Course on Free Electron Lasers

Peter Schmüser
University of Hamburg

Part 1

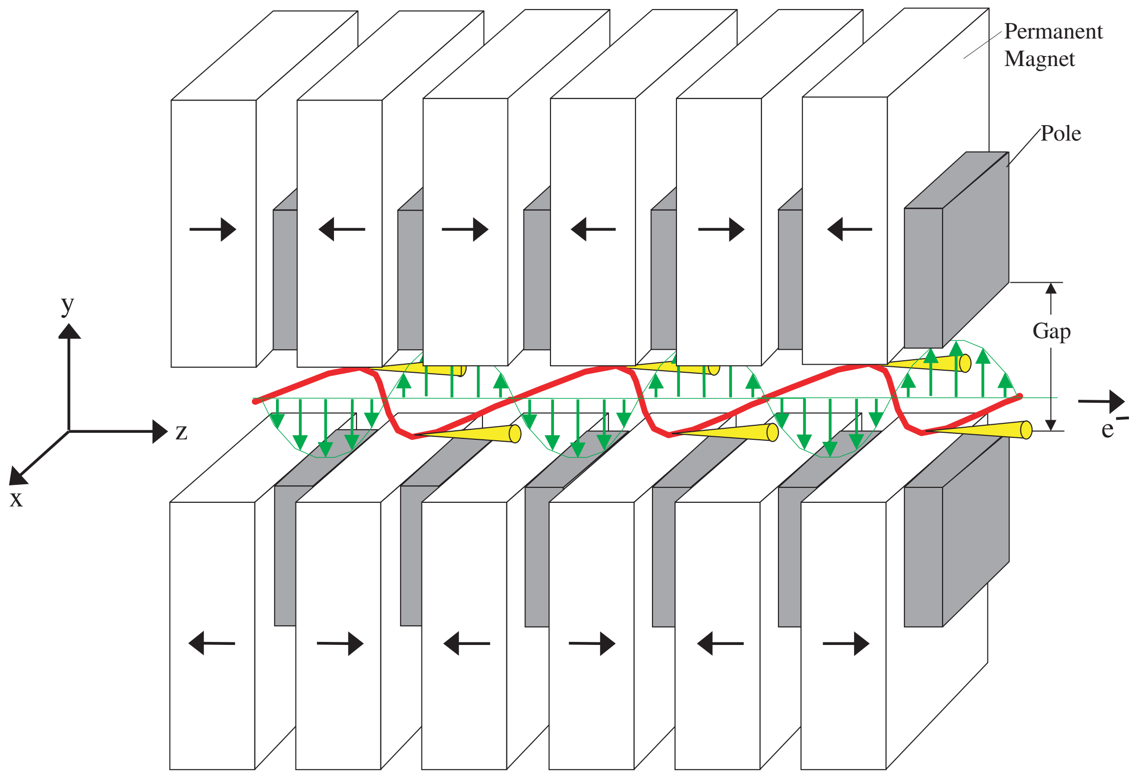
1. Undulator Radiation

- 1.1 Magnetic field of undulator
- 1.2 Electron motion in undulator, undulator parameter K
- 1.3 Motion in second order
- 1.4 Lorentz transformation to moving coordinate system
- 1.5 Transformation of radiation into laboratory system
- 1.6 Line shape of undulator radiation

2. Low-Gain Free Electron Laser

- 2.1 Energy transfer from electron to light wave
- 2.2 Quantitative treatment
- 2.3 FEL with optical resonator, pendulum equation
- 2.4 Computation of gain in FEL (low-gain case), Madey theorem

1.1 Magnetic Field of Undulator



Beam along z direction, magnetic field in y direction (vertical)
λ_u period of the magnet arrangement

Assume width of pole shoes larger than λ_u $\Rightarrow$ x dependence of field can be neglected
Field on the axis approximately harmonic

$$B_y(0,0,z) = B_0 \cos(k_u z) \quad \text{with} \quad k_u = 2\pi/\lambda_u \tag{1}$$

In vacuum we have $\vec{\nabla} \times \vec{B} = 0$, hence magnetic field can be written as gradient of scalar magnetic potential

$$\vec{B} = \nabla \varphi$$

Ansatz

$$\varphi(y, z) = f(y) \cos(k_u z)$$

potential φ fulfills Laplace equation

$$\nabla^2 \varphi = 0 \quad \Rightarrow \quad \frac{d^2 f}{dy^2} - k_u^2 f = 0$$

General solution

$$f(y) = c_1 \sinh(y) + c_2 \cosh(y)$$

$$B_y(y,z) = \frac{\partial\varphi}{\partial\varphi} = k_u(c_1 \cosh(y) + c_2 \sinh(y)) \cos(k_u z)$$

B_y is symmetric with respect to the plane $y = 0 \quad \Rightarrow \quad c_2 = 0$ and $k_u c_1 = B_0$

$$\varphi(x,y,z) = \frac{B_0}{k_u} \sinh(k_u y) \cos(k_u z) \tag{2}$$

For $y \neq 0$: field has also a z component

$$\begin{aligned} B_x &= 0 \\ B_y &= B_0 \cosh(k_u y) \cos(k_u z) \\ B_z &= -B_0 \sinh(k_u y) \sin(k_u z) \end{aligned} \tag{3}$$

In the following we restrict ourselves to the symmetry plane $y = 0$.

1.2 Electron Motion in Undulator

Call $W = E_{kin} + m_e c^2$ the total relativistic energy of the electron. The transverse acceleration by the Lorentz force is

$$\gamma m_e \dot{\vec{v}} = -e\vec{v} \times \vec{B} \quad \text{with} \quad \gamma = \frac{W}{m_e c^2} \tag{4}$$

Two coupled equations in symmetry plane $y = 0$

$$\ddot{x} = \frac{e}{\gamma m_e} B_y \dot{z} \qquad\qquad \ddot{z} = -\frac{e}{\gamma m_e} B_y \dot{x} \tag{5}$$

First-order solution: $v_z = \dot{z} \approx \beta\, c = const$, $v_x \ll v_z$

$$x(t) \approx -\frac{eB_0}{\gamma m_e \beta c k_u^2} \cos(k_u \beta c t) \qquad z(t) \approx \beta c t \tag{6}$$

Cosinelike trajectory $x(z)$ as a function of longitudinal position

$$x(z) = -A\cos(k_u z) \quad \text{with} \quad A = \frac{eB_0}{\gamma m_e \beta c k_u^2}$$

Maximum divergence angle

$$\theta max \approx \left[\frac{dx}{dz}\right]_{max} = \frac{eB_0}{\gamma m_e \beta c k_u} = \frac{K}{\beta\gamma}$$

Definition of undulator parameter

$$\boxed{K = \frac{eB_0}{m_e c k_u} = \frac{eB_0 \lambda_u}{2\pi m_e c}} \tag{7}$$

The emission of synchrotron radiation is inside a cone with opening angle $1/\gamma$
Undulator: $K \leq 1$, electron trajectory inside radiation cone
Wiggler: $K > 1$
Note: $\beta = v/c$ is very close to 1

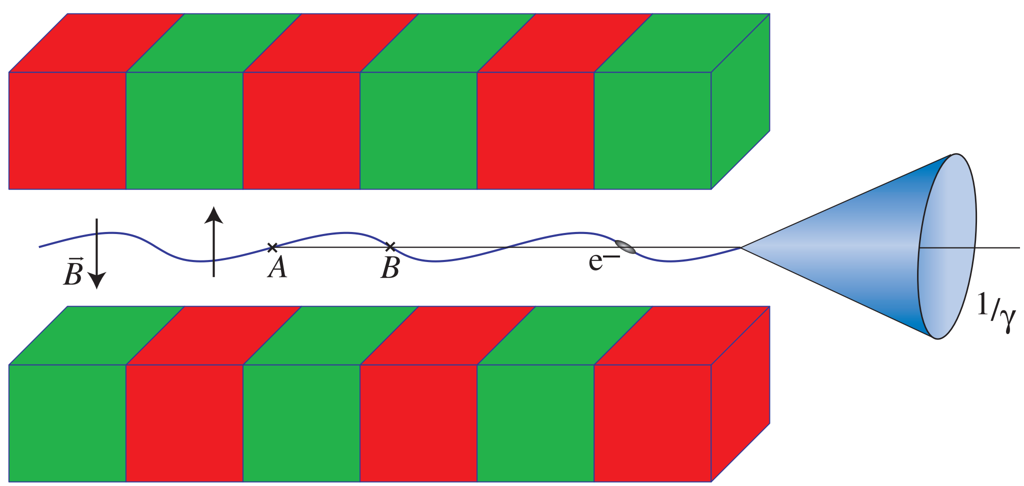


Figure 2.1.: Emission of radiation in an undulator.

1.3 Motion in second order

$$\dot{z} = \sqrt{\beta^2 c^2 - \dot{x}^2} \approx c\left(1 - \frac{1}{2\gamma^2}\left(1 + \gamma^2 \dot{x}^2/c^2\right)\right)$$

insert for $\dot{x}(t)$ first order solution, then average z velocity is

$$\bar{v}_z = c\left(1 - \frac{1}{2\gamma^2}\left(1 + K^2/2\right)\right) \equiv \bar{\beta}c \tag{8}$$

z velocity oscillates

$$\dot{z}(t) = \bar{\beta}c + \frac{cK^2}{4\gamma^2}\cos(2\omega_u t) \quad \text{with} \quad \omega_u = \bar{\beta}ck_u$$

trajectory in second order

$$x(t) = -\frac{cK}{\gamma\omega_u}\cos(\omega_u t) \qquad z(t) = \bar{\beta}ct + \frac{cK^2}{8\gamma^2\omega_u}\sin(2\omega_u t) \tag{9}$$

1.4 Lorentz transformation to moving coordinate system

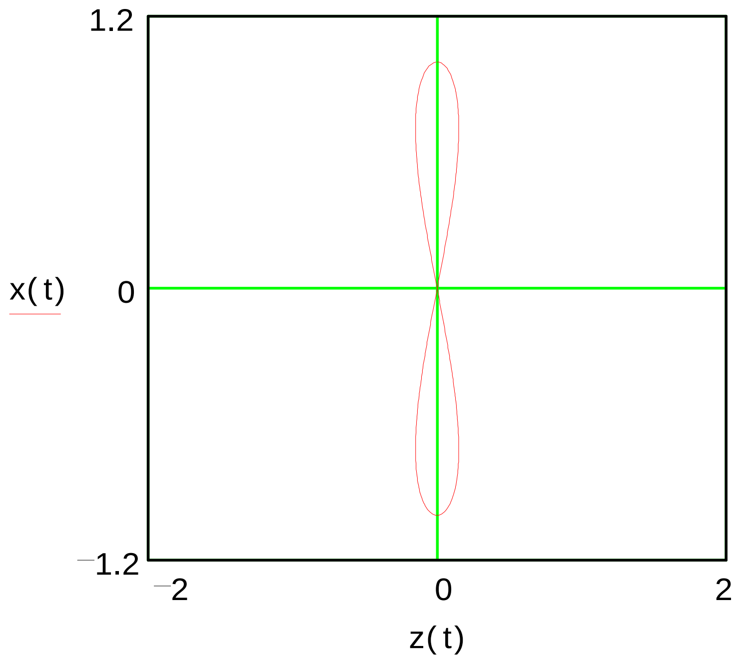
Consider coordinate system (x^*, y^*, z^*) moving with the average z velocity of electron: $v = \bar{v}_z = \bar{\beta}c, \quad \bar{\gamma} \approx \gamma = W/(m_e c^2)$. The Lorentz transformation reads

$$
\begin{aligned}
t^* &= \bar{\gamma}(t - \bar{\beta}z/c) = \bar{\gamma}t(1 - \bar{\beta}^2) \approx t/\gamma \\
x^* &= x = -\frac{cK}{\gamma\omega_u}\cos(\omega_u t) \\
z^* &= \bar{\gamma}(z - \bar{\beta}ct) \approx \frac{cK^2}{8\gamma\omega_u}\sin(2\omega_u t)
\end{aligned}
$$

The orbit in moving system (we introduce $\omega^* = \gamma\omega_u$, then $\omega_u t = \omega^* t^*$) is:

$$
x^*(t^*) = -\frac{cK}{\gamma\omega_u}\cos(\omega^* t^*) \qquad z^*(t^*) = \frac{cK^2}{8\gamma\omega_u}\sin(2\omega^* t^*) \tag{10}
$$

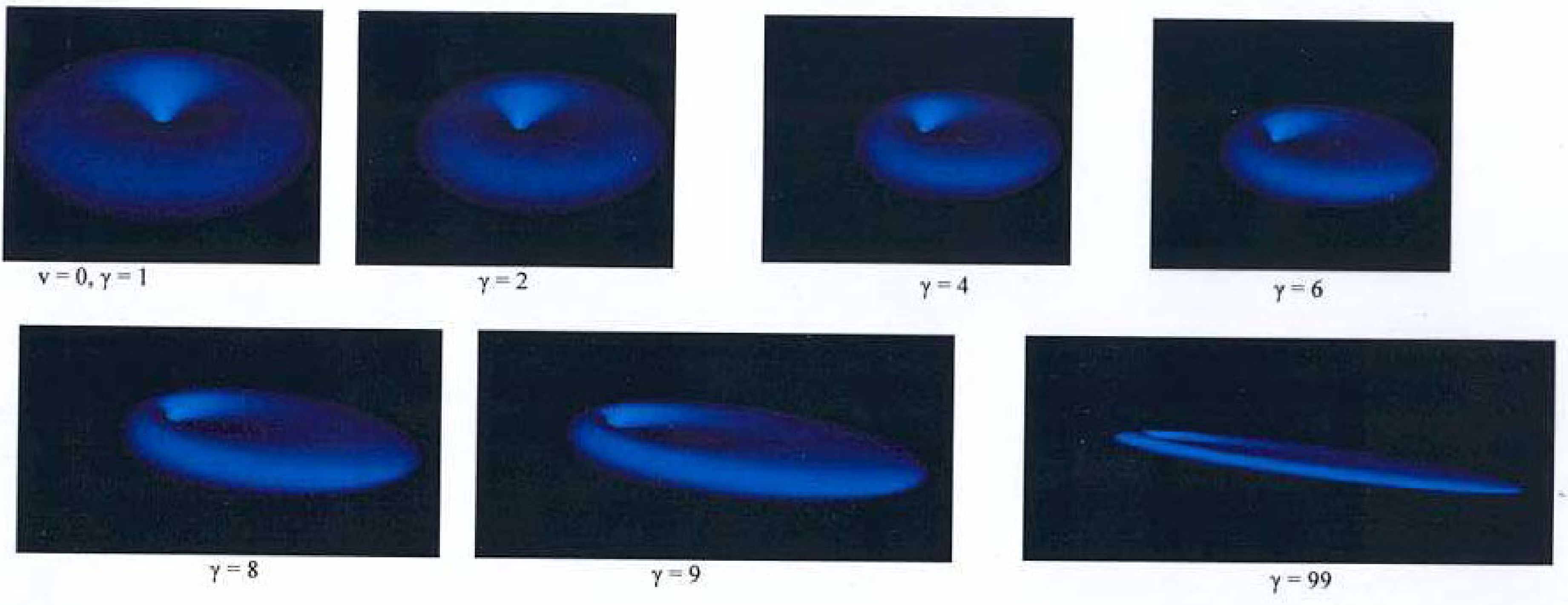
This is mainly a transverse harmonic oscillation with the frequency $\omega^* = \gamma\omega_u$. Superimposed is a small longitudinal oscillation. This will be ignored here, it leads to higher harmonics in the radiation.



oscillation of electron in
co-moving coordinate system

In co-moving system: electron emits dipole radiation:
frequency $\omega^* = \gamma\omega_u$ and wavelength $\lambda^* = \lambda_u/\gamma$
Remember: λ_u is the undulator period, i.e. the distance between two north poles.
Typical value: $\lambda_u = 25$ mm.

1.5 Transformation of radiation into laboratory system



Angular distribution of dipole radiation for a moving dipole (computed by Sven Reiche)

We are interested in the light wavelength as function of the angle θ with respect to the beam axis
Lorentz transformation of the photon energy

$$\hbar\omega^* = \bar{\gamma}\hbar\omega_\ell(1 - \bar{\beta}\cos\theta)$$

$$\Rightarrow \quad \lambda_\ell = \frac{2\pi c}{\omega_\ell} = \frac{2\pi c \bar{\gamma}}{\omega^*}(1 - \bar{\beta}\cos\theta) = \lambda_u(1 - \bar{\beta}\cos\theta)$$

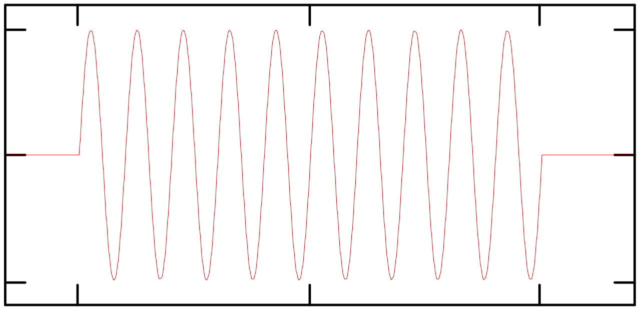
Use $\bar{\gamma} \approx \gamma$, $\bar{\beta} = \left(1 - \frac{1}{2\gamma^2}(1 + K^2/2)\right)$ and $\cos\theta \approx 1 - \theta^2/2$
we obtain for the wavelength of undulator radiation

$$\boxed{\lambda_\ell = \frac{\lambda_u}{2\gamma^2}(1 + K^2/2 + \gamma^2\theta^2)} \tag{11}$$

1.6 Line shape of undulator radiation

An electron passing an undulator with N_u periods produces a wavetrain with N_u oscillations. Electric field of light wave:

$$\Rightarrow \quad E_\ell(t) = \begin{cases} E_0 \mathrm{e}^{i\omega_\ell t} & \text{if} - T/2 < t < T/2 \\ 0 & \text{otherwise} \end{cases}$$



Finite wave train
(here with 10 periods)

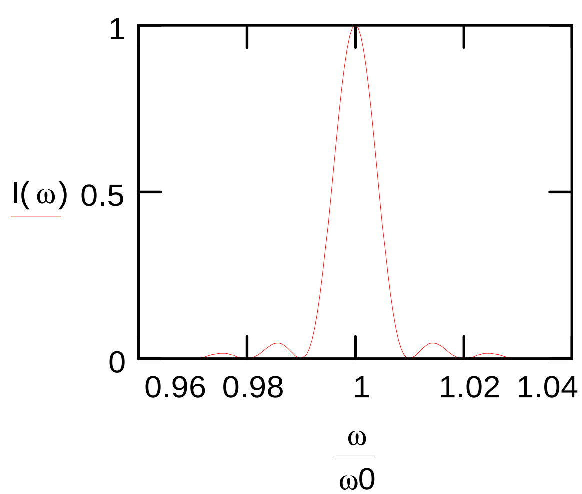
Time duration of wave train $T = N_u \lambda_\ell / c$
The wave train contains a frequency spectrum which is obtained by Fourier transformation

$$\begin{aligned} A(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} E_L(t) e^{-i\omega t} dt = \frac{E_0}{\sqrt{2\pi}} \int_{-T/2}^{+T/2} e^{i(\omega_\ell - \omega)t} dt \\ &= \frac{2E_0}{\sqrt{2\pi}} \cdot \frac{\sin(\Delta\omega T/2)}{\Delta\omega} \quad \text{with} \quad \Delta\omega = \omega - \omega_\ell \end{aligned}$$

The spectral intensity is

$$I(\omega) \propto \left(\frac{\sin \xi}{\xi} \right)^2 \quad \text{with} \quad \xi = \Delta\omega T/2 = \frac{\pi \, N_u(\omega - \omega_\ell)}{\omega_\ell}$$

It has a maximum at $\omega = \omega_\ell$ and a width proportional to $1/N_u$.



Spectral intensity for a wave train with N_u = 100 periods

2. Low-Gain FEL

2.1 Energy transfer from electron to light wave

Consider light wave co-propagating with relativistic electron beam (provided for instance by "seed laser") plane electromagnetic wave polarised in x direction

$$E_x(z,t) = E_0 \cos(k_\ell z - \omega_\ell t) \quad \text{with} \quad k_\ell = \omega_\ell / c$$

Question: can there by continuous energy transfer from electron beam to light wave?
Electron energy is $W = \gamma m_e c^2$, it changes in time dt by

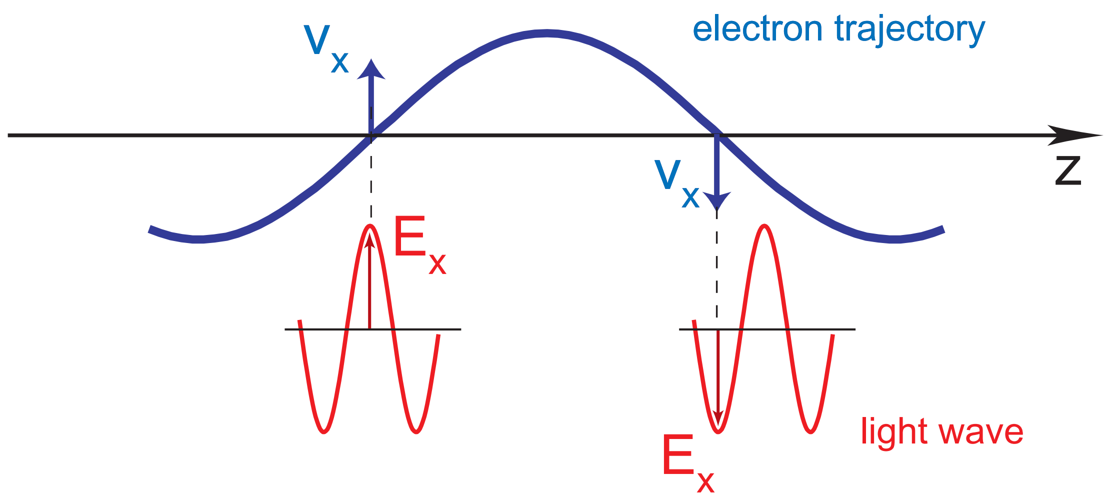
$$dW = \vec{v} \cdot \vec{F} = -e v_x(t) E_x(t) dt$$

The average electron speed in z direction is $\bar{v}_z = c\left(1 - \frac{1}{2\gamma^2}(1 + K^2/2)\right) \; < \; c$

electron and light travel times for half period of undulator:

$$t_{el} = \lambda_u / (2\bar{v}_z), \quad t_{light} = \lambda_u / (2c)$$

Continuous energy transfer happens if $\omega(t_{el} - t_{light}) = \pi$
(Remark: also $3\pi, 5\pi \ldots$ are possible, leading to higher harmonics of the radiation)



Using

$$1/\bar{v}_z - 1/c \approx 1/(2\gamma^2)(1 + K^2/2)$$

ones finds for the light wavelength

$$\boxed{\lambda_\ell = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)} \qquad (12)$$

This is equal to the undulator radition wavelength at $\theta = 0$.

2.2 Quantitative treatment

Energy transfer from electron to light wave ($W = \gamma m_e c^2$ total energy of electron):

$$\begin{aligned} \frac{dW}{dt} &= -ev_x(t)E_x(t) \\ &= -e\frac{cK}{\gamma}\sin(k_u z)E_0\cos(k_\ell z - \omega_\ell t) \\ &= -\frac{ecKE_0}{2\gamma}\left[\sin((k_\ell + k_u)z - \omega_\ell t) - \sin((k_\ell - k_u)z - \omega_\ell t)\right] \end{aligned}$$

The argument of first sine function is called the ponderomotive phase:

$$\psi \equiv (k_\ell + k_u)z - \omega_\ell t \tag{13}$$

One can show that the second sine term oscillates rapidly, it will be neglected here.

$$\Longrightarrow \quad m_e c^2 \frac{d\gamma}{dt} \equiv \frac{dW}{dt} = -\frac{e\,cE_0 K}{2\gamma}\sin\psi \tag{14}$$

If $dW/dt < 0 \quad \Leftrightarrow 0 < \psi < \pi$: energy is transferred from the electron to the radiation field, the light wave is amplified

If we keep the phase ψ constant during the passage through undulator, then we get continuous energy transfer

$$\psi = const \quad \Leftrightarrow \quad \frac{d\psi}{dt} = (k_\ell + k_u)\bar{v}_z - k_\ell c = 0$$

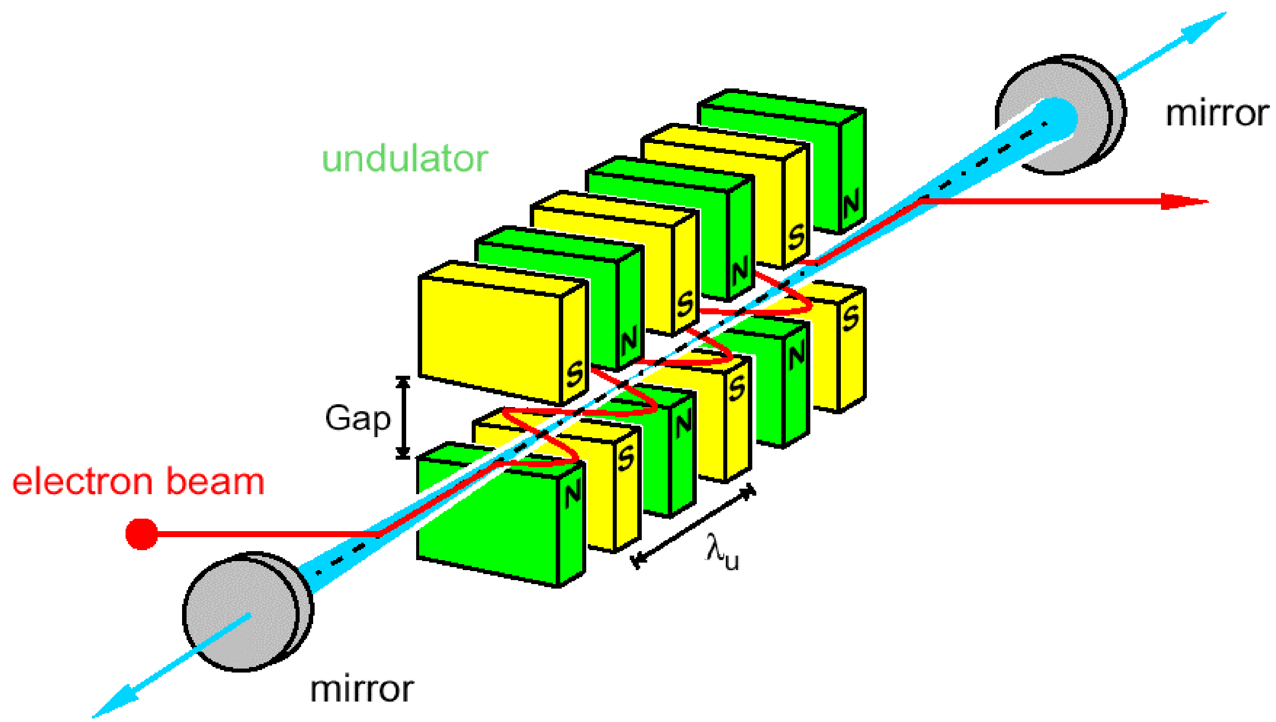
Insertion of $\bar{v}_z$ yields for the light wavelength

$$\boxed{\lambda_\ell = \frac{\lambda_u}{2\gamma^2}\left(1 + \frac{K^2}{2}\right)}$$

Consequence: the condition for resonant energy transfer yields the same light wavelength as in undulator radiation at $\theta = 0$.

2.3 FEL with optical resonator

"Seeding" by external light source with wavelength λ_ℓ



Resonant energy $\gamma_r m_e c^2$ defined by

$$\lambda_\ell = \frac{\lambda_u}{2\gamma_r^2}\left(1 + \frac{K^2}{2}\right) \tag{15}$$

Let electron energy be slightly larger, $\gamma > \gamma_r$

$$0 < \frac{\Delta\gamma}{\gamma_r} = \frac{\gamma - \gamma_r}{\gamma_r} \ll 1$$

Energy deviation $\Delta\gamma$ and ponderomotive phase ψ will both change due to the interaction with the radiation field

Remark: in Low-gain FEL: field amplitude $E_0 \approx const$ during one passage of undulator

The time derivative of the ponderomotive phase is no longer zero for $\gamma > \gamma_r$

$\dot{\psi} = k_u c - k_\ell c \frac{1+K^2/2}{2\gamma^2}$,   subtract $0 = k_u c - k_\ell c \frac{1+K^2/2}{2\gamma_r^2}$ (see eq. (12))

$$\Rightarrow \quad \frac{d\psi}{dt} = \frac{k_\ell c}{2}\left(1 + \frac{K^2}{2}\right)\left(\frac{1}{\gamma_r^2} - \frac{1}{\gamma^2}\right)$$

It follows

$$\frac{d\psi}{dt} \approx 2k_u c \frac{\Delta\gamma}{\gamma_r} \tag{16}$$

The time derivative of gamma is

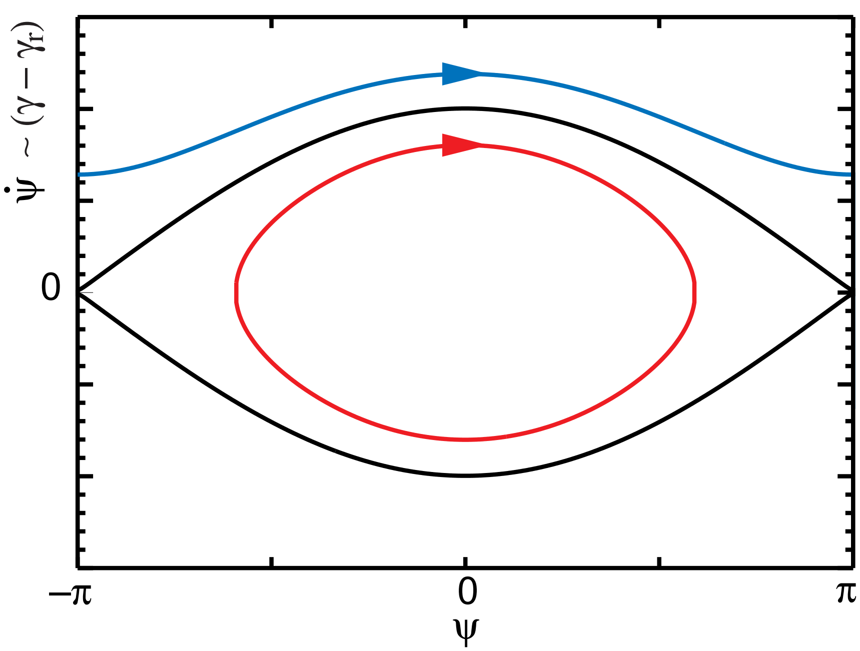
$$\frac{d\gamma}{dt} = -\frac{eE_0K}{2m_e c\gamma_r^2}\sin\psi \tag{17}$$

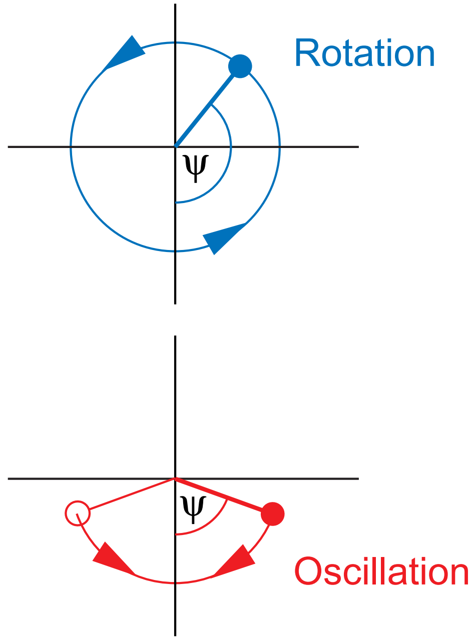
Combination of eq. (16) and (17) yields the "Pendulum Equation" of the low-gain FEL

$$\boxed{\ddot{\psi} + \Omega^2 \sin\psi = 0} \quad \text{with} \quad \Omega^2 = \frac{eE_0Kk_u}{m_e\gamma_r^2} \tag{18}$$

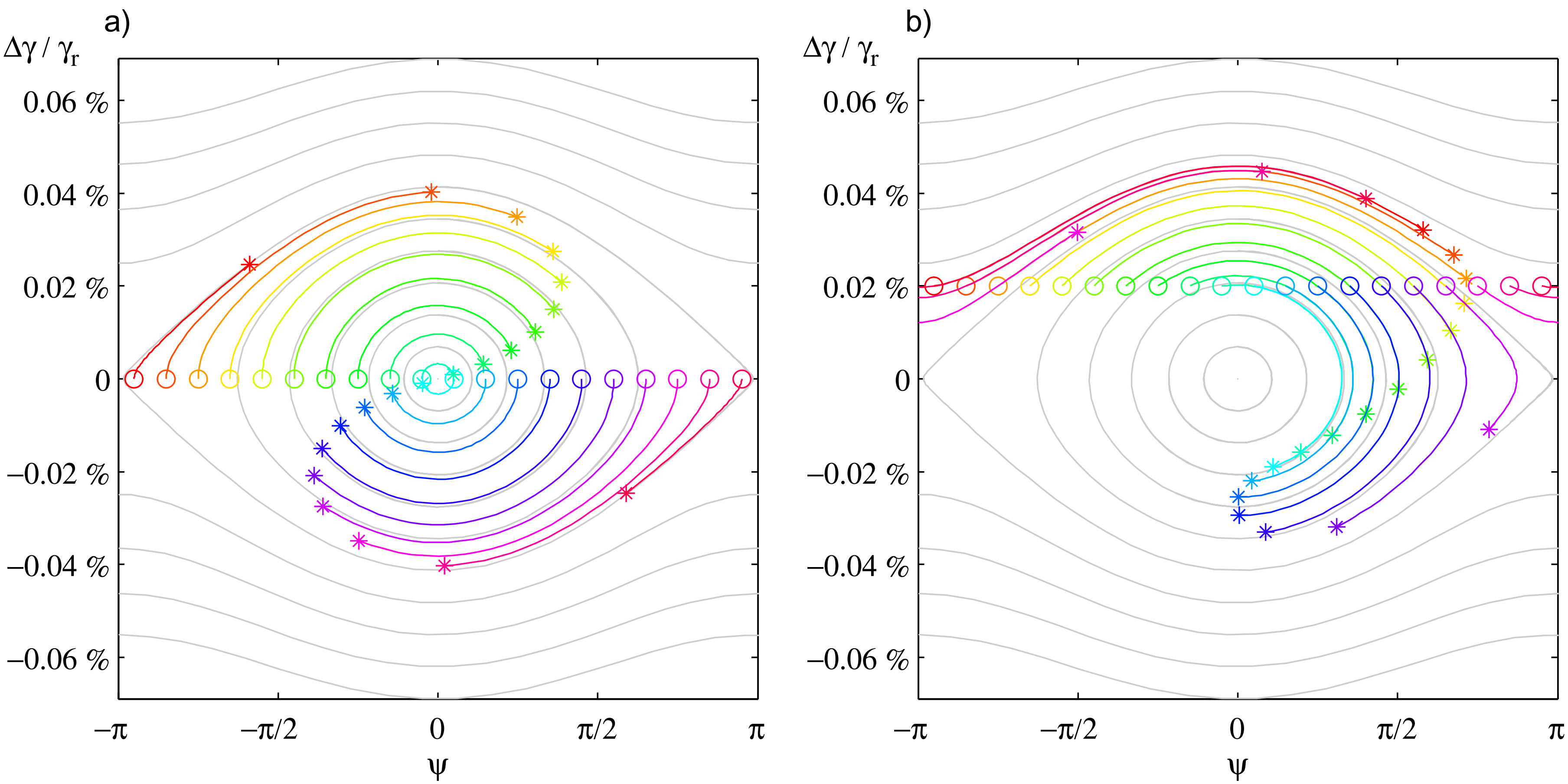
Phase space representation

There is a complete analogy with the motion of a mathematical pendulum. At small amplitude we get a harmonic oscillation. With increasing angular momentum the motion becomes unharmonic. At very large angular momentum one gets a rotation (unbounded motion).





Phase space trajectories for many electrons with $\gamma = \gamma_r$ and with $\gamma > \gamma_r$



In the next chapter we will show that for $\gamma > \gamma_r$ energy is transferred from the electron beam to the light wave.

2.4 Computation of Gain in FEL (Low-Gain Case)

The energy (per unit volume) of the laser field is

$$W_\ell = \frac{\varepsilon_0}{2} E_0^2$$

The energy increase and relative gain caused by 1 electron is

$$\Delta W_\ell = -m_e c^2 \Delta\gamma \qquad G_1 = \frac{\Delta W_\ell}{W_\ell} = -\frac{2 m_e c^2}{\varepsilon_0 E_0^2}\Delta\gamma$$

Considering all electrons in bunch and using eq. (16) the total gain becomes

$$\boxed{G = -\frac{m_e c^2 \gamma_r n_e}{\varepsilon_0 E_0^2 k_u} \cdot < \dot{\psi} >} \tag{19}$$

So we have to compute the quantity $< \dot{\psi} >$.

Phase change in undulator

Multiply pendulum equation $\ddot{\psi} + \Omega^2 \sin\psi = 0$ with $2\dot{\psi}$ and integrate over time

$$\dot{\psi}^2 - 2\Omega^2 \cos\psi = const \quad \Rightarrow \quad \dot{\psi}(t)^2 = \dot{\psi}_0^2 + 2\Omega^2[\cos\psi(t) - \cos\psi_0]$$

From eq. (16)

$$\dot\psi_0 = \dot\psi(0) = 2c\,k_u\,\frac{\gamma_0-\gamma_r}{\gamma_r} = \omega$$

$$\dot\psi(t) = \omega\sqrt{1+2(\Omega/\omega)^2[\cos\psi(t)-\cos\psi_0]} \tag{20}$$

For weak laser field one has $(\Omega/\omega)^2 E_0 \ll 1$, expand square root up to second order $\sqrt{1+x} = 1 + x/2 - x^2/8\ldots$

$$\dot\psi(t) = \omega + (\Omega^2/\omega)[\cos\psi(t)-\cos\psi_0] - \Omega^4/(2\omega^3)[\cos\psi(t)-\cos\psi_0]^2 \tag{21}$$

This equation is solved iteratively

Zeroth order: $\psi_0(t) = \psi_0 = const \quad \dot\psi_0 = \omega$

First order: get phase $\psi(t)$ in first order by integrating $\dot\psi_0$:

$$\psi_1(t) = \psi_0 + \dot\psi_0\cdot t = \psi_0 + \omega\cdot t$$

Insert this in eq. (21) to get $\dot\psi$ in first order

$$\dot\psi_1(t) = \omega + (\Omega^2/\omega)[\cos(\psi_0+\omega t)-\cos\psi_0] \tag{22}$$

According to eq. (19) the gain is obtained by averaging $\dot{\psi}$ over all particles in the bunch, i.e. by averaging over all initial phases ψ_0. This yields

$$< \dot{\psi}_1 >= 0$$

$\Longrightarrow$ FEL gain is zero in first order

Reason: the phase space distribution is almost symmetric

Second order: integrate (22) to get ψ in second order

$$\psi_2(t) = \underbrace{\psi_0 + \omega \cdot t}_{\psi_1(t)} + \underbrace{(\Omega/\omega)^2[\sin(\psi_0 + \omega t) - \sin\psi_0 - \omega t \cos\psi_0]}_{\delta\psi_2(t)} \tag{23}$$

Insert in eq. (21) to get $\dot{\psi}$ in second order

$$\begin{aligned} \dot{\psi}_2(t) &= \omega + (\Omega^2/\omega)[\cos(\psi_0 + \omega t + \delta\psi_2) - \cos\psi_0] \\ &- \Omega^4/(2\omega^3)[\cos(\psi_0 + \omega t + \delta\psi_2) - \cos\psi_0]^2 \end{aligned} \tag{24}$$

$$\delta\psi_2 \ll 1 \quad \Rightarrow \quad \cos(\psi_0 + \omega t + \delta\psi_2) \approx \cos(\psi_0 + \omega t) - \delta\psi_2 \sin(\psi_0 + \omega t)$$

$$\begin{aligned} \cos(\psi_0 + \omega t + \delta\psi_2) &\approx \cos(\psi_0 + \omega t) \\ &- (\Omega/\omega)^2 \sin(\psi_0 + \omega t)[\sin(\psi_0 + \omega t) - \sin\psi_0 - \omega t \cos\psi_0] \end{aligned}$$

Averaging over all start phases ψ_0 yields
$< \cos(\psi_0 + \omega t + \delta\psi_2) >= (1/2)(1 - \cos(\omega t) - \omega t \sin(\omega t))$

$$< \dot{\psi}_2 >= -(\Omega^4/\omega^3)[1 - \cos(\omega t) - (\omega t/2)\sin(\omega t)]$$

Remember $T = N_u \lambda_u / c$ flight time through undulator and $\xi = \Delta\omega T/2$ then

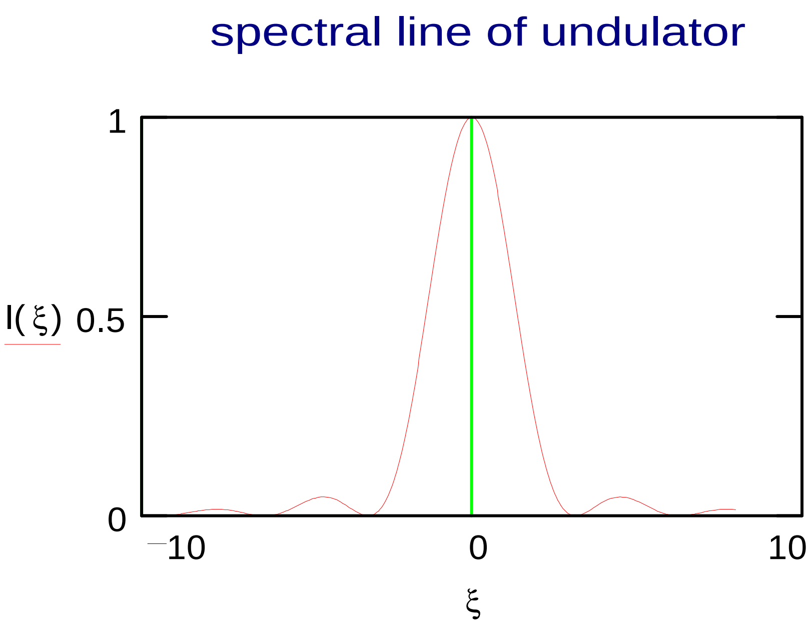
$$\begin{aligned} < \dot{\psi}_2(T) > &= -\frac{\Omega^4}{\omega^3}[1 - \cos(\omega T) - (\omega T/2)\sin(\omega T)] \\ &= -\frac{\Omega^4}{\omega^3}[1 - \cos(2\xi) - \xi\ \sin(2\xi)] \\ &= \frac{N_u^3 \lambda_u^3 \Omega^4}{8c^3} \cdot \frac{d}{d\xi}\left(\frac{\sin\xi}{\xi}\right)^2 \end{aligned}$$

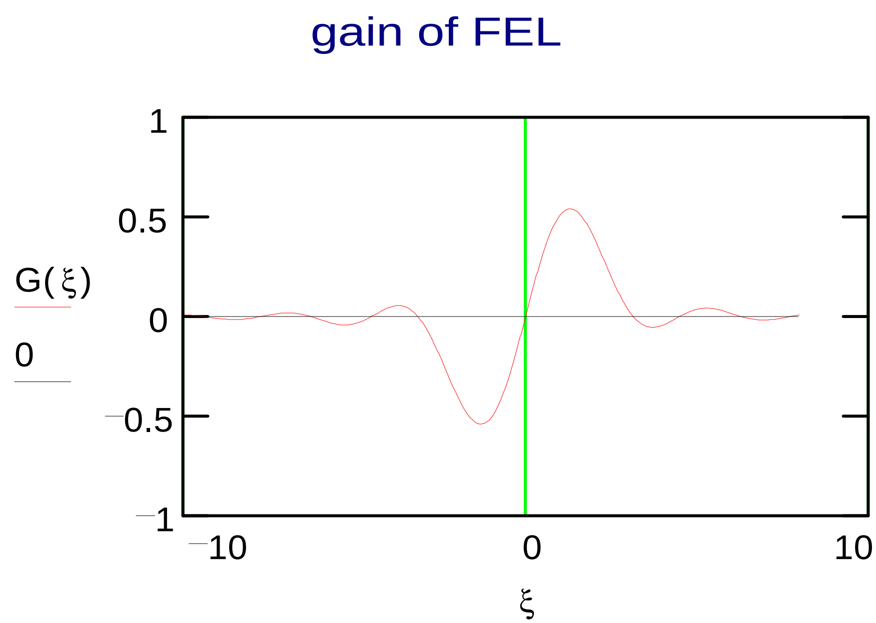
FEL gain function (19) is hence

$$G(\xi) = -\frac{\pi\, e^2 K^2 N_u^3 \lambda_u^2\, n_e}{4\varepsilon_0 m_e c^2 \gamma_r^3} \cdot \frac{d}{d\xi}\left(\frac{\sin^2\xi}{\xi^2}\right) \qquad (25)$$

Madey Theorem

The FEL gain curve is obtained by taking the negative derivative of the line-shape curve of undulator radiation.





$$\xi = \pi N_u \frac{\omega - \omega_\ell}{\omega_\ell}$$