CAS September 2007, Daresbury, UK

Low Emittance Machines

Lecture 3

Emittance Computation and Tuning in Coupled Lattices

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Lectures 1 and 2 summary

In Lecture 1, we:

- discussed the effect of synchrotron radiation on the (linear) motion of particles in storage rings;
- derived expressions for the damping times of the vertical, horizontal and longitudinal emittances;
- discussed the effects of quantum excitation, and derive expressions for the equilibrium horizontal and longitudinal beam emittances in an electron storage ring in terms of the synchrotron radiation integrals.

In Lecture 2, we:

- derived expressions for the natural emittance in different types of lattice (FODO, DBA, multi-bend achromats, TME):

$$\varepsilon_0 \approx F C_q \gamma^2 \theta^3$$

- considered how the natural emittance in an achromat could be reduced by "detuning" from the achromat conditions.

Lecture 3 objectives: emittance computation and tuning

In this lecture, we shall:

- learn how to compute the emittance in a lattice with coupling;
- discuss different sources of vertical emittance, and some of the issues involved in tuning a lattice for ultra-low vertical emittance.

Emittance computation in practice

The formulae for the natural emittance using the synchrotron radiation integrals are useful for ideal (error-free) lattices without betatron coupling.

When coupling is present, things get more cumbersome, though the same principles still apply. Usually, we turn to more numerical methods for computing the emittance in practical cases.

There are at least two common methods used for computing the equilibrium emittances in coupled lattices:

- Chao's method
  - A. Chao, "Evaluation of beam distribution parameters in an electron storage ring," Journal of Applied Physics, 50, 595-598 (1979).
- The "envelope" method
  - e.g. K. Ohmi, K. Hirata, K. Oide, "From the beam-envelope matrix to synchrotron radiation integrals," Phys. Rev. E 49, 751-765 (1994).

In this lecture, we shall discuss only the envelope method.

Back to basics: the sigma matrix and the beam emittances

The sigma matrix is defined as the matrix of second-order moments of the beam distribution:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle & \langle xy \rangle & \langle x p_y \rangle & \langle xz \rangle & \langle x\delta \rangle \\ \langle p_x x \rangle & \langle p_x^2 \rangle & \langle p_x y \rangle & \langle p_x p_y \rangle & \langle p_x z \rangle & \langle p_x \delta \rangle \\ \langle yx \rangle & \langle y p_x \rangle & \langle y^2 \rangle & \langle y p_y \rangle & \langle yz \rangle & \langle y\delta \rangle \\ \langle p_y x \rangle & \langle p_y p_x \rangle & \langle p_y y \rangle & \langle p_y^2 \rangle & \langle p_y z \rangle & \langle p_y \delta \rangle \\ \langle zx \rangle & \langle z p_x \rangle & \langle zy \rangle & \langle z p_y \rangle & \langle z^2 \rangle & \langle z\delta \rangle \\ \langle x\delta \rangle & \langle p_x \delta \rangle & \langle y\delta \rangle & \langle p_y \delta \rangle & \langle z\delta \rangle & \langle \delta^2 \rangle \end{pmatrix}$$

This can be conveniently written as:

$$\Sigma_{ij} = \langle x_i x_j \rangle \qquad \vec{x}^{\mathrm{T}} = \begin{pmatrix} x & p_x & y & p_y & z & \delta \end{pmatrix}$$

where the brackets $\langle\ \rangle$ indicate an average over all particles in the bunch.

In the absence of coupling, the sigma matrix will be block diagonal. We are interested in the more general case, where coupling is present.

Back to basics: the sigma matrix and the beam emittances

The emittances and the lattice functions can be calculated from the sigma matrix, and vice-versa.

Consider the (simpler) case of one degree of freedom. The sigma matrix in this case is:

$$\Sigma = \begin{pmatrix} \langle x^2 \rangle & \langle x p_x \rangle \\ \langle p_x x \rangle & \langle p_x^2 \rangle \end{pmatrix} = \begin{pmatrix} \beta_x & -\alpha_x \\ -\alpha_x & \gamma_x \end{pmatrix} \varepsilon_x$$

Note that given a sigma matrix, we can compute the emittance as follows.

First, define the matrix S:

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Then:

$\boxed{\text{the eigenvalues of } \Sigma \cdot S \text{ are } \pm i\varepsilon_x}$

The proof is left as an exercise for the student!

Back to basics: the sigma matrix and the beam emittances

Now, we can show that, under certain assumptions, the emittance is conserved as a bunch is transported along a beam line as follows.

The linear transformation in phase space coordinates of a particle in the bunch between two points in the beam line can be represented by a matrix M:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ p_x \end{pmatrix}$$

If (for the moment) we neglect radiation and certain other effects, and consider only the Lorentz force on the particles from the external electromagnetic fields, then the transport is *symplectic*.

Physically, this means that the phase-space volume of the bunch is conserved as the bunch moves along the beam line.

Mathematically, this means that M is a *symplectic matrix*, i.e. M satisfies:

$$M^{\mathrm{T}} \cdot S \cdot M = S$$

Now consider how the sigma matrix transforms. Since it is written as the product of the phase-space coordinates averaged over the bunch, we have:

$$\begin{pmatrix} x \\ p_x \end{pmatrix} \mapsto M \cdot \begin{pmatrix} x \\ p_x \end{pmatrix}$$

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathrm{T}}$$

Since S is a constant matrix, we can write:

$$\Sigma \cdot S \mapsto M \cdot \Sigma \cdot M^{\mathrm{T}} \cdot S$$

Then, using the fact that M is symplectic, we have:

$$\Sigma \cdot S \mapsto M \cdot \Sigma \cdot S \cdot M^{-1}$$

But the eigenvalues of $\Sigma \cdot S$ are conserved under a transformation of this type. Therefore, since the eigenvalues are just the bunch emittance, the eigenvalues are conserved under *linear, symplectic* transport.

Back to basics: the sigma matrix and the beam emittances

All the above immediately generalises to three degrees of freedom.

If we define the matrix S in three degrees of freedom by:

$$S = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

In three degrees of freedom, the six eigenvalues of $\Sigma \cdot S$ are just:

$$\pm i\varepsilon_x \qquad \pm i\varepsilon_y \qquad \pm i\varepsilon_z$$

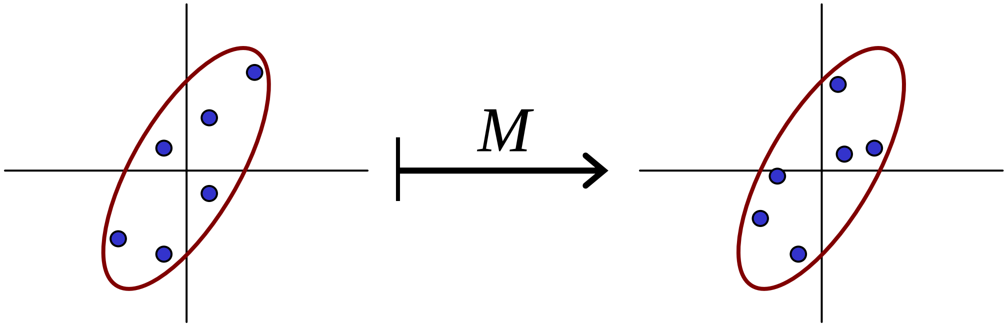
and these quantities are conserved under linear symplectic transport.

Even if, as is generally the case, the sigma matrix is not block-diagonal (i.e. there is coupling present), we can still find three conserved emittances using this method, without any modification.

The matched distribution in a storage ring

If M is a matrix that represents the linear single-turn transformation at some point in a storage ring, then an invariant or "matched" distribution is one that satisfies:

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathrm{T}} = \Sigma$$



This is not sufficient to determine the beam emittances – though this condition will determine the lattice functions (which can be obtained from the *eigenvectors* of $\Sigma \cdot S$).

In other words, the matched distribution condition determines the *shape* of the bunch, but not the size of the bunch. This makes sense: after all, in a proton storage ring, we can have a matched beam with any emittance.

However, in an electron storage ring, we know that radiation effects will damp the emittances to some equilibrium values.

How can we apply the concept of a matched distribution to find the equilibrium emittance values?

The matched distribution in a storage ring

In an electron storage ring, we must make two modifications to the single-turn transformation to account for radiation effects:

1. The matrix M will no longer be symplectic: this accounts for radiation damping.
2. As well as first-order terms in the transformation (represented by the matrix M), there will be zeroth-order terms: these will turn out to correspond to the quantum excitation.

The condition for a matched distribution should then be written:

$$\Sigma = M \cdot \Sigma \cdot M^{\mathrm{T}} + D$$

where M and D are constant (non-symplectic) matrices that represent the first-order and zeroth-order terms in the single-turn transformation, respectively.

This equation is sufficient to determine the sigma matrix uniquely – in other words, using just this equation (with known M and D) we can find the bunch emittances and the matched lattice functions.

The envelope method

The envelope method for finding the equilibrium emittances in a storage ring consists of three steps:

1. Find the first-order terms M and zeroth-order terms D in the single-turn transformation:

$$\Sigma \mapsto M \cdot \Sigma \cdot M^{\mathrm{T}} + D$$

2. Use the matching condition:

$$\Sigma = M \cdot \Sigma \cdot M^{\mathrm{T}} + D$$

   to determine the sigma matrix.

3. Find the equilibrium emittances from the eigenvalues of $\Sigma \cdot S$

Note: strictly speaking, since M is not symplectic, the emittances are not conserved as the bunch moves around the ring. Therefore, we may expect to find a different emittance at each point around the ring. However, if radiation effects are fairly small, the variations in the emittances will be small.

The envelope method: finding the transformation matrices M and D

As an illustration of the transformation matrices M and D, we shall consider a thin "slice" of a dipole.

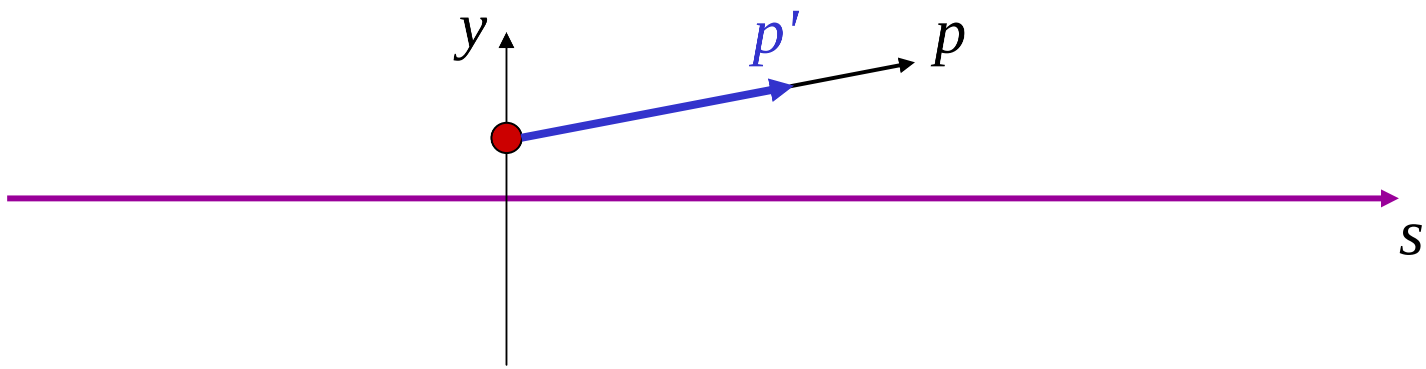
The details of the calculation are given in Appendix A; in the main part of the lecture, we just indicate the principles, and state the results.

The thin slice of dipole is an important case:

- in most storage rings, radiation effects are only significant in dipoles;
- "complete" dipoles can be constructed by concatenating the maps for a number of slices.

Once we have the map for a thin slice of a dipole, we simply need to concatenate the maps for all the elements in the ring, to construct the map for a complete turn starting at any given point.

The transformation matrices M and D in a thin slice of a dipole



Recall (from Lecture 1) the transformation of the phase space variables in the emission of radiation carrying momentum dp is:

$$x \mapsto x \qquad y \mapsto y \qquad z \mapsto z$$

$$p_x \mapsto \left(1 - \frac{dp}{P_0}\right) p_x \qquad p_y \mapsto \left(1 - \frac{dp}{P_0}\right) p_y \qquad \delta \mapsto \delta - \frac{dp}{P_0}$$

where P_0 is the reference momentum. In general, dp is a function of the coordinates.

To find the transformation matrices M and D, we find an explicit expression for dp/P_0, and then write down the above transformations to first order…

The transformation matrices M and D in a thin slice of a dipole

For a thin slice of dipole, of length ds, the radiation effects can be represented by the matrices (see Appendix A):

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2}+2k_1\right)\frac{E_0^3}{\rho}ds & 0 & 0 & 0 & 0 & 1-2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_q\gamma^2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^3}ds \end{pmatrix}$$

Concatenating the transformations

To construct the full transformation for a dipole (or for an entire lattice) we need to concatenate the maps.

It is straightforward to do this numerically using a computer. We only need to be careful about how we handle the D matrices.

For example, given the sigma matrix at a location s_0, we find the sigma matrix at a location $s_1 = s_0 + ds$ from:

$$\Sigma(s_1) = M(s_1; s_0) \cdot \Sigma(s_0) \cdot M^{\mathrm{T}}(s_1; s_0) + D(s_1; s_0)$$

Then the sigma matrix at s_2 is given by:

$$\Sigma(s_2) = M(s_2; s_1) \cdot \Sigma(s_1) \cdot M^{\mathrm{T}}(s_2; s_1) + D(s_2; s_1)$$

$$= M(s_2; s_0) \cdot \Sigma(s_0) \cdot M^{\mathrm{T}}(s_2; s_0) + M(s_2; s_1) \cdot D(s_1; s_0) \cdot M^{\mathrm{T}}(s_2; s_1) + D(s_2; s_1)$$

Hence:

$$M(s_2; s_0) = M(s_2; s_1) \cdot M(s_1; s_0)$$

$$D(s_2; s_0) = M(s_2; s_1) \cdot D(s_1; s_0) \cdot M^{\mathrm{T}}(s_2; s_1) + D(s_2; s_1)$$

Concatenating the transformations

Continuing the process, we find we can write:

$$M(s_n;s_0) = M(s_n;s_{n-1}) \cdot M(s_{n-1};s_{n-2}) \cdot \ldots \cdot M(s_1;s_0)$$

$$D(s_n;s_0) = \sum_{i=1}^{n} M(s_n;s_i) \cdot D(s_i;s_{i-1}) \cdot M^{\mathrm{T}}(s_n;s_i)$$

Using a computer, it's actually not too difficult to concatenate the maps. In a dipole, we have to remember to "interleave" the radiation maps with the usual symplectic transport map for thin slice of dipole.

Solving the matching condition

Having obtained the maps M and D for an entire ring, we now need to solve the equation:

$$\Sigma = M \cdot \Sigma \cdot M^{\mathrm{T}} + D$$

to find the sigma matrix for the equilibrium distribution.

To solve this equation for the sigma matrix, we make use of the eigenvectors U and the (diagonal matrix of) eigenvalues Λ of M:

$$M \cdot U = \Lambda \cdot U$$

Defining $\tilde{\Sigma}$ and $\tilde{D}$ by:

$$\Sigma = U \cdot \tilde{\Sigma} \cdot U^{\mathrm{T}}$$

$$D = U \cdot \tilde{D} \cdot U^{\mathrm{T}}$$

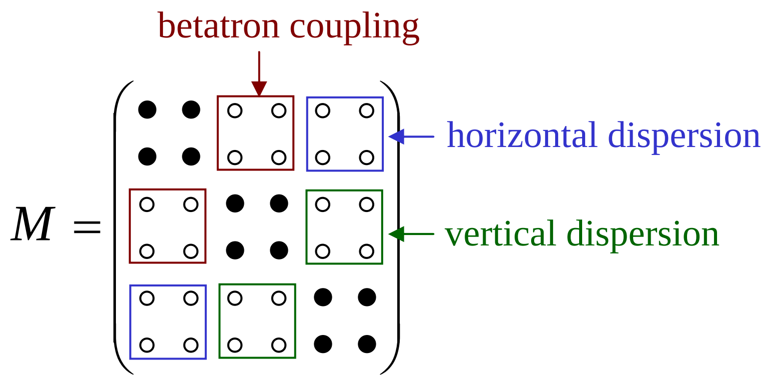
the solution for the sigma matrix can be written:

$$\tilde{\Sigma}_{ij} = \frac{\tilde{D}_{ij}}{1 - \Lambda_i \Lambda_j}$$

Vertical emittance can be generated by:

- Coupling between the vertical and longitudinal planes in regions where radiation is emitted; i.e. by vertical dispersion *in dipoles*.
- Coupling between the vertical and horizontal planes in regions where radiation is emitted; i.e. by betatron coupling *in dipoles*.

Here, we need to be very careful in how we use the word "coupling". In this context, coupling means the presence of non-zero off-block-diagonal components in the single-turn matrix, M.



Full characterisation of the coupling requires complete specification of all these off-block-diagonal components.

It is quite possible to have coupling in a storage ring, and not generate any vertical emittance…

Comments on the envelope method: (2) Applications

Numerical computational procedures (such as the envelope method) for finding the equilibrium beam distribution in a storage ring are important because they provide a means to calculate the equilibrium emittances in complex, coupled lattices.

Note that a variety of non-symplectic effects (including, for example, intrabeam scattering) can be included in the computation: not just synchrotron radiation.

Often, coupling comes from magnet alignment errors (as we shall discuss next), which are not completely known in an operating machine.

However, at the design stage, it is important to characterise the *sensitivity* of a lattice to magnet alignment errors, particularly regarding the vertical emittance.

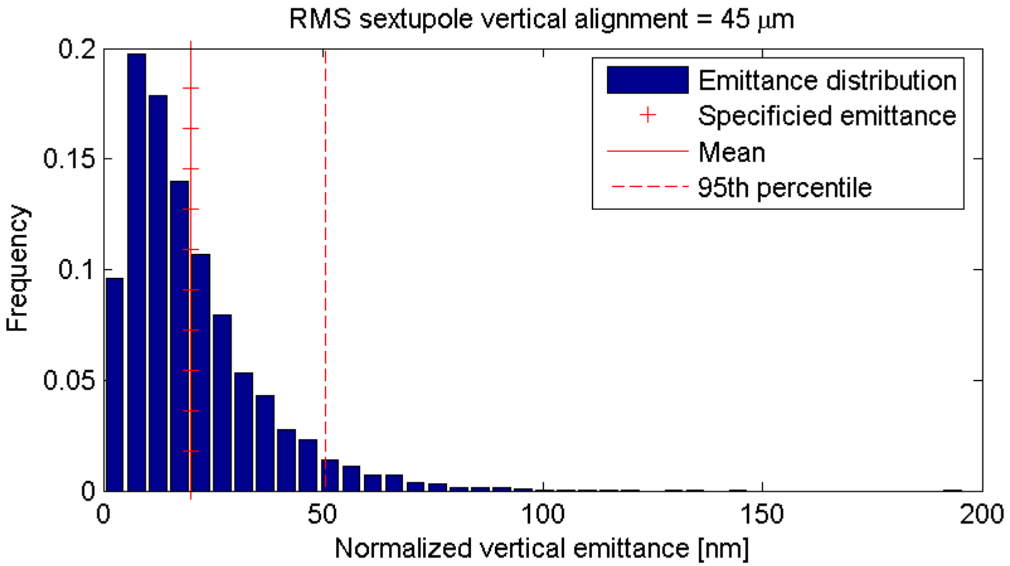
Having a means to compute the beam emittances in a storage ring with coupling errors present allows us to simulate the effects of various types and sizes of alignment error – and (we hope) to optimise the lattice design to minimise the sensitivity to likely errors.

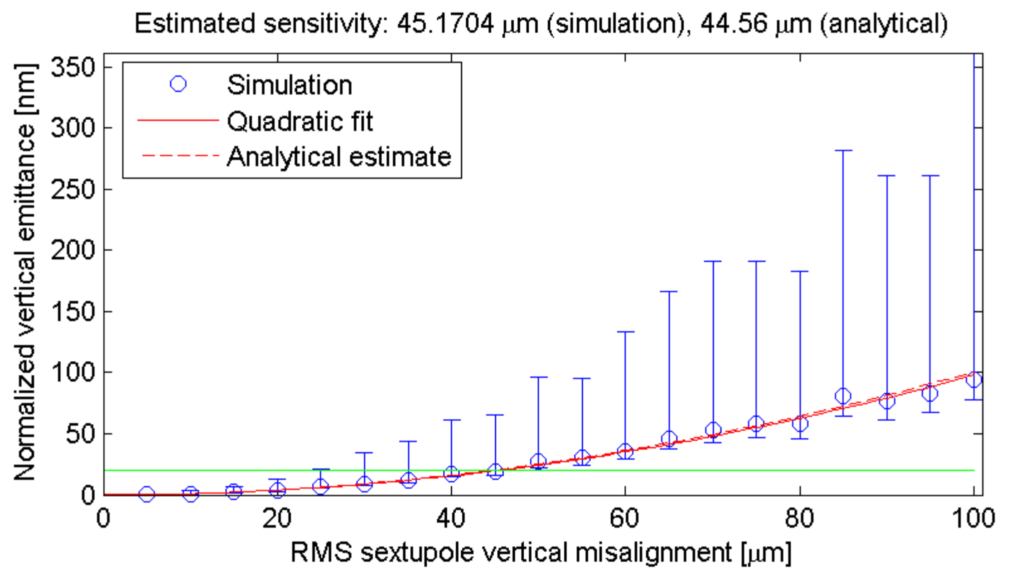
Example: ε_y sensitivity to sextupole alignment in the ILC damping rings

To achieve the machine luminosity goals, the damping rings of the International Linear Collider will need to produce beams with 2 pm vertical emittance.

2 pm is more than a factor of two smaller than the smallest vertical emittance so far demonstrated in an electron storage ring.

Sensitivity to coupling errors must be well-understood, and effective techniques for correcting or compensating for coupling errors will have to be applied.

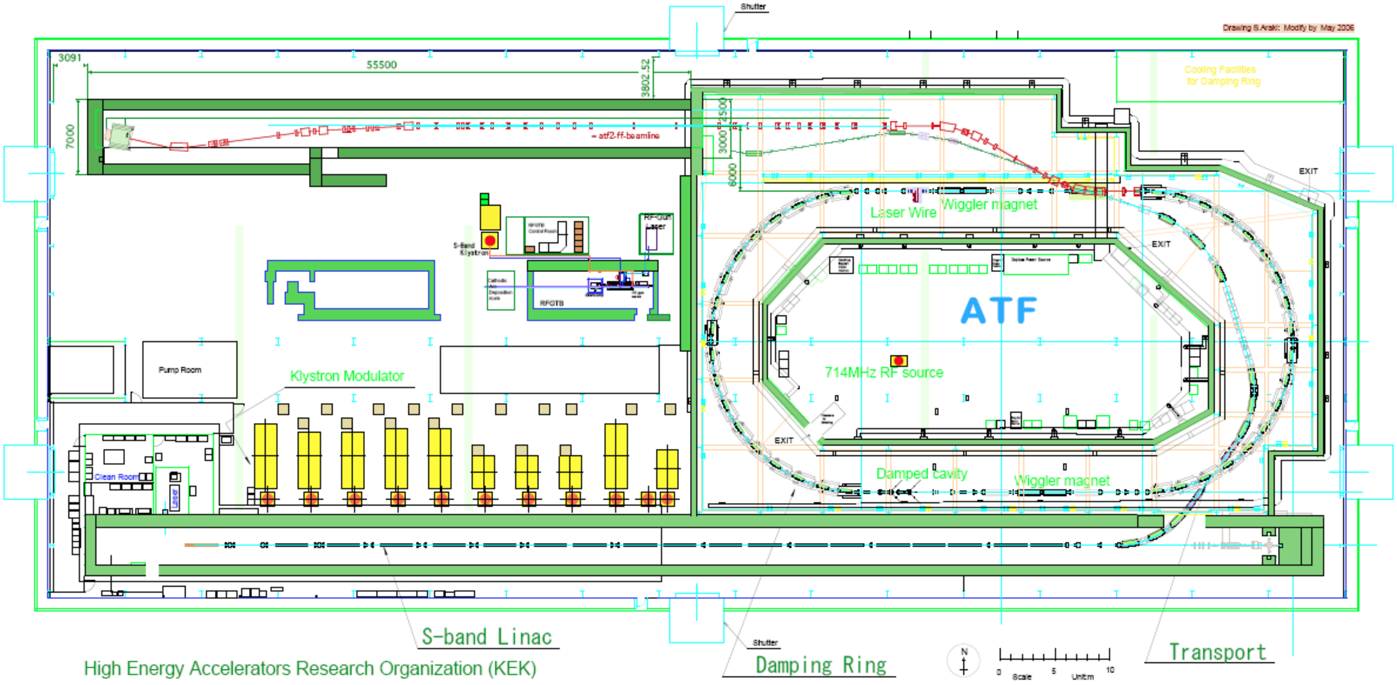




Case study: the KEK-ATF prototype linear collider damping ring

Vertical emittances in light sources are typically of order several 10's of pm, corresponding (usually) to 1% of the horizontal emittance.

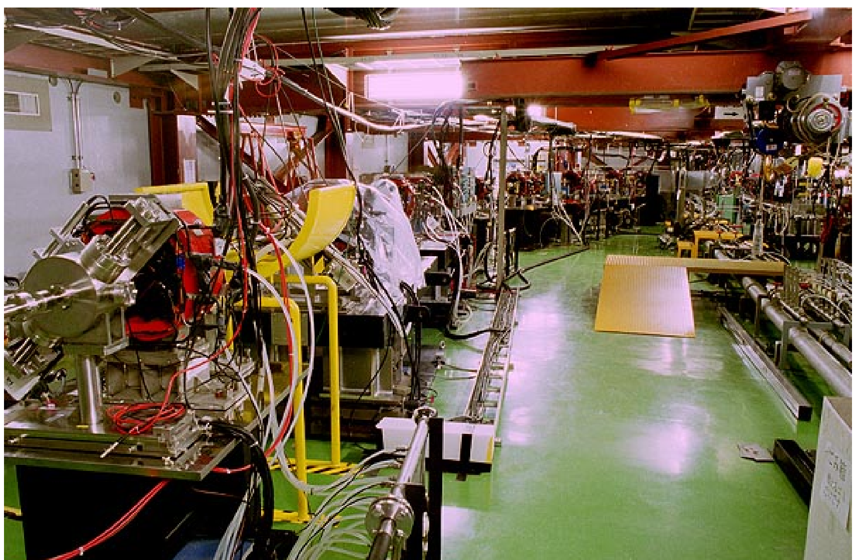
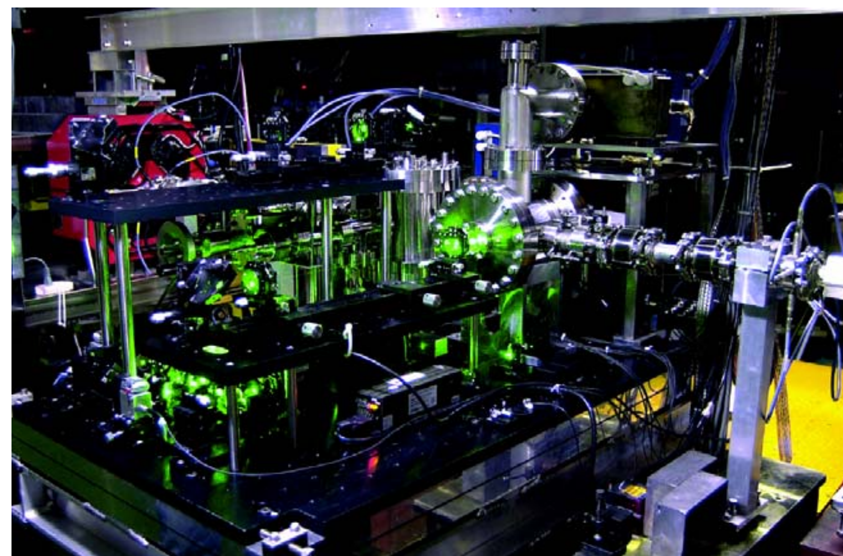
The KEK-ATF presently holds the record for the smallest vertical emittance achieved in an accelerator: 4.5 pm.



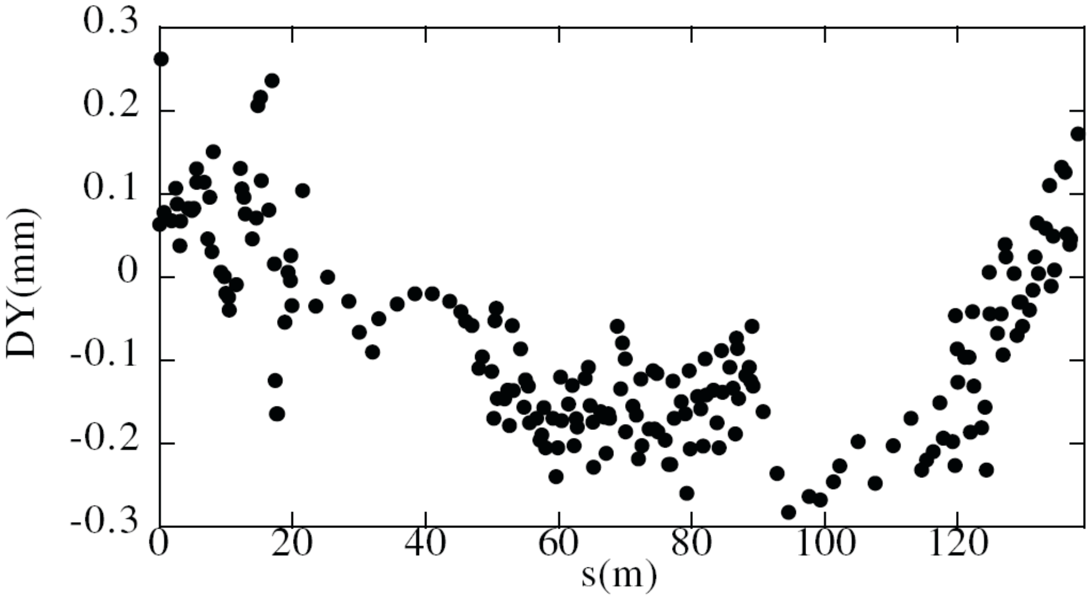
The main components are a 1.28 GeV S-band linac, a 1.28 GeV storage ring, and an extraction line. The extraction line is presently being extended (ATF2) to provide a test facility for linear collider beam delivery systems.

(Clockwise) ATF injector; damping ring; laser wire; extraction line



Achieving a low emittance starts with achieving good magnet alignment

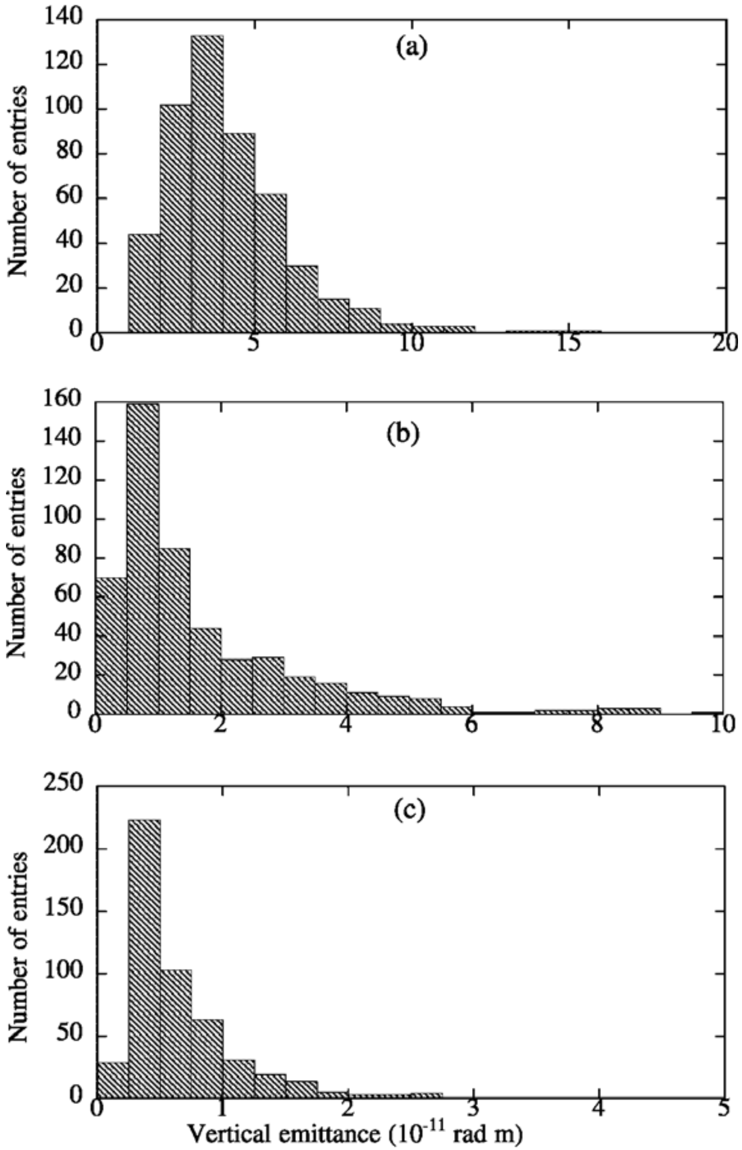


Vertical alignment of the magnets is critical:

- Vertical alignment errors on the quadrupoles generates vertical orbit distortion and vertical dispersion.
- Vertical orbit offset in the sextupoles (by orbit distortion or sextupole alignment errors) generates vertical dispersion and betatron coupling.

Rotational (tilt) alignment of the quadrupoles is also critical, to avoid skew quadrupole components that will generate coupling.

Tuning for low emittance proceeds in stages



Simulations of emittance tuning in the ATF damping ring.

Top: after correction of vertical closed orbit distortion.

Middle: after further correction (combined vertical closed orbit and vertical dispersion).

Bottom: after correction of betatron coupling.

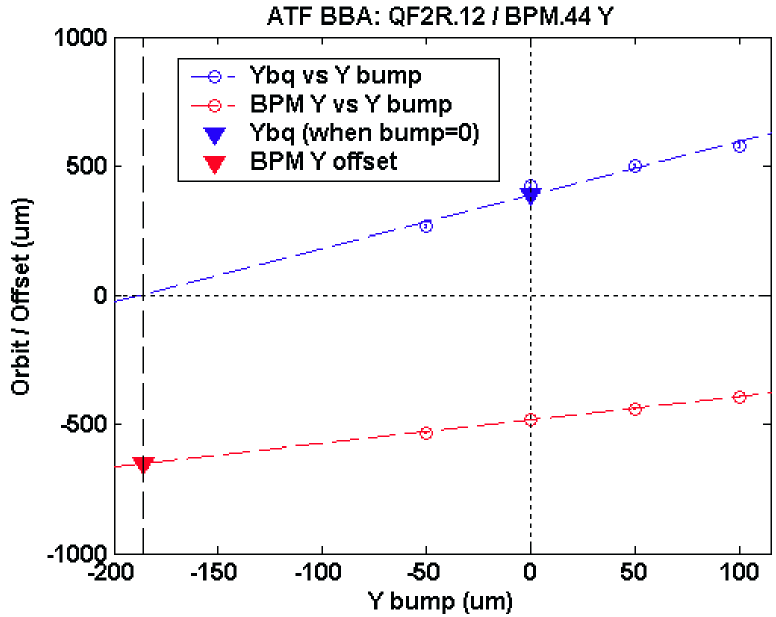
Each plot shows a histogram of 500 cases with random errors:

Additional magnet offset	30 μm
Magnet rotation	300 μrad
BPM offset	300 μm
BPM rotation	20 mrad

Errors in the diagnostics dominate over errors in the magnets!

K.Kubo, "Simulation Study of Low Emittance Tuning of the Accelerator Test Facility Damping Ring at KEK," Phys.Rev.ST-AB 6, 092801 (2003).

Understanding the BPM offsets is essential for low emittance tuning



Beam-based alignment in the KEK-ATF damping ring:

Red line: vertical BPM reading as a function of vertical bump amplitude.

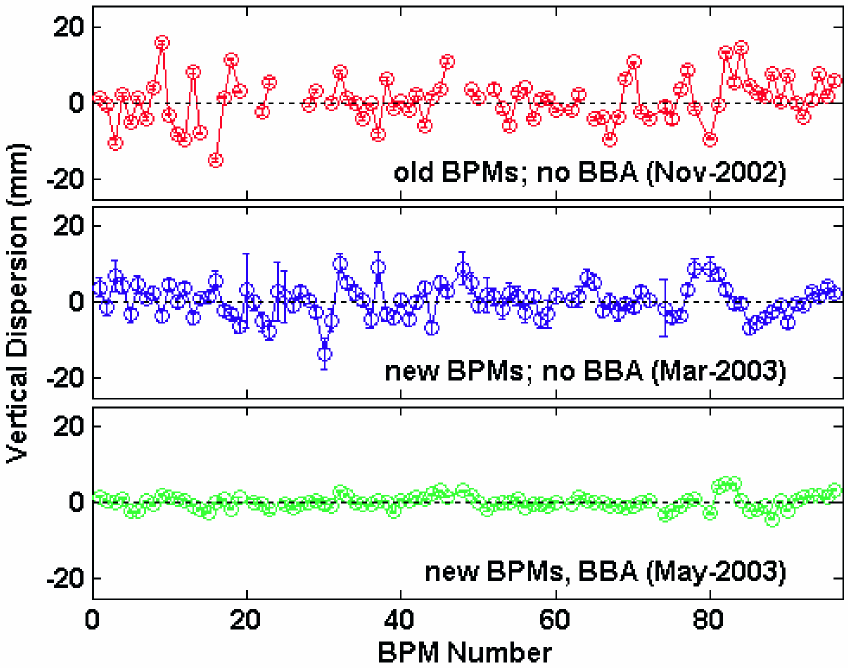
Blue line: beam-quad vertical offset as a function of bump amplitude.

The beam-quad offset at each bump amplitude is determined by changing the strength of the quadrupole, and measuring the resulting change in the closed orbit.

In this case, when the beam is centered in the quadrupole, the adjacent BPM reads a beam position of -650 μm.

M.D.Woodley et al, "Beam Based Alignment at the KEK ATF Damping Ring," Proc. EPAC 2004.

Steering to quadrupole centres helps reduce vertical dispersion in the ATF



rms η_y = 5.8 mm

rms η_y = 4.7 mm

rms η_y = 1.7 mm

If the vertical dispersion is generated by random errors, the contribution of the vertical dispersion to the vertical emittance may be estimated from (see Appendix B):

$$\varepsilon_y \approx 2 j_z \frac{\langle \eta_y^2 \rangle}{\langle \beta_y \rangle} \sigma_\delta^2 \qquad \text{i.e.} \qquad \varepsilon_y \propto \langle \eta_y^2 \rangle$$

With rms vertical dispersion 1.7 mm, the contribution of the vertical dispersion to the vertical emittance in the ATF is of order 0.5 pm, which is much less than 4.5 pm…

Correction of betatron coupling

In the KEK-ATF, after correcting the vertical closed orbit distortion and vertical dispersion, most of the remaining emittance is generated by betatron coupling.

Correction of betatron coupling is achieved using skew quadrupoles distributed around the ring. There are various techniques used to determine the optimum settings for the skew quadrupoles.

Generally, one measures the changes in the vertical closed orbit in response to changes in horizontal steering magnet strengths, and vice-versa. Analysis of the data yields settings for the skew quadrupoles to minimise the cross-plane orbit response.

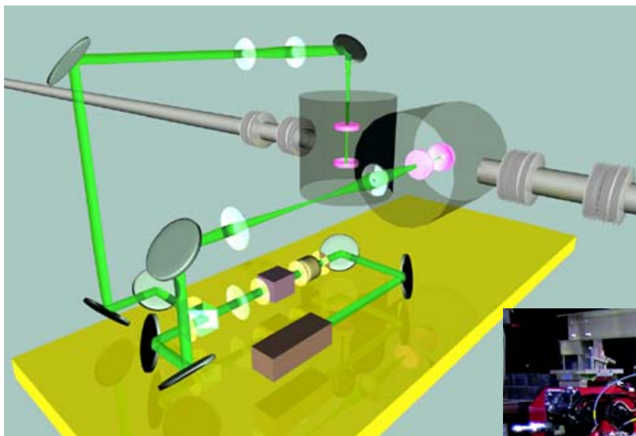
The effectiveness of this technique depends on rotational alignment accuracy of the steering magnets and the BPMs. If this alignment is not precise, one can "measure" cross-plane orbit response even when no coupling is in fact present.

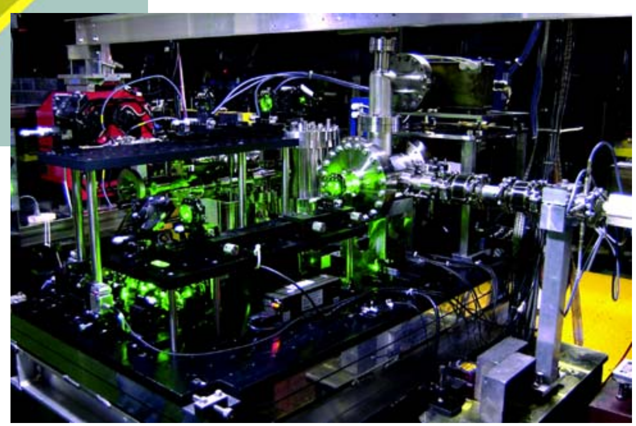
Measurement of picometer emittances

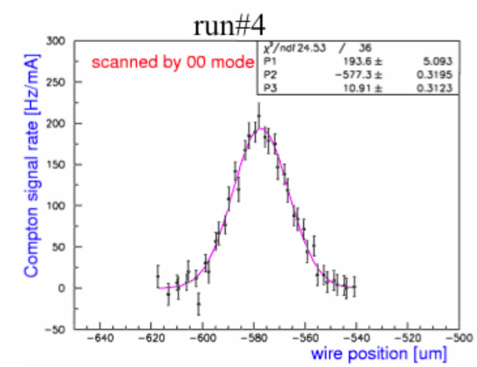
Beams with vertical emittance of a few picometers will generally have vertical beam size of a few microns.

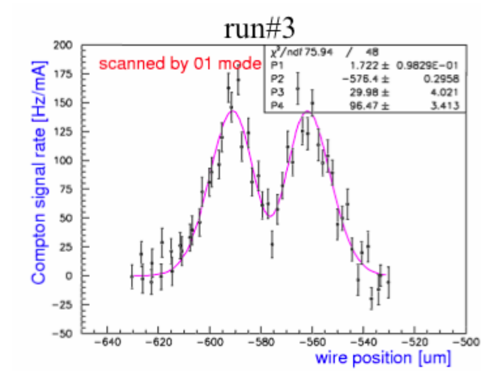
This presents a challenge for the instrumentation used to measure beam sizes and emittances. However, there are various types of instrument, such as X-ray synchrotron radiation monitors, that provide the necessary resolution.

At the KEK-ATF, a laser wire is used to measure the vertical beam size.

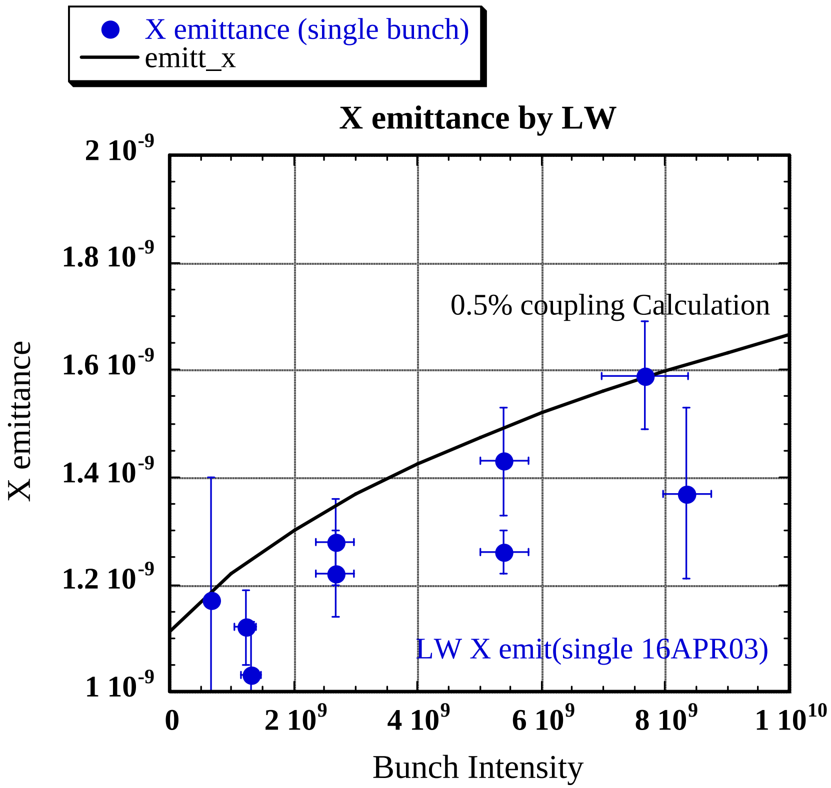


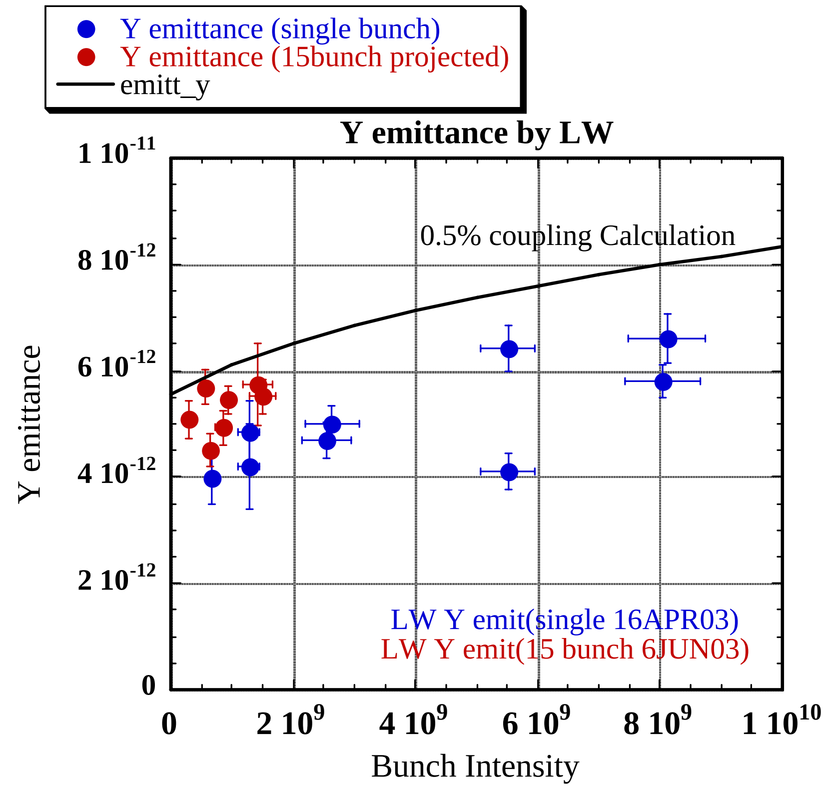






ATF beam shows emittance growth from IBS with ε_y ~ 5 pm





VOLUME 92, NUMBER 5 PHYSICAL REVIEW LETTERS week ending 6 FEBRUARY 2004

Achievement of Ultralow Emittance Beam in the Accelerator Test Facility Damping Ring

Y. Honda,[1] K. Kubo,[2] S. Anderson,[3] S. Araki,[2] K. Bane,[3] A. Brachmann,[3] J. Frisch,[3] M. Fukuda,[6] K. Hasegawa,[14] H. Hayano,[2] L. Hendrickson,[3] Y. Higashi,[2] T. Higo,[2] K. Hirano,[13] T. Hirose,[15] K. Iida,[12] T. Imai,[9] Y. Inoue,[7] P. Karataev,[6] M. Kuriki,[2] R. Kuroda,[8] S. Kuroda,[2] X. Luo,[11] D. McCormick,[3] M. Matsuda,[10] T. Muto,[2] K. Nakajima,[2] Takashi Naito,[2] J. Nelson,[3] M. Nomura,[13] A. Ohashi,[6] T. Omori,[2] T. Okugi,[2] M. Ross,[3] H. Sakai,[12] I. Sakai,[13] N. Sasao,[1] S. Smith,[3] Toshikazu Suzuki,[2] M. Takano,[13] T. Taniguchi,[2] N. Terunuma,[2] J. Turner,[3] N. Toge,[2] J. Urakawa,[2] V. Vogel,[2] M. Woodley,[3] A. Wolski,[4] I. Yamazaki,[8] Yoshio Yamazaki,[2] G. Yocky,[3] A. Young,[3] and F. Zimmermann[5]

Summary 1: computing equilibrium emittances

In a lattice with coupling errors:

- the analytical formulae using the synchrotron radiation integrals are not the most useful for calculating the equilibrium emittances;
- various methods do exist for computing the equilibrium beam distributions, from which the emittances can be found.

The envelope method is based on computing the zeroth-order (D) and first-order (M) terms in the single-turn transfer map (including the effects of radiation, and – possibly – other non-symplectic processes), and then finding the matched distribution:

$$\Sigma = M \cdot \Sigma \cdot M^{\mathrm{T}} + D$$

The emittances are the eigenvalues of $\Sigma \cdot S$.

Summary 2: ultra-low vertical emittance in electron storage rings

Vertical emittance in synchrotron storage rings is generated by coupling with the longitudinal motion (vertical dispersion) and horizontal motion (betatron coupling).

Generally, synchrotron light sources operate with vertical emittances of some 10's of picometers, corresponding to ~ 1% of the horizontal emittance.

Some applications – notably linear colliders – demand much smaller vertical emittances, of order 2 pm. Issues involved in achieving such emittances include:

- sensitivity of the lattice to a range of coupling errors (including vertical alignment of sextupoles, and rotational alignment of quadrupoles);
- accuracy and precision of magnet alignment;
- performance of instrumentation, particularly BPMs and beam-size monitors;
- use of a range of beam-based techniques for characterising and compensating for the coupling errors.

At ultra-low vertical emittances, collective effects which are not normally relevant in ultra-relativistic beams can start to impact performance. Examples include intrabeam scattering and space charge,

Appendices

Appendix A: the transformation matrices M and D in a thin slice of a dipole

For an ultra-relativistic particle, the momentum lost through radiation can be expressed in terms of the synchrotron radiation power, P_γ (energy loss per unit time):

$$\frac{dp}{P_0} \approx \frac{P_\gamma}{E_0} dt \approx \frac{P_\gamma}{E_0} \left(1 + \frac{x}{\rho}\right) \frac{ds}{c}$$

where ρ is the radius of curvature of the reference trajectory.

Recall (from Lecture 1) that the radiation power from a particle of charge e and energy E in a magnetic field B is given by:

$$P_\gamma = \frac{C_\gamma}{2\pi} c^3 e^2 B^2 E^2$$

The dipole may have a quadrupole gradient: $B = B_0 + B_1 x$

The particle may have some energy deviation: $E = E_0(1 + \delta)$

Substituting these expressions, we find (after some manipulation)…

Appendix A: the transformation matrices M and D in a thin slice of a dipole

$$P_\gamma = c\frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\frac{x}{\rho}\right)(1+\delta)^2 E_0^4$$

where k_1 is the normalised quadrupole gradient in the dipole:

$$k_1 = \frac{e}{P_0}B_1$$

Hence, the normalised momentum loss may be written:

$$\frac{dp}{P_0} \approx \frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\frac{x}{\rho}\right)\left(1 + \frac{x}{\rho}\right)(1+\delta)^2 E_0^3 ds$$

Expanding to first order in the phase space variables:

$$\frac{dp}{P_0} \approx \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds + \frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\right)\frac{E_0^3}{\rho}x \cdot ds + 2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}\delta \cdot ds + O(x^2) + O(\delta^2)$$

Appendix A: the transformation matrices M and D in a thin slice of a dipole

Given the expression for dp/P_0 on the previous slide, the transformations of the phase space variables become:

$$x \mapsto x \qquad p_x \mapsto \left(1 - \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds\right)p_x$$

$$y \mapsto y \qquad p_y \mapsto \left(1 - \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds\right)p_y$$

$$z \mapsto z \qquad \delta \mapsto \left(1 - 2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds\right)\delta - \frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2} + 2k_1\right)\frac{E_0^3}{\rho^2}x \cdot ds - \frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds$$

The first-order terms give us components in M.

There is already a zeroth-order term that will contribute to D, in the (6,6) component, but we have not yet taken proper account of the quantum nature of the radiation…

Note that the zeroth-order term in the map is going to be found from:

$$D_{66} = \left\langle \left(\frac{dp}{P_0} \right)^2 \right\rangle \approx \frac{\langle u^2 \rangle}{E_0^2}$$

where $\langle u^2 \rangle$ is the mean square of the photon energy.

We use the results (quoted in Lecture 1):

$$\dot{N} \langle u \rangle = P_\gamma$$

$$\dot{N} \langle u^2 \rangle = 2 C_q \gamma^2 E_0 \frac{P_\gamma}{\rho}$$

to find that, to zeroth-order in the phase space variables:

$$\left\langle \left(\frac{dp}{P_0} \right)^2 \right\rangle \approx 2 C_q \gamma^2 \frac{C_\gamma}{2\pi} \frac{E_0^3}{\rho^3} ds$$

Note that this term is first-order in ds, whereas the first contribution we found is second-order in ds; hence the first contribution vanishes in the limit $ds \to 0$.

Appendix A: the transformation matrices M and D in a thin slice of a dipole

Hence, we find that, for a thin slice of dipole of length ds, the radiation effects can be represented by the matrices:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -\frac{C_\gamma}{2\pi}\left(\frac{1}{\rho^2}+2k_1\right)\frac{E_0^3}{\rho}ds & 0 & 0 & 0 & 0 & 1-2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^2}ds \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_q\gamma^2\frac{C_\gamma}{2\pi}\frac{E_0^3}{\rho^3}ds \end{pmatrix}$$

In the absence of betatron coupling, the vertical emittance may be calculated from:

$$\varepsilon_y = C_q \gamma^2 \frac{I_{5y}}{j_y I_2}$$

where:

$$I_{5y} = \oint \frac{\mathcal{H}_y}{\left|\rho^3\right|} ds$$

Assuming that the vertical dispersion is generated by random errors around the machine, we can make the approximation:

$$I_{5y} \approx \langle \mathcal{H}_y \rangle \oint \frac{1}{\left|\rho^3\right|} ds = \langle \mathcal{H}_y \rangle I_3$$

Hence, the expression for the vertical emittance becomes:

$$\varepsilon_y \approx C_q \gamma^2 \langle \mathcal{H}_y \rangle \frac{I_3}{j_y I_2} = \langle \mathcal{H}_y \rangle \frac{j_z}{j_y} \sigma_\delta^2$$

Now, compare the definition of the curly-H function:

$$\mathcal{H}_y = \gamma_y \eta_y^2 + 2\alpha_y \eta_y \eta_{py} + \beta_y \eta_{py}^2$$

with the action of a particle performing betatron oscillations:

$$2J_y = \gamma_y y^2 + 2\alpha_y y p_y + \beta_y p_y^2$$

Just as we can write the vertical coordinate in terms of action-angle variables:

$$y = \sqrt{2\beta_y J_y} \cos\varphi_y$$

so we can write the dispersion:

$$\eta_y = \sqrt{\beta_y \mathcal{H}_y} \cos\varphi_y$$

Hence:

$$\left\langle \eta_y^2 \right\rangle = \left\langle \beta_y \mathcal{H}_y \cos^2\varphi_y \right\rangle \approx \tfrac{1}{2} \left\langle \beta_y \right\rangle \left\langle \mathcal{H}_y \right\rangle \qquad \therefore \qquad \left\langle \mathcal{H}_y \right\rangle \approx 2 \frac{\left\langle \eta_y^2 \right\rangle}{\left\langle \beta_y \right\rangle}$$

Thus, we have:

$$\varepsilon_y \approx 2 \frac{j_z}{j_y} \frac{\left\langle \eta_y^2 \right\rangle}{\left\langle \beta_y \right\rangle} \sigma_\delta^2$$