = -B_0 + gx$$

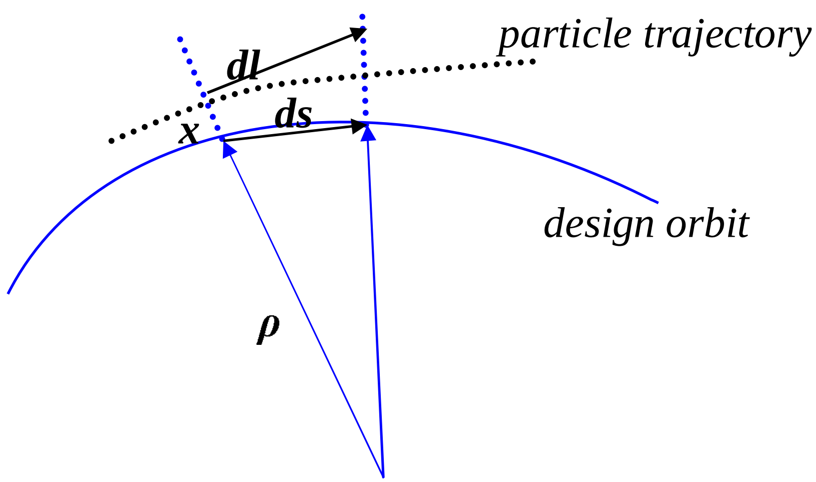
$$n = \frac{\partial B_z}{\partial x} \cdot \frac{\rho}{B_0} = g\frac{\rho}{B_0} = k\rho^2$$

Nota bene: Synchrotron magnet can be considered as a shifted quadrupole lens „off center quadrupole“.

III.) The equation of motion

Pre-requisites: *** consider particles with ideal momentum or at least with only small momentum error***
**** neglect terms of second order in x,z,and $\Delta p/p$ → linear approximation***
**** independent variable "s", write derivative with respect to s as …´***

θ_0 angle of ideal orbit
θ angle of particle trajectory
x´ derivative of particle amplitude with resp. to s



$$x' = \theta - \theta_0$$

$$x'' = \frac{d(\theta - \theta_0)}{ds}$$

For any circular orbit path we get:

$$d\theta_0 = -\frac{ds}{\rho} \quad \rightarrow \frac{d\theta_0}{ds} = \frac{-1}{\rho}$$

... quite clear, but what is d θ/ds ?

$$(a) \qquad d\theta = -\frac{dl}{\rho} = -dl\,\frac{Be}{p}$$

as for any circular orbit we know: $$\frac{1}{\rho} = \frac{B}{p/e}$$

as long as the angle x´ is small dl *is related to s by:*

$$(b) \qquad dl = \frac{\rho + x}{\rho}\,ds$$

$$dl = (1 + \frac{x}{\rho})ds$$

Magnetic field: assume only dipole and quadrupole terms

$$B = B_0 + \frac{\partial B}{\partial x}x = B_0 - gx$$

remember: definition of field gradient

$$B_x = -gx$$

$$B_z = -gz$$

$$(c) \qquad B = \frac{p_0}{e\rho} - kx\frac{p_0}{e} = \frac{p_0}{e}\left\{\frac{1}{\rho} - kx\right\}$$

normalised strength $$k = \frac{g}{p_0/e}$$

putting the term (b) and (c) into the expression for the angle dθ …

$$d\theta = -dl\,\frac{Be}{p} = -(1+\frac{x}{\rho})ds\cdot\frac{eB}{p_0+\Delta p}$$

$$d\theta = -(1+\frac{x}{\rho})ds\,\frac{e\cdot\frac{p_0}{e}\left(\frac{1}{\rho}-kx\right)}{p_0+\Delta p}$$

$$d\theta = -\frac{p_0}{p_0+\Delta p}\left\{\frac{1}{\rho}-kx+\frac{x}{\rho^2}-\frac{kx^2}{\rho}\right\}ds$$

develop the momentum
p *for small* **Δp**

$$\frac{p_0}{p_0+\Delta p}\approx 1-\frac{\Delta p}{p_0}$$

$$d\theta = -ds\left\{\frac{1}{\rho}-kx+\frac{x}{\rho^2}-\cancel{\frac{kx^2}{\rho}}-\frac{\Delta p}{p_0\rho}+\cancel{kx\frac{\Delta p}{p_0}}-\cancel{\frac{x}{\rho^2}\frac{\Delta p}{p_0}}+\cancel{kx^2\frac{\Delta p}{\rho p_0}}\right\}$$

and keep only first order terms in x, z, **Δp**

!

$$d\theta = -ds\left\{\frac{1}{\rho} - kx + \frac{x}{\rho^2} - \frac{\Delta p}{p_0\rho}\right\}$$

... do you still remember the beginning ?

we were looking for ...

$$x'' = \frac{d\theta}{ds} - \frac{d\theta_0}{ds} \qquad x'' = -\frac{1}{\rho} + kx - \frac{x}{\rho^2} + \frac{\Delta p}{p_0}\frac{1}{\rho} - \frac{d\theta_0}{ds}$$

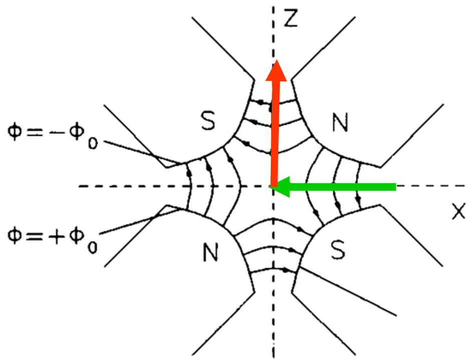
$$x'' + \left(\frac{1}{\rho^2} - k\right)x = \frac{1}{\rho}\frac{\Delta p}{p}$$

vertical direction:

* ***no bending (... in general) → no $1/\rho^2$ term***
* ***vertical gradient:***

$$\nabla \times B = 0 \quad \Leftrightarrow \quad \frac{\partial B_z}{\partial x} = \frac{\partial B_x}{\partial z}$$

* ***Lorentz force gets a "-":*** F=q (v x B)



$$z'' + kz = 0$$

IV.) Solution of trajectory equations

horizontal plane:

$$x'' + \left(\frac{1}{\rho^2(s)} - k(s)\right)x = \frac{1}{\rho(s)}\frac{\Delta p}{p}$$

define:

$$K(s) = -k(s) + 1/\rho^2(s)$$

$$x'' + K(s) * x = \frac{1}{\rho}\frac{\Delta p}{p}$$

2 Problems: **** inhomogeneous equation*** ***→ set for the moment $\Delta p/p=0$***
i.e. consider particles of ideal momentum

**** K(s) is not constant but varies as a function of the azimuth***
K(s) is a "time dependent" restoring force
→ the differential equation can only be solved numerically

K(s) is prescribed by the storage ring design:
given by the magnet parameters

remember: hard edge model:
K = const within a magnet

SPS Lattice

$$x'' + K * x = 0$$

differential equation for the transverse oscillation of a particle in a magnetic element of the storage ring. (... harmonic oscillator)

** second order* → *two independent solutions,*
** linear in x* *any linear combination of these „principal solutions“ will again be a solution.*

we choose for K > 0:

$$C(s) = \cos(\sqrt{K}s) \,, \qquad S(s) = \frac{1}{\sqrt{K}} \sin(\sqrt{K}s)$$

with the initial conditions:

$$C(0) = 1 \,, \qquad S(0) = 0$$

$$C'(0) = 0 \,, \qquad S'(0) = 1$$

for K < 0:

$$C(s) = \cosh(\sqrt{K}s) \,, \qquad S(s) = \frac{1}{\sqrt{K}} \sinh(\sqrt{K}s)$$

Arbitrary solution of any particle:

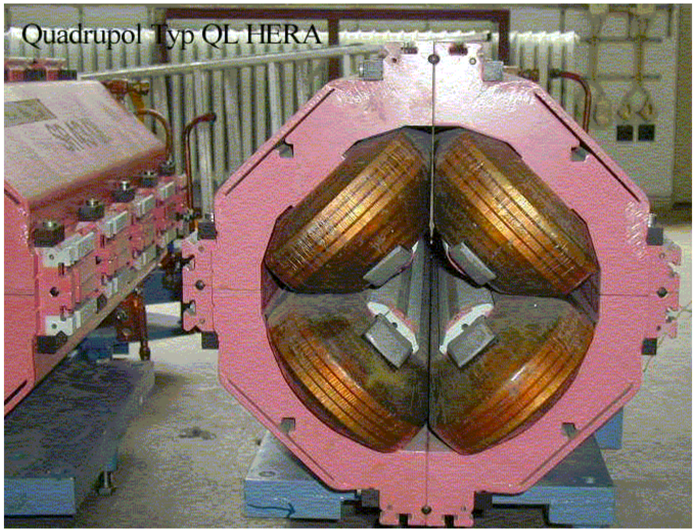
$$x(s) = x_0 \cdot C(s) + x_0' \cdot S(s)$$

Matrix formalism for beam transfer in a lattice:

$$\left.\begin{aligned} x(s) &= x_0 \cdot C(s) + x_0' \cdot S(s) \\ x'(s) &= x_0 \cdot C'(s) + x_0' \cdot S'(s) \end{aligned}\right\} \Rightarrow \boxed{\begin{pmatrix} x(s) \\ x'(s) \end{pmatrix} = M * \begin{pmatrix} x(s) \\ x'(s) \end{pmatrix}}$$

where $M = \begin{pmatrix} C(s) & S(s) \\ C'(s) & S'(s) \end{pmatrix}$

HERA standard type quadrupole lens



... depending on the value of K we can establish a transfer matrix for any (linear) lattice element in the ring.

horizontal focusing quadrupole: K > 0

$$M = \begin{pmatrix} \cos\sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}\,l \\ -\sqrt{|K|}\sin\sqrt{|K|}\,l & \cos\sqrt{|K|}\,l \end{pmatrix}$$

vertical focusing quadrupole: K < 0

$$M = \begin{pmatrix} \cosh \sqrt{|K|}\,l & \frac{1}{\sqrt{|K|}} \sinh \sqrt{|K|}\,l \\ \sqrt{|K|} \sinh \sqrt{|K|}\,l & \cosh \sqrt{|K|}\,l \end{pmatrix}$$

drift space: K = 0

$$M = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}$$

particle motion in the vertical plane:

in general storage rings are built in the horizontal plane.

no vertical bending dipoles → $1/\rho = 0$

define: $K = k$ *→ same matrices as in x-plane.*

! with the assumptions made, the motion in the horizontal and vertical planes are independent „ ... the particle motion in x & z is uncoupled“

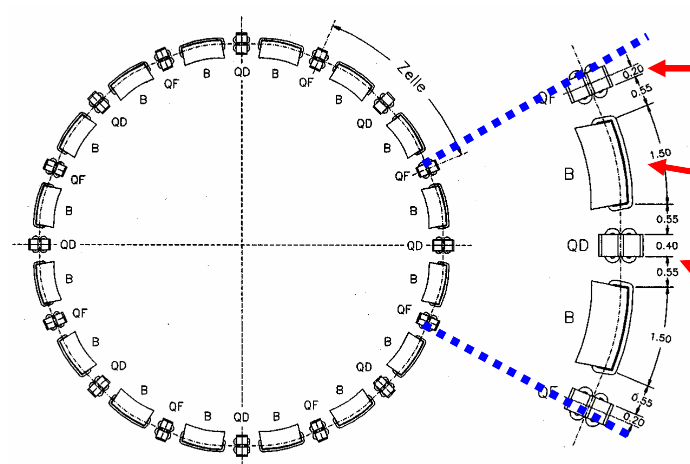
!! dont´t forget the inhomogeneous equation

transformation through a system of lattice elements

combine the single element solutions by multiplication of the matrices

$$M_{total} = M_{QF} * M_{D} * M_{QD} * M_{Bend} * M_{D*.....}$$

$$\begin{pmatrix} x \\ x' \end{pmatrix}_{s2} = M(s2,s1) * \begin{pmatrix} x \\ x' \end{pmatrix}_{s1}$$

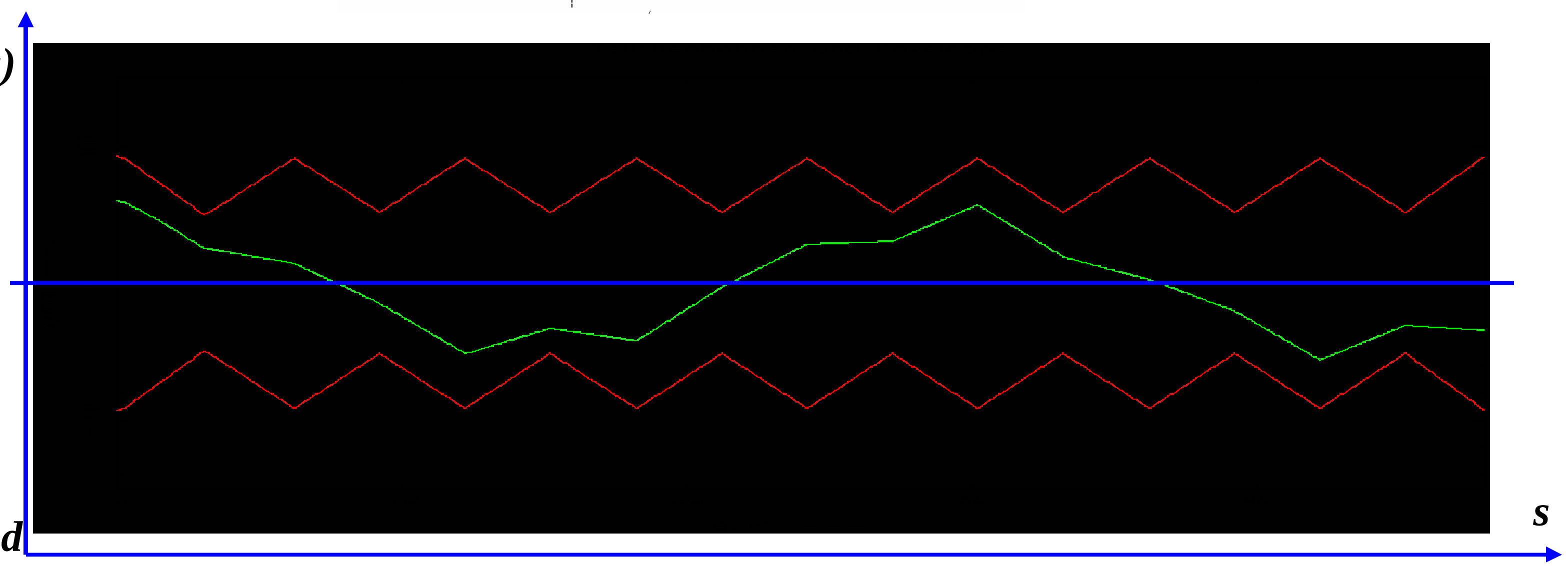


focusing lens

dipole magnet

defocusing lens

K. Wille



typical values
in a strong
foc. machine:
$x \approx mm$, $x' \leq mrad$

Dispersion:

inhomogeneous equation

$$x'' + K(s) * x = \frac{1}{\rho}\frac{\Delta p}{p}$$

general solution = complete solution of the homogeneous equation
+ particular solution of inhomogeneous equation

$$x(s) = x_h(s) + x_i(s)$$

with $x_h'' + K(s) * x_h = 0$ $\qquad x_i'' + K(s) * x_i = \frac{1}{\rho}\frac{\Delta p}{p}$

normalise with respect to Δp/p: $D(s) = \frac{x_i(s)}{\Delta p / p}$

$$D''(s) + K(s) * D(s) = \frac{1}{\rho}$$

initial conditions: $D_0 = D_0' = 0$

$$\boxed{x(s) = x_0 \cdot C(s) + x_0' \cdot S(s) + D(s)\frac{\Delta p}{p}}$$

Dispersion:

for convenience:
expand the matrix formalism:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \end{pmatrix}_{s0} + \frac{\Delta p}{p} \begin{pmatrix} D \\ D' \end{pmatrix}$$

or even more convenient

$$\begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_s = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ x' \\ \frac{\Delta p}{p} \end{pmatrix}_{s0}$$

Determine the Dispersion from the lattice parameters:

remember: C and S are independent solutions
of the equation of motion → the Wronski determinant

$$W = \begin{vmatrix} C & S \\ C' & S' \end{vmatrix} \neq 0$$

even more, we get:

$$\frac{dW}{ds} = \frac{d}{ds}(CS' - SC') = CS'' - SC''$$

$$= -K(CS - SC) = 0$$

Dispersion:

$\rightarrow \quad W = const.$

choose the position $s = s_0$ where

$$C_0 = 1, \quad C_0' = 0$$
$$S_0 = 0, \quad S_0' = 1$$

$W = 1$

the dispersion trajectory can be calculated from the cosine and sinelike solutions:

$$D(s) = S(s)\int_{s0}^{s} \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C(s)\int_{s0}^{s} \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

proof: D(s) has to fulfil the equation of motion

$$D'(s) = S'(s)\int_{s0}^{s} \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C'(s)\int_{s0}^{s} \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s}$$

$$D''(s) = S''(s)\int_{s0}^{s} \frac{1}{\rho(\tilde{s})} C(\tilde{s}) d\tilde{s} - C''(s)\int_{s0}^{s} \frac{1}{\rho(\tilde{s})} S(\tilde{s}) d\tilde{s} + \frac{1}{\rho}\underbrace{(CS' - SC')}_{=1}$$

$$D'' = -K(s)D(s) + \frac{1}{\rho}$$

Example:

$$\boxed{D(s) = S(s)\int_{s0}^{s} \frac{1}{\rho(\tilde{s})} C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s} \frac{1}{\rho(\tilde{s})} S(\tilde{s})d\tilde{s}}$$

drift:

$$M_{Drift} = \begin{pmatrix} 1 & l & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$1/\rho = k = 0,$$
$$\rightarrow D(s) = 0,\ D'(s) = 0$$

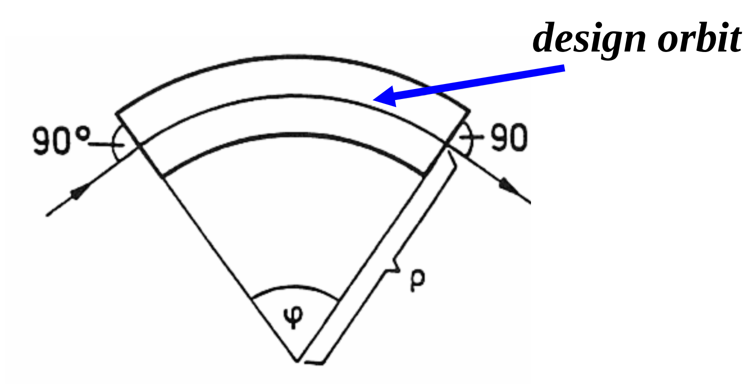
foc. Quadrupole:

$$M_{Qfoc} = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l & 0 \\ \sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$1/\rho = 0,\ K = -k$$
$$\rightarrow D(s) = 0,\ D'(s) = 0$$

dipole sector magnet:

angle at entrance and exit: 90°



dipole sector magnet:

$$\frac{1}{\rho} = const,\ k = 0$$

$$K = 1/\rho^2$$

2x2 matrix

$$M = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\cdot\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

$$D(s) = S(s)\int_{s0}^{s}\frac{1}{\rho(\tilde{s})}C(\tilde{s})d\tilde{s} - C(s)\int_{s0}^{s}\frac{1}{\rho(\tilde{s})}S(\tilde{s})d\tilde{s}$$

$$D(s) = \rho\sin\frac{l}{\rho}\cdot\int_0^l\frac{1}{\rho}\cdot\cos\frac{s}{\rho}ds - \cos\frac{l}{\rho}\cdot\int_0^l\frac{1}{\rho}\cdot\rho\sin\frac{s}{\rho}ds$$

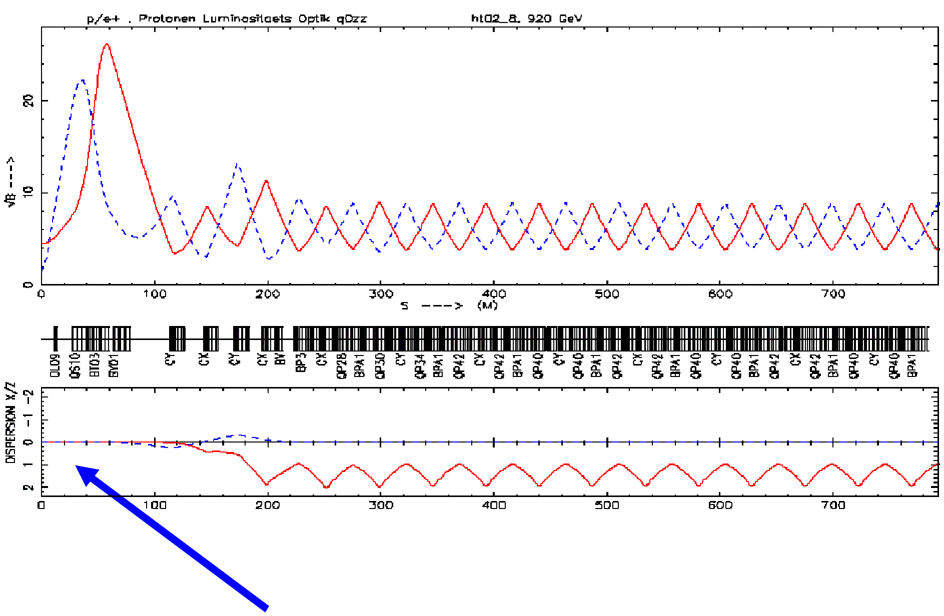
$$D(s) = \rho\cdot\sin^2\frac{l}{\rho} + \cos\frac{l}{\rho}\cdot\left[\cos\frac{l}{\rho} - 1\right]\cdot\rho$$

$$D(s) = \rho(1-\cos\frac{l}{\rho}) \qquad D'(s) = \sin\frac{l}{\rho})$$

dipole sector magnet:

$$M_{Qfoc} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} & \rho\left(1-\cos\frac{l}{\rho}\right) \\ \frac{-1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} & \sin\frac{l}{\rho} \\ 0 & 0 & 1 \end{pmatrix}$$

Example: HERA Interaction region



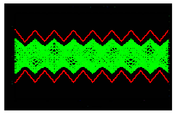
typical values: $\Delta p/p \approx 10^{-3}$

$D \approx 1 \ldots 2m$

$x_i = D^* \Delta p/p \approx 1\text{-}2\ mm$

start value: $D_0 = D_0' = 0$

dispersion is generated as soon as we enter the dipole magnets where $1/\rho \neq 0$



Remarks on Magnet Matrices:

1.) thin lens approximation:

matrix of a quadrupole lens

$$M = \begin{pmatrix} \cos\sqrt{|k|}l & \frac{1}{\sqrt{|k|}}\sin\sqrt{|k|}l \\ -\sqrt{|k|}\sin\sqrt{|k|}l & \cos\sqrt{|k|}l \end{pmatrix}$$

in many practical cases we have the situation:

$$f = \frac{1}{kl_q} >> l_q$$

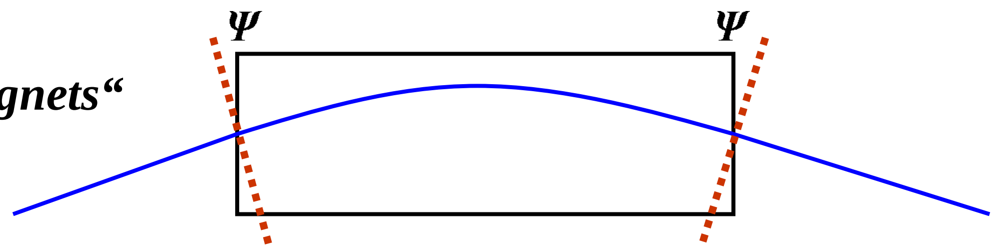
... focal length of the lens is much bigger than the length of the magnet

limes: $l \rightarrow 0$ *while keeping :* $kl = const$

$$\boxed{M_x = \begin{pmatrix} 1 & 0 \\ \frac{1}{f} & 1 \end{pmatrix} \qquad M_z = \begin{pmatrix} 1 & 0 \\ \frac{-1}{f} & 1 \end{pmatrix}}$$

... usefull for fast (and in large machines still quite accurate) „back on the envelope calculations“ ... and for the guided studies !

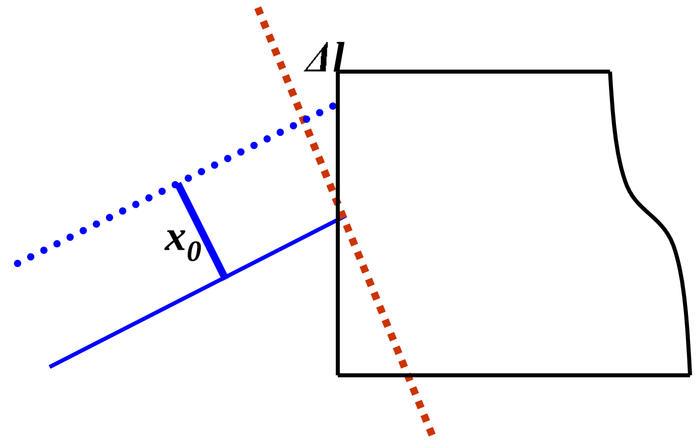
3.) edge focusing: ... dipole „box-magnets"



*** horizontal plane:**

particle at distance x_0 to the design orbit sees a „shorter magnetic field"

$$\Delta l = x_0 \cdot \tan\psi$$



error in the bending angle of the dipole

$$\Delta\alpha = \frac{\Delta l}{\rho} = x_0 \cdot \frac{\tan\psi}{\rho}$$

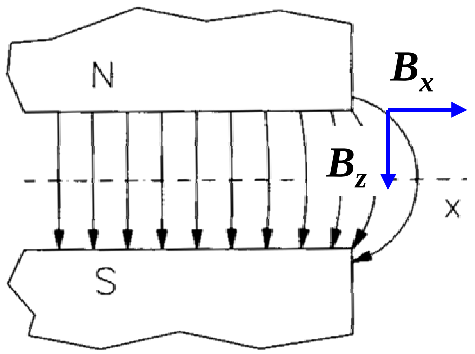
→*corresponds to a horizontal defocusing effect*

... in the approximation of Δl = small

$$\left.\begin{aligned} x &= x_0 \\ x' &= x_0' + x_0 \frac{\tan\psi}{\rho} \end{aligned}\right\} \Rightarrow M = \begin{pmatrix} 1 & 0 \\ \frac{\tan\psi}{\rho} & 1 \end{pmatrix}$$

3.) edge focusing: vertical plane

particle trajectory crosses the field lines at the dipole edge.
→horizontal field component
vertical focusing effect



fringe field effect at the edge of a dipole

$$M_z \approx \begin{pmatrix} 1 & 0 \\ \frac{-\tan\psi}{\rho} & 1 \end{pmatrix}$$

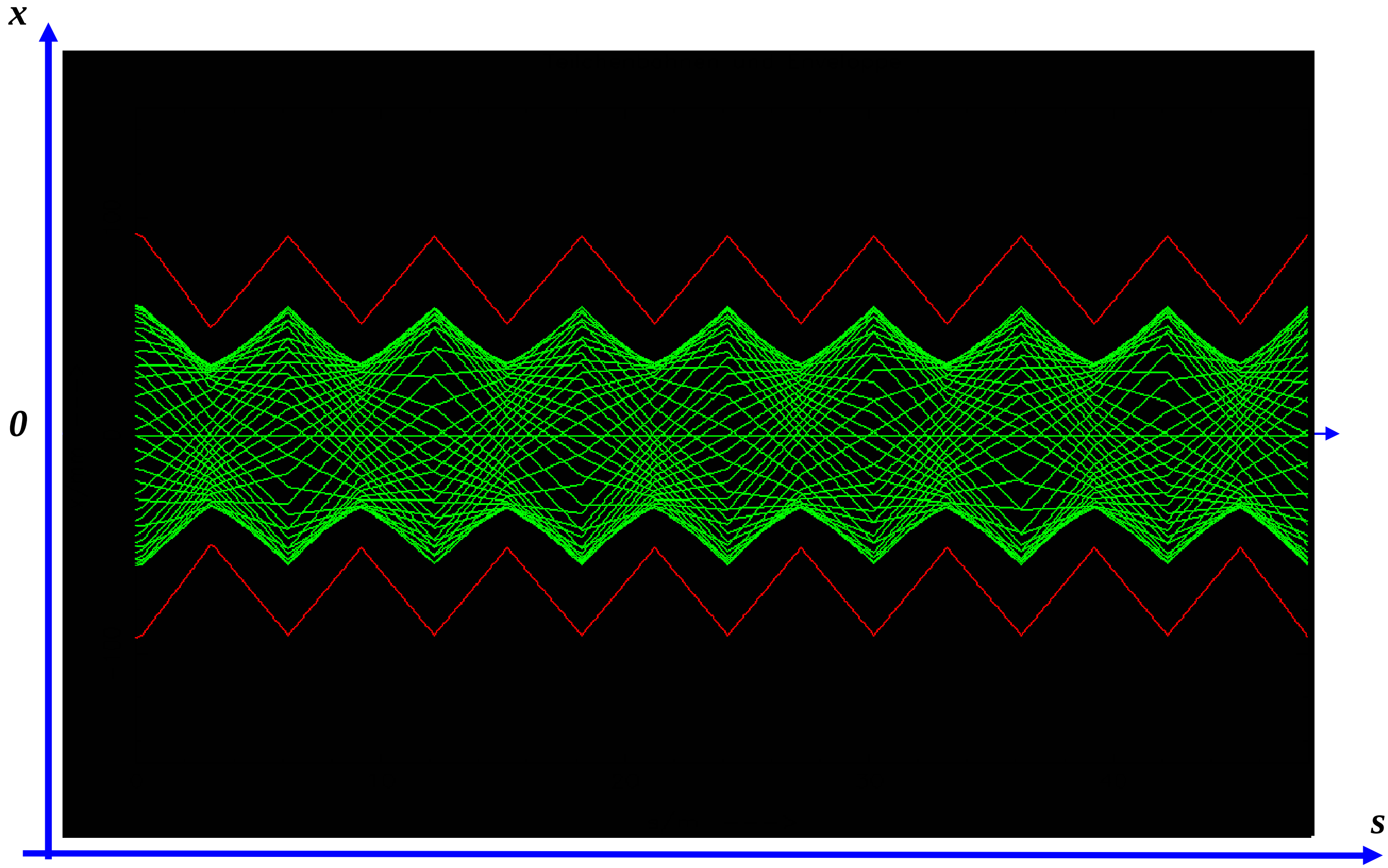
for purists only: vertical edge effect depends on the exact form of the dipole fringe field

$$M_z \approx \begin{pmatrix} 1 & 0 \\ \frac{1}{\rho^2}\frac{b}{6\cos\psi} - \frac{\tan\psi}{\rho} & 1 \end{pmatrix}$$

where b = distance over which the fringe field drops to zero

Question: ***what will happen, if the particle performs a second turn ?***

... or a third one or ... 10^{10} turns



Answer: ***... will be discussed in the evening having a good glass of red wine***

... or tomorrow in the next lecture.

V.) Résumé:

beam rigidity: $B \cdot \rho = p/q$

bending strength of a dipole: $\frac{1}{\rho}\left[m^{-1}\right] = \frac{0.2998 \cdot B_0(T)}{p(GeV/c)}$

focusing strength of a quadrupole: $k\left[m^{-2}\right] = \frac{0.2998 \cdot g}{p(GeV/c)}$

$$k\left[m^{-2}\right] = \frac{0.2998}{p(GeV/c)} \frac{2\mu_0 nI}{a_r^2}$$

focal length of a quadrupole: $f = \frac{1}{k \cdot l_q}$

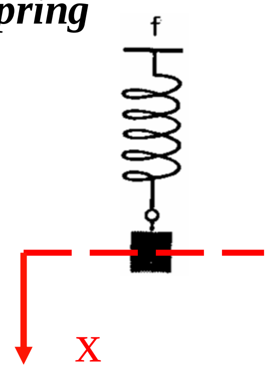
equation of motion: $x'' + Kx = \frac{1}{\rho}\frac{\Delta p}{p}$

matrix of a foc. quadrupole: $x_{s2} = M \cdot x_{s1}$

$$M = \begin{pmatrix} \cos\sqrt{|K|}l & \frac{1}{\sqrt{|K|}}\sin\sqrt{|K|}l \\ -\sqrt{|K|}\sin\sqrt{|K|}l & \cos\sqrt{|K|}l \end{pmatrix}$$

VI.) Appendix: Equation of motion in the case of weak focusing

restoring forces are linear in the deviations x, z from the ideal orbit. Example: harmonic oscillation of a spring



restoring force: $F_r = -c * x$

equation of motion: $\ddot{x} + c * x = 0 \quad \rightarrow \quad x(t) = x_0 * \cos \omega t$

$$\omega = \sqrt{c/m}$$

in our case: $F_r = -\frac{Bev}{\rho}(1-n)x = -\frac{Be}{m}\frac{mv}{\rho}(1-n)x$

ω_0, the angular revolution- (or cyclotron-) frequency is obtained from:

$$\left.\begin{array}{l} mv^2/\rho = evB \quad \rightarrow \quad mv/\rho = eB \\ \omega_0 = v/\rho = eB/m \end{array}\right\} F_r = -\omega_0^2 (1-n) * x$$

$$\omega = \sqrt{c/m} = \omega_0 * \sqrt{1-n}$$

As 0 < n < 1 is required for stability the frequency of the transverse oscillations ω is smaller than the revolution frequency ω_0.

Maxwell's equations

in vacuum

$$\vec{\nabla}\cdot\vec{E}=\rho$$
$$\vec{\nabla}\times\vec{B}=\vec{j}+\frac{\delta\vec{E}}{\delta t}$$
$$\vec{\nabla}\times\vec{E}=-\frac{\delta\vec{B}}{\delta t}$$
$$\vec{\nabla}\cdot\vec{B}=0$$

and in matter

$$\vec{\nabla}\cdot\vec{D}=\rho$$
$$\vec{\nabla}\times\vec{H}=\vec{j}+\frac{\delta\vec{D}}{\delta t}$$
$$\vec{\nabla}\times\vec{E}=-\frac{\delta\vec{B}}{\delta t}$$
$$\vec{\nabla}\cdot\vec{B}=0$$

Stokes integral theorem:

$$\int_S (\vec{\nabla}\times\vec{A})\vec{n}\,da=\oint_C \vec{A}\,d\vec{l}$$

Gauß´ integral theorem:

$$\int_V \vec{\nabla}\cdot\vec{A}\,dx^3=\int_S \vec{A}\vec{n}\,da$$

where	$\vec{A}$	*Vectorfield,*
	V	*Volume*
	S	*Surface surrounding the Volume V*

da	*Surface Element of the Surface S*
$\vec{n}$	*Normvector on the Surface S*

Solution of the equation of motion:

$$x'' + kx = 0$$

k>0 → foc. quadrupole in the horizontal plane

Ansatz:

$$x(t) = a_1 \sin(\omega t) + a_2 \cos(\omega t)$$

with the derivatives:

$$x'(t) = a_1\omega \cos(\omega t) - a_2\omega \sin(\omega t)$$

$$x''(t) = -a_1\omega^2 \sin(\omega t) - a_2\omega^2 \cos(\omega t)$$

$$= -\omega^2 x(t)$$

and we get for the differential equation:

$$x(t) = a_1 \cos(\sqrt{k}t) + a_2 \sin(\sqrt{k}t) \qquad \text{with} \qquad \omega = \sqrt{k}$$

the constants a_1 and a_2 are determined by boundary (i.e. initial) conditions:

at $t = 0$ *we require* $x(0) = x_0, \quad x'(0) = x'_0$

$$x(0) = a_1 \cos(0) + a_2 \sin(0) \quad \rightarrow \quad a_1 = x_0$$

$$x'(0) = -a_1\sqrt{k} \sin(0) + a_2\sqrt{k} \cos(0) \quad \rightarrow \quad a_2 = \frac{x_0'}{\sqrt{k}}$$

$$x(t) = x_0 \cos(\sqrt{k}t) + \frac{x_0'}{\sqrt{k}} \sin(\sqrt{k}t)$$

$$x'(t) = -x_0\sqrt{k} \sin(\sqrt{k}t) + x_0' \cos(\sqrt{k}t)$$

or expressed for convenience in matrix form:

$$\begin{pmatrix} x \\ x' \end{pmatrix}_s = M * \begin{pmatrix} x_0 \\ x_0' \end{pmatrix},$$

$$M = \begin{pmatrix} \cos\sqrt{k}t & \frac{1}{\sqrt{k}} \sin\sqrt{k}t \\ -\sqrt{k} \sin\sqrt{k}t & \cos\sqrt{k}t \end{pmatrix}$$